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Convergence of an Iterative Scheme for Multifunctions on Fuzzy Metric Spaces

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Abstract. Recently, Reich and Zaslavski have studied a new inexact iterative scheme for fixed points of contractive and nonexpansive multifunctions. In 2011, Aleomraninejad, et. al. generalized some of their results to Suzuki-type multifunctions. The study of iterative schemes for various classes of contractive and nonexpansive mappings is a central topic in fixed point theory. The importance of Banach contraction principle is that it also gives the convergence of an iterative scheme to a unique fixed point. In this paper, we consider $(X, M, *)$ to be fuzzy metric spaces in Park's sense and we show our results for fixed points of contractive and nonexpansive multifunctions on Hausdorff fuzzy metric space.

1. Introduction

The study of iterative schemes for various classes of contractive and nonexpansive mappings is a central topic in metric fixed point theory. The study is started in 1922, with the work of Banach who proved a classical theorem, known as the Banach contraction principle, for the existence of a unique fixed point for a contraction [3]. The importance of this result is that it also gives the convergence of an iterative scheme to a unique fixed point. Many works have been published about fixed point theory for different kinds of contractions on some spaces such as quasi-metric spaces [5, 10], cone metric spaces [2, 21], partially ordered metric spaces [1, 4, 20], Menger spaces [14], and fuzzy metric spaces [8, 9]. The concept of fuzzy sets introduced by Zadeh in 1965 [25]. In 1975, Kramosil and Michalek introduced the notion of fuzzy metric

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spaces [12], and George and Veeramani modified the concept in 1994 [7]. Some researchers have been provided different fixed point results in fuzzy metric spaces [6, 11, 15, 16]. In this paper, we consider $(X, M, *)$ to be fuzzy metric spaces in Park's sense and by using their idea provide some fixed point results for the contractive mappings on complete fuzzy metric spaces.

2. Preliminaries

Here, we recall some basic notions.

A continuous, commutative and associative map $* : [0,1]^2 \to [0,1]$ is called a continuous t-norm whenever $a * 1 = a$ for all $a \in [0,1]$ and $a * b \leq c * d$ for all $a, b, c, d \in [0,1]$ with $a \leq c$ and $b \leq d$ [16]. For example, $a * b = ab$, $a * b = \min\{a, b\}$, $a * b = \max\{a + b - 1, 0\}$ and

$$a * b = \frac{ab}{\max\{a, b, \lambda\}}, \quad 0 < \lambda < 1,$$

are continuous t-norms.

Definition 2.1 ([16])**.** Let X be a non-empty set, $*$ a continuous be t-norm and M be a fuzzy set on $X^2 \times [0, \infty)$ such that $M(x, y, 0) = 0$, $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$, $M(x, y, t) = M(y, x, t)$,

$$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s),$$

for all $x, y, z \in X,\ s, t > 0$, $M(x, y, .) : [0, \infty) \longrightarrow [0, 1]$ is continuous, and

$$\lim_{t \to \infty} M(x, y, t) = 1,$$

for all $x, y \in X$. Then $(X, M, *)$ is called a fuzzy metric space.

Let $(X, M, *)$ be a fuzzy metric space. For each $x \in X$, $t > 0$ and $0 < r < 1$, set

$$B(x, r, t) = \{y \in X \,:\, M(x, y, t) > 1 - r\}.$$

Denote the generated topology by the sets $B(x, r, t)$ by τ_M. It has been proved that in a fuzzy metric space every compact set is closed and bounded [16]. A sequence $\{x_n\}$ in $(X, M, *)$ is said to be Cauchy whenever for each $\varepsilon > 0$ and $t > 0$, there exists a natural number n_0 such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$. Also, $(X, M, *)$ is called complete whenever every Cauchy sequence is convergent with respect to τ_M. The fuzzy metric M is triangular whenever

$$\frac{1}{M(x, y, t)} - 1 \leq \frac{1}{M(x, z, t)} - 1 + \frac{1}{M(z, y, t)} - 1,$$

for all $x, y, z \in X$ and $t > 0$. A self map f on a fuzzy metric space $(X, M, *)$ is called a Banach fuzzy contraction whenever there exists $k \in (0,1)$ such that

$$M(f(x), f(y), kt) \geq M(x, y, t),$$

for all $x, y \in X$ and $t > 0$ [18]. Let B be a nonempty subset of a fuzzy metric space $(X, M, *)$. According to [24], for $x \in X$ and $t > 0$, define

$$M(x, B, t) = \sup_{b \in B} M(x, b, t).$$

For a fuzzy metric space $(X, M, *)$, denote by $\mathcal{C}(X)$, $\mathcal{CB}(X)$ and $\mathcal{H}(X)$ the set of nonempty closed subsets, the set of nonempty closed bounded subsets and the set of nonempty compact subsets of (X, τ_M), respectively. Let B be a nonempty subset of a fuzzy metric space $(X, M, *)$, $x \in X$ and $t > 0$. In this case, H_M stands for the Hausdorff fuzzy metric space on $\mathcal{H} \times \mathcal{H} \times (0, \infty)$ which is defined by

$$H_M(A, B, t) = \min\left\{\inf_{a \in A} M(a, B, t), \inf_{b \in B} M(b, A, t)\right\},$$

for all $A, B \in \mathcal{H}$ and $t > 0$ [22].

3. Main Results

Now, we are ready to state and prove our main results. Throughout this paper, we suppose that 2^X is the family of all nonempty subsets of a fuzzy metric space $(X, M, *)$.

Theorem 3.1. *Let $(X, M, *)$ be a complete fuzzy metric space, $T : X \to \mathcal{C}(X)$ be a multifunction, and $\{\varepsilon_i\}_{i=0}^{\infty}$ and $\{\delta_i\}_{i=0}^{\infty}$ be two sequences in $(0, \infty)$ such that*

$$\sum_{i=0}^{\infty} \varepsilon_i < \infty,$$

and

$$\sum_{i=0}^{\infty} \delta_i < \infty.$$

Suppose that there exist $\alpha, \beta \in (0,1)$ such that $\alpha(3 - 2\alpha + \beta) \leq 1$ and

$$M\left(x, Tx, \frac{t}{\alpha}\right) \geq M(x, y, t) \quad \Rightarrow \quad H_M(Tx, Ty, t) \geq M\left(x, y, \frac{t}{\beta}\right),$$

for all $x, y \in X$. Let $T_i : X \to 2^X$ satisfies, for each integer $i \geq 0$, $H_M(Tx, T_i x, t) \geq 1 - \varepsilon_i$ for all $x \in X$. Assume that $x_0 \in X$ and for each integer $i \geq 0$,

$$\frac{\varepsilon_i}{t(1-\alpha)} \leq \frac{1}{M(x_i, x_{i+1}, t)} - 1$$

$$\leq \frac{1}{M(x, T_i x_i, t)} - 1 + \frac{\delta_i}{t},$$

for $x_{i+1} \in T_i x_i$. *Then* $\{x_i\}_{i=0}^{\infty}$ *converges to a fixed point of* T.

Proof. We first show that $\{x_i\}_{i=0}^{\infty}$ is a Cauchy sequence. To this end, let $i \geq 0$ be an integer. Then, we have

$$\begin{aligned}
\frac{1}{M(x_{i+1}, x_{i+2}, t)} - 1 &\leq \frac{1}{M(x_{i+1}, T_{i+1}x_{i+1}, t)} - 1 + \frac{\delta_{i+1}}{t} \\
&\leq \frac{1}{M(x_{i+1}, Tx_{i+1}, t)} - 1 \\
&\quad + \frac{1}{H_M(x_{i+1}, T_{i+1}x_{i+1}, t)} - 1 + \frac{\delta_{i+1}}{t} \\
&\leq \frac{1}{H_M(T_i x_i, Tx_{i+1}, t)} - 1 + \frac{\varepsilon_{i+1}}{t} + \frac{\delta_{i+1}}{t} \\
&\leq \frac{1}{H_M(T_i x_i, Tx_i, t)} - 1 \\
&\quad + \frac{1}{H_M(Tx_i, Tx_{i+1}, t)} - 1 + \frac{\varepsilon_{i+1}}{t} + \frac{\delta_{i+1}}{t}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{M(x_{i+1}, x_{i+2}, t)} - 1 &\leq \frac{1}{H_M(Tx_i, Tx_{i+1}, t)} - 1 \\
&\quad + \frac{\varepsilon_i + \varepsilon_{i+1} + \delta_{i+1}}{t},
\end{aligned} \tag{3.1}$$

for all $i \geq 0$. Since $\alpha(2-\alpha) < 1$,

$$\varepsilon_i \leq t(1-\alpha)\left(\frac{1}{M(x_i, x_{i+1}, t)} - 1\right),$$

and

$$\begin{aligned}
\frac{1}{M(Tx_i, x_i, t)} - 1 &\leq \frac{1}{M(x_i, T_i x_i, t)} - 1 + \frac{1}{H_M(T_i x_i, Tx_i, t)} - 1 \\
&\leq \frac{1}{M(x_i, x_{i+1}, t)} - 1 + \frac{\varepsilon_i}{t} \\
&\leq \frac{1}{M(x_i, x_{i+1}, t)} - 1 + (1-\alpha)\left(\frac{1}{M(x_i, x_{i+1}, t)} - 1\right) \\
&= (2-\alpha)\left(\frac{1}{M(x_i, x_{i+1}, t)} - 1\right).
\end{aligned}$$

We have

$$\alpha(\frac{1}{M(x_i, Tx_i, t)} - 1) < \frac{1}{M(x_i, x_{i+1}, t)} - 1,$$

and so

$$\frac{1}{H_M(Tx_i, Tx_{i+1}, t)} - 1 \leq \beta \left(\frac{1}{M(x_i, x_{i+1}, t)} - 1 \right). \tag{3.2}$$

Now, by using (3.1) and (3.2) we obtain

$$\begin{aligned} \frac{1}{M(x_{i+1}, x_{i+2}, t)} - 1 \leq \beta \left(\frac{1}{M(x_i, x_{i+1}, t)} - 1 \right) \\ + \frac{\varepsilon_i + \varepsilon_{i+1} + \delta_{i+1}}{t}, \end{aligned} \tag{3.3}$$

for all $i \geq 0$. Thus,

$$\begin{aligned} \frac{1}{M(x_1, x_2, t)} - 1 \leq \beta \left(\frac{1}{M(x_0, x_1, t)} - 1 \right) \\ + \frac{\varepsilon_0 + \varepsilon_1 + \delta_1}{t}, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \frac{1}{M(x_2, x_3, t)} - 1 \leq \beta^2 \left(\frac{1}{M(x_0, x_1, t)} - 1 \right) \\ + \beta \left(\frac{\varepsilon_0 + \varepsilon_1 + \delta_1}{t} \right) + \frac{\varepsilon_1 + \varepsilon_2 + \delta_2}{t}. \end{aligned} \tag{3.5}$$

Now, we show by induction that for each $n \geq 1$, we have

$$\begin{aligned} \frac{1}{M(x_n, x_{n+1}, t)} - 1 \leq \beta^n \left(\frac{1}{M(x_0, x_1, t)} - 1 \right) \\ + \sum_{i=0}^{n-1} \frac{\beta^i}{t} (\varepsilon_{n-i} + \varepsilon_{n-i-1} + \delta_{n-i}). \end{aligned} \tag{3.6}$$

In view of (3.4) and (3.5), inequality (3.6) holds for $n = 1, 2$. Assume that $k \geq 1$ is an integer and (3.6) holds for $n = k$. By using 3.3, we have

$$\begin{aligned} \frac{1}{M(x_{k+1}, x_{k+2}, t)} - 1 &\leq \beta \left(\frac{1}{M(x_k, x_{k+1}, t)} - 1 \right) + \frac{\varepsilon_k + \varepsilon_{k+1} + \delta_{k+1}}{t} \\ &\leq \beta^{k+1} \left(\frac{1}{M(x_0, x_1, t)} - 1 \right) \\ &\quad + \beta \sum_{i=0}^{k-1} \frac{\beta^i}{t} (\varepsilon_{k-i} + \varepsilon_{k-i-1} + \delta_{k-i}) \\ &\quad + \frac{\varepsilon_k + \varepsilon_{k+1} + \delta_{k+1}}{t} \\ &= \beta^{k+1} \left(\frac{1}{M(x_0, x_1, t)} - 1 \right) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{k}\frac{\beta^i}{t}(\varepsilon_{k-i+1}+\varepsilon_{k-i}+\delta_{k-i+1})\\
&+\frac{\varepsilon_k+\varepsilon_{k+1}+\delta_{k+1}}{t}\\
=\ &\beta^{k+1}\left(\frac{1}{M(x_0,x_1,t)}-1\right)\\
&+\sum_{i=0}^{k}\frac{\beta^i}{t}(\varepsilon_{k-i+1}+\varepsilon_{k-i}+\delta_{k-i+1}).
\end{aligned}
$$

This implies that (3.6) holds for all $n\geq 1$. Now, by using (3.6) we obtain

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{M(x_n,x_{n+1},t)}-1 &\leq \sum_{n=1}^{\infty}\beta^n\left(\frac{1}{M(x_0,x_1,t)}-1\right)\\
&\quad+\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\frac{\beta^i}{t}(\varepsilon_{n-i}+\varepsilon_{n-i-1}+\delta_{n-i})\\
&=\sum_{n=1}^{\infty}\beta^n\left(\frac{1}{M(x_0,x_1,t)}-1\right)\\
&\quad+\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{\beta^{n-i}}{t}(\varepsilon_i+\varepsilon_{i-1}+\delta_i)\\
&\leq\sum_{n=1}^{\infty}\beta^n\left(\frac{1}{M(x_0,x_1,t)}-1\right)\\
&\quad+\frac{\beta^0}{t}(\varepsilon_1+\varepsilon_0+\delta_1)+\frac{\beta^0}{t}(\varepsilon_2+\varepsilon_1+\delta_2)\\
&\quad+\frac{\beta^1}{t}(\varepsilon_1+\varepsilon_0+\delta_1)+\frac{\beta^0}{t}(\varepsilon_3+\varepsilon_2+\delta_3)\\
&\quad+\frac{\beta^1}{t}(\varepsilon_2+\varepsilon_1+\delta_2)+\frac{\beta^2}{t}(\varepsilon_1+\varepsilon_0+\delta_1)+\cdots\\
&=\sum_{n=1}^{\infty}\beta^n\left(\frac{1}{M(x_0,x_1,t)}-1\right)\\
&\quad+\left(\frac{\beta^0+\beta^1+\beta^2+\cdots}{t}\right)(\varepsilon_1+\varepsilon_0+\delta_1)\\
&\quad+\left(\frac{\beta^0+\beta^1+\beta^2+\cdots}{t}\right)(\varepsilon_2+\varepsilon_1+\delta_2)\\
&\quad+\left(\frac{\beta^0+\beta^1+\beta^2+\cdots}{t}\right)(\varepsilon_3+\varepsilon_2+\delta_3)+\cdots
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\sum_{n=1}^{\infty}\beta^n\right)\left(\frac{1}{M(x_0,x_1,t)}-1\right) \\
&\quad + \sum_{i=1}^{\infty}\left(\frac{\varepsilon_i+\varepsilon_{i-1}+\delta_i}{t}\right) \\
&< \infty.
\end{aligned}
$$

Thus, $\{x_i\}_{i=0}^{\infty}$ is a Cauchy sequence and so there exists $x \in X$ such that $x = \lim_{n\to\infty} x_n$. Now, we claim that for each $n \geq 1$ either

$$M\left(x_n, Tx_n, \frac{t}{\alpha}\right) \geq M(x_n, x, t),$$

or

$$M\left(x_{n+1}, Tx_{n+1}, \frac{t}{\alpha}\right) \geq M(x_{n+1}, x, t),$$

holds. If $M\left(x_n, Tx_n, \frac{t}{\alpha}\right) \geq M(x_n, x, t)$ and

$$M\left(x_{n+1}, Tx_{n+1}, \frac{t}{\alpha}\right) \geq M(x_{n+1}, x, t),$$

for some $n \geq 1$, then we obtain

$$
\begin{aligned}
\frac{1}{M(x_{n+1},x_n,t)}-1 &\leq \frac{1}{M(x_{n+1},x,t)}-1+\frac{1}{M(x,x_n,t)}-1 \\
&< \alpha\left(\frac{1}{M(x_{n+1},Tx_{n+1},t)}-1\right) \\
&\quad + \alpha\left(\frac{1}{M(x_n,Tx_n,t)}-1\right) \\
&\leq \alpha\left[\left(\frac{1}{H_M(T_nx_n,Tx_{n+1},t)}-1\right)\right. \\
&\quad \left. +(2-\alpha)\left(\frac{1}{M(x_n,x_{n+1},t)}-1\right)\right] \\
&\leq \alpha\left[\left(\frac{1}{H_M(T_nx_n,Tx_n,t)}-1\right)\right. \\
&\quad + \left(\frac{1}{H_M(Tx_n,Tx_{n+1},t)}-1\right) \\
&\quad \left. +(2-\alpha)\left(\frac{1}{M(x_n,x_{n+1},t)}-1\right)\right] \\
&\leq \alpha\left[\frac{\varepsilon_n}{t}+\beta\left(\frac{1}{M(x_n,x_{n+1},t)}-1\right)\right. \\
&\quad \left. +(2-\alpha)\left(\frac{1}{M(x_n,x_{n+1},t)}-1\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le \alpha \left[(1-\alpha)\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right)\right. \\
&\quad + \beta \left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right) \\
&\quad \left. +(2-\alpha)\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right)\right] \\
&= \alpha(3 - 2\alpha + \beta)\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right),
\end{aligned}
$$

because

$$\varepsilon_n \le t(1-\alpha)\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right).$$

It implies that $\alpha(3 - 2\alpha + \beta) > 1$, which is a contradiction. Hence, our claim is proved. Thus, by using the assumption of the theorem, for each $n \ge 1$, either

$$H_M(Tx_n, Tx, t) \ge M\left(x_n, x, \frac{t}{\beta}\right),$$

or

$$H_M(Tx_{n+1}, Tx, t) \ge M\left(x_{n+1}, x, \frac{t}{\beta}\right),$$

holds Therefore, one of the following cases holds.

(i) There exists an infinite subset $I \subseteq \mathbb{N}$ such that

$$H_M(Tx_n, Tx, t) \ge M\left(x_n, x, \frac{t}{\beta}\right),$$

for all $n \in I$.

(ii) There exists an infinite subset $J \subseteq \mathbb{N}$ such that

$$H_M(Tx_{n+1}, Tx, t) \ge M\left(x_{n+1}, x, \frac{t}{\beta}\right),$$

for all $n \in J$.

In case (i), we obtain

$$
\begin{aligned}
\frac{1}{M(x, Tx, t)} - 1 &\le \frac{1}{M(x, x_n, t)} - 1 + \frac{1}{M(x_n, Tx, t)} - 1 \\
&\le \frac{1}{M(x, x_n, t)} - 1 + \frac{1}{M(x_n, Tx_n, t)} - 1 \\
&\quad + \frac{1}{H_M(Tx_n, Tx, t)} - 1 \\
&\le \frac{1}{M(x, x_n, t)} - 1 + \frac{1}{\alpha}\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right)
\end{aligned}
$$

$$+\beta\left(\frac{1}{M(x,x_n,t)}-1\right),$$

for all $n \in I$, and so $M(x,Tx,t)=1$. Hence $x \in Tx$. Similar to (i), we obtain $x \in Tx$, from case (ii). This completes the proof. $\square$

The following example shows that there are some multifunctions which satisfy the assumption of Theorem 3.1 while there are not contractive multifunctions.

Example 3.2. Let $X=[-4,3]\cup\{0\}\cup[3,4]$, $M(x,y,t)=\frac{t}{t+|x-y|}$ and $T: X \to \mathcal{C}(X)$ be defined by

$$T(x)=\begin{cases}\left[3,\frac{5(-x)-6}{-x}\right], & -4 \le x < -3.4,\\ \{0\}, & x \in [-3.4,-3]\cup\{0\}\cup[3,3.4],\\ \left\{-\frac{5x-6}{x}\right\}, & 3.4 < x \le 4.\end{cases}$$

We show that T satisfies the assumption of Theorem 3.1 for $\alpha=\frac{2}{7}$ and $\beta=\frac{90}{91}$ while T is not a contractive multifunction. If $3.4 < x \le 4$, then

$$3 < 3+\frac{2x-6}{x}=\frac{5x-6}{x}\le\frac{7}{2},$$

and

$$\frac{90}{91}x-\frac{5x-6}{x}>0.$$

If $-4 \le x < -3.4$, then

$$M(0,Tx,t)=\frac{t}{t+3}>\frac{t}{t-\frac{90}{91}x}=\frac{t}{t+\frac{90}{91}|x|},$$

$Tx \subset [3,3.5]$ and

$$H_M(\{0\},Tx,t)=M(0,Tx,t)=\frac{t}{t+\frac{5(-x)-6}{-x}}>\frac{t}{t-\frac{90}{91}x}=\frac{t}{t+\frac{90}{91}|x|}.$$

If $3.4 < x \le 4$, then $Tx \subset [-3.5,-3)$ and

$$H_M(\{0\},Tx,t)=M(0,Tx,t)=\frac{t}{t+\frac{5x-6}{x}}>\frac{t}{t+\frac{90}{91}x}=\frac{t}{t+\frac{90}{91}|x|}.$$

Thus,

$$H_M\left(Tx,Ty,\frac{90}{91}t\right)>M(x,y,t),$$

whenever $x=0$ and $y \ne 0$, or $y=0$ and $x \ne 0$. If $x \in [3,4]$ and $y \in [-4,-3]$, then

$$M\left(x,Tx,\frac{7}{2}t\right)\ge\frac{t}{t+\frac{2\times*6.5}{7}}>\frac{t}{t+6}\ge M(x,y,t),$$

and so

$$\frac{1}{H_M(Tx,Ty,t)}-1 \leq \frac{1}{M(Tx,0,t)}-1+\frac{1}{H_M(Ty,\{0\},t)}-1$$
$$\leq \frac{90}{91}\left(\frac{|x|+|y|}{t}\right)$$
$$= \frac{90}{91}\left(\frac{1}{M(x,y,t)}-1\right).$$

If $x,y \in [3,3.4]$ or $x,y \in [-3.4,-3]$, then

$$M\left(x,Tx,\frac{7}{2}t\right) \geq \frac{t}{t+\frac{2*3}{7}} > M(x,y,t).$$

If $x \in [3,3.4]$ and $y \in [3.4,6]$, or $x \in [-3.4,-3]$ and $y \in [-4,-3.4]$, or $y \in [3,3.4]$ and $x \in [3.4,4]$, or $y \in [-3.4,-3]$ and $x \in [-4,-3.4]$, then we have

$$M\left(x,Tx,\frac{7}{2}t\right) \geq \frac{t}{t+\frac{2*3}{7}} > \frac{t}{t+1} \geq M(x,Tx,t).$$

If $x=3$ and $y=4$, then

$$H_M(Tx,Ty,t)=M(x,y,t)=\frac{t}{t+\frac{7}{2}} < M\left(x,y,\frac{91}{90}t\right).$$

Lemma 3.3. *Let $x \in X$, F be a nonempty closed subset of X, p be a natural number, $\delta > 0$, $\{x_0,x_1,\dots,x_p\} \subset X$ such that $x_0 = x$ and $x_{i+1} \in T_i x_i$ $(i=0,1,\dots,p-1)$. Then there is a natural number $q>p$ and $\{x_p,x_{p+1},\dots,x_q\} \subset X$ such that $x_{i+1} \in T_i x_i$ $(i=p,\dots,q-1)$ and $M(x_q,F,t) \geq 1-\delta$.*

Proof. Choose a natural number $p_1 > p$ such that $\sum_{i=p_1}^{\infty} \varepsilon_i < \frac{\delta}{8}$ and a sequence $\{x_p,x_{p+1},\dots,x_{p_1}\} \subset X$ such that $x_{i+1} \in T_i x_i$ $(i=p,\dots,p_1-1)$. By using the assumptions of the lemma, there is a sequence $\{y_i\}_{i=p_1}^{\infty} \subset X$, such that $y_{p_1}=x_{p_1}, y_{i+1} \in Ty_i$ for all $i \geq p_1$ and $\lim_{i\to\infty} M(y_i,F,t)=1$. Now, we define by induction a sequence $\{x_i\}_{i=p_1}^{\infty} \subset X$. To this end, assume that $k \geq p_1$ is an integer and we have already defined $x_i \in X$, $i=p_1,\dots,k$, such that $x_{i+1} \in T_i x_i$ $(i=p_1,\dots,k-1)$ and

$$\frac{1}{\max\{M(x_k,y_k,t),M(x_k,y_{k+1},t)\}}-1 \leq \sum_{i=p_1}^{k-1}\frac{\varepsilon_i}{t}.$$

Clearly this assumption holds for $k=p_1$. Since $y_{k+1} \in Ty_k$, by using a similar proof as the one for Theorem 3.1, it is easy to show that for each k, one of the following cases holds:

(i) $M(y_k,Ty_k,\frac{t}{\alpha}) \geq M(y_k,x_k,t)$
(ii) $M\left(y_{k+1},Ty_{k+1}.\frac{t}{\alpha}\right) \geq M(y_{k+1},x_k,t)$.

By case (i), we obtain $H_M(Ty_k, Tx_k, t) \geq M(x_k, y_k, t)$ and so

$$M(y_{k+1}, Tx_k, t) \geq M(x_k, y_k, t).$$

Hence, there exists $\widetilde{y}_{k+1} \in Tx_k$ such that

$$\frac{1}{M(y_{k+1}, \widetilde{y}_{k+1}, t)} - 1 \leq \frac{1}{M(x_k, y_k, t)} - 1 + \frac{\varepsilon_k}{t}.$$

This implies that

$$\frac{1}{M(\widetilde{y}_{k+1}, T_k x_k, , t)} - 1 \leq \frac{1}{H_M(Tx_k, T_k x_k, t)} - 1 \leq \frac{\varepsilon_k}{t},$$

and so there exists $x_{k+1} \in T_k x_k$ such that

$$\frac{1}{M(\widetilde{y}_{k+1}, x_{k+1}, t)} - 1 \leq \frac{2\varepsilon_k}{t}.$$

Thus, we have

$$\begin{aligned}
\frac{1}{M(x_{k+1}, y_{k+1}, t)} - 1 &\leq \frac{1}{M(x_{k+1}, \widetilde{y}_{k+1}, t)} - 1 + \frac{1}{M(y_{k+1}, \widetilde{y}_{k+1}, t)} - 1 \\
&\leq \frac{2\varepsilon_k}{t} + \left(\frac{1}{M(x_k, y_k, t)} - 1\right) + \frac{\varepsilon}{t} \\
&= \frac{3\varepsilon_k}{t} + \left(\frac{1}{M(x_k, y_k, t)} - 1\right).
\end{aligned}$$

By case (ii), we obtain

$$H_M(Ty_{k+1}, Tx_k, t) \geq M(x_k, y_{k+1}, t),$$

and so $M(y_{k+2}, Tx_k, t) \geq M(x_k, y_{k+1}, t)$. Hence, there exists $\widetilde{y}_{k+1} \in Tx_k$ such that

$$\frac{1}{M(y_{k+2}, \widetilde{y}_{k+1}, t)} - 1 \leq \frac{1}{M(x_k, y_{k+1}, t)} - 1 + \frac{\varepsilon_k}{t}.$$

This implies that

$$\frac{1}{M(\widetilde{y}_{k+1}, T_k x_k, t)} - 1 \leq \frac{1}{H_M(Tx_k, T_k x_k, t)} - 1 \leq \frac{\varepsilon_k}{t},$$

and so there exists $x_{k+1} \in T_k x_k$ such that

$$\frac{1}{M(\widetilde{y}_{k+1}, x_{k+1}, t)} - 1 \leq \frac{2\varepsilon_k}{t}.$$

Thus, we have

$$\begin{aligned}
\frac{1}{M(x_{k+1}, y_{k+2}, t)} - 1 &\leq \frac{1}{M(x_{k+1}, \widetilde{y}_{k+1}, t)} - 1 + \frac{1}{M(y_{k+1}, \widetilde{y}_{k+2}, t)} - 1 \\
&\leq \frac{2\varepsilon_k}{t} + \left(\frac{1}{M(x_k, y_{k+1}, t)} - 1\right) + \frac{\varepsilon}{t}
\end{aligned}$$

$$= \frac{3\varepsilon_k}{t} + \left(\frac{1}{M(x_k, y_{k+1}, t)} - 1\right).$$

Thus, by considering the above two cases, we have

$$\begin{aligned}
&\frac{1}{\max\{M(x_{k+1}, y_{k+1}, t), M(x_{k+1}, y_{k+2}, t)\}} - 1 \\
&\qquad \leq \frac{1}{\max\{M(x_k, y_k, t), M(x_k, y_{k+1}, t)\}} - 1 + \frac{3\varepsilon}{t} \\
&\qquad \leq \sum_{i=p_1}^{k-1} \frac{\varepsilon_i}{t} + \frac{3\varepsilon_k}{t} \\
&\qquad = 3 \sum_{i=p_1}^{k} \frac{\varepsilon_i}{t}.
\end{aligned}$$

Therefore, we have indeed defined by induction a sequence $\{x_i\}_{i=p_1}^{\infty} \subset X$ such that $x_{i+1} \in Tx_i$ $(i = p_1, \ldots)$ and

$$\frac{1}{\max\{M(x_k, y_k, t), M(x_k, y_{k+1}, t)\}} - 1 \leq \sum_{i=p_1}^{k-1} \frac{\varepsilon_i}{t}.$$

Hence, there exists an integer $q > p_1 + 2$ such that $M(y_q, F, t) > 1 - \frac{\delta}{4}$ and $M(y_{q+1}, F, t) > 1 - \frac{\delta}{4}$. Thus, we obtain

$$\begin{aligned}
\frac{1}{M(x_q, F, t)} - 1 &\leq \frac{1}{M(x_q, y_q, t)} - 1 + \frac{1}{M(y_q, F, t)} - 1 \\
&\leq \frac{1}{M(x_q, y_q, t)} - 1 + \frac{\delta}{4t},
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
\frac{1}{M(x_q, F, t)} - 1 &\leq \frac{1}{M(x_q, y_{q+1}, t)} - 1 + \frac{1}{M(y_{q+1}, F, t)} - 1 \\
&\leq \frac{1}{M(x_q, y_{q+1}, t)} - 1 + \frac{\delta}{4t}.
\end{aligned} \tag{3.8}$$

Combining (3.7) with (3.8) implies that

$$\begin{aligned}
\frac{1}{M(x_q, F, t)} - 1 &\leq \frac{1}{\max\{M(x_q, y_q, t), M(x_q, y_{q+1}, t)\}} - 1 + \frac{\delta}{4t} \\
&\leq \sum_{i=p_1}^{k-1} \frac{\varepsilon_i}{t} + \frac{\delta}{4t} \leq \frac{\delta}{8t} + \frac{\delta}{4t}.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 3.4. *Let* $\{x_i\}_{i=0}^{\infty}$ *be a sequence in* X, $x_{i+1} \in T_i x_i$ *for all* $i \geq 0$, $\delta > 0$, p *be a natural number,* F *be a nonempty closed subset of* X, $M(x_p, F, t) \geq 1 - \delta$, *and* $\sum\limits_{i=p}^{\infty} \varepsilon_i < \delta$. *Then* $M(x_i, F, t) \geq 1 - 3\delta$ *for all* $i \geq p$.

Proof. We intend to show by induction that

$$\frac{1}{M(x_n, F, t)} - 1 \leq \frac{\delta}{t} + \sum_{i=p}^{n-1} \frac{2\varepsilon_i}{t}, \tag{3.9}$$

for all $n \geq p$. Clearly, (3.9) holds for $n = p$. Assume that (3.9) holds for $n \geq p$. Then there exists $y_n \in F$ such that

$$\frac{1}{M(x_n, y_n, t)} - 1 \leq \frac{\delta}{t} + \sum_{i=p}^{n-1} \frac{2\varepsilon_i}{t} + \frac{2\varepsilon_n}{4t}.$$

By assumption, $M(x_{n+1}, Tx_n, t) \geq 1 - \varepsilon_n$ and so there exists $\widetilde{x}_{n+1} \in Tx_n$ such that $M(x_{n+1}, \widetilde{x}_{n+1}, t) \geq 1 - \frac{3\varepsilon_n}{2}$. If $x_n \in F$ then

$$\frac{1}{M(x_{n+1}, F, t)} - 1 \leq \frac{1}{M(x_{n+1}, Tx_n, t)} - 1 \leq \frac{\varepsilon_n}{t} \leq \frac{\delta}{t} + \sum_{i=p}^{n} \frac{2\varepsilon_i}{t}.$$

On the other hand,

$$\begin{aligned}
\alpha \left(\frac{1}{M(y_n, Ty_n, t)} - 1 \right) &\leq \alpha \left[\frac{1}{M(x_n, y_n, t)} - 1 + \frac{1}{M(x_n, Ty_n, t)} - 1 \right] \\
&\leq 2\alpha \left(\frac{1}{M(x_n, y_n, t)} - 1 \right) \\
&< \frac{1}{M(x_n, y_n, t)} - 1,
\end{aligned}$$

and so $H_M(Tx_n, Ty_n, t) \geq M(x_n, y_n, t)$. Hence, there exists $\widetilde{y}_{n+1} \in Ty_n$ such that

$$\frac{1}{M(\widetilde{y}_{n+1}, \widetilde{x}_{n+1}, t)} - 1 \leq \frac{1}{M(x_n, y_n, t)} - 1 + \frac{\varepsilon_n}{4},$$

and so

$$\begin{aligned}
\alpha \left(\frac{1}{M(x_{n+1}, F, t)} - 1 \right) &\leq \frac{1}{M(x_{n+1}, \widetilde{y}_{n+1}, t)} - 1 \\
&\leq \frac{1}{M(x_{n+1}, \widetilde{x}_{n+1}, t)} - 1 + \frac{1}{M(\widetilde{y}_{n+1}, \widetilde{x}_{n+1}, t)} - 1 \\
&\leq \frac{7\varepsilon_n}{4t} + \frac{1}{M(x_n, y_n, t)} - 1
\end{aligned}$$

$$\leq \frac{\delta}{t} + \sum_{i=p}^{n} \frac{2\varepsilon_i}{t}.$$

This implies that

$$\frac{1}{M(x_n, F, t)} - 1 \leq \frac{\delta}{t} + 2\sum_{i=p}^{n-1} \frac{\varepsilon_i}{t} \leq \frac{\delta}{t} + 2\sum_{i=p}^{\infty} \frac{\varepsilon_i}{t} \leq \frac{\delta}{t} + \frac{2\delta}{t} = \frac{3\delta}{t} < 3\delta,$$

for all $n > p$. □

Theorem 3.5. *Let $(X, M, *)$ be a complete fuzzy metric space, F be a nonempty closed subset of X, $T : X \to \mathcal{C}(X)$ be a multifunction, and $\{\varepsilon_i\}_{i=0}^{\infty}$ be a sequence in $(0, \infty)$ such that $\sum_{i=0}^{\infty} \varepsilon_i < \infty$. Suppose that $T(F) \subset F$, $M(x, Ty, t) \geq M(x, y, t)$ for every $x \in F^c$, $y \in F$, and there exists $\alpha \in (0, \frac{1}{2})$ such that*

$$M(x, Tx, \frac{t}{\alpha}) \geq M(x, y, t) \quad \Rightarrow \quad H_M(Tx, Ty, t) \geq M(x, y, t),$$

for all $x, y \in X$. Let $T_i : X \to 2^X$ satisfy, for each integer $i \geq 0$, $H_M(Tx, T_i x, t) \geq 1 - \varepsilon_i$ for all $x \in X$. Assume that for each $x \in X$ there exists a sequence $\{x_i\}_{i=0}^{\infty}$ in X such that $x_0 = x$, $x_{i+1} \in Tx_i$ for all $i \geq 0$ and $\lim_{i\to\infty} M(x_i, F, t) = 1$. Then for each $x \in X$ there exists a sequence $\{x_i\}_{i=0}^{\infty}$ in X such that $x_0 = x$, $x_{i+1} \in T_i x_i$ for all $i \geq 0$ and $\lim_{i\to\infty} M(x_i, F, t) = 1$.

Proof. Let $x \in X$. By using Lemma 3.3, there exist a sequence $\{x_i\}_{i=0}^{\infty}$ and a strictly increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that $x_{i+1} \in T_i x_i$ for all $i \geq 0$ and for each $k \geq 1$,

$$M(x_{n_k}, F, t) \geq 1 - 2^{-k},$$

and $\sum_{i=n_k}^{\infty} \varepsilon_i < 2^{-k}$. Hence, by using Lemma 3.4, we have

$$\lim_{n\to\infty} M(x_n, F, t) = 1.$$

This completes the proof. □

The following example of Suzuki [23] shows that T satisfies the assumptions of Theorem 3.5 for $\alpha = \frac{1}{2}$ while T is not a nonexpansive multifunction.

Example 3.6. Let $X = \{(0, 0), (6, 0), (0, 6), (6, 7), (7, 6)\}$,

$$M(x, y, t) = \frac{t}{t + d(x, y)},$$

where the metric d is defined by $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$. Define T on X by

$$T(x) = \begin{cases} \{(0, x_1)\}, & x_1 \leq x_2, \\ \{(x_2, 0)\}, & x_2 < x_1. \end{cases}$$

Put $F = \{(0,0)\}$. Then T Satisfies the assumptions of Theorems 3.5 while T is not a nonexpansive multifunction. First note that,

$$H_M(Tx, Ty, t) \geq M(x, y, t),$$

if $(x, y) \neq ((3,4),(4,3))$ and $(x, y) \neq ((4,3),(3,4))$. Also, for each $x \in X$ there exists a sequence $\{x_i\}_{i=0}^{\infty}$ in X such that $x_0 = x$, $x_{i+1} \in Tx_i$ for all $i \geq 0$, $\lim_{i\to\infty} \frac{1}{M(x_i,F,t)} - 1 = 0$ and $M(x, Ty, t) \geq M(x, y, t)$ for all $x \in F^c$ and $y \in F$. Thus,

$$M\left((6,7), T((6,7)), \frac{1}{2}t\right) < \frac{t}{t+\frac{7}{2}} < \frac{t}{t+2} = M\left((6,7),(7,6),t\right),$$
$$M\left((7,6), T((7,6)), \frac{1}{2}t\right) < \frac{t}{t+\frac{7}{2}} < \frac{t}{t+2} = M\left((7,6),(6,7),t\right).$$

Hence, T satisfies the assumptions of Theorem 3.5 while T is not a nonexpansive multifunction because

$$H_M(T((6,7)), T((7,6)), t) = \frac{t}{t+12} < \frac{t}{t+2} = M((6,7),(7,6),t).$$

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