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Topological Centers and Factorization of Certain Module Actions

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Abstract. Let A be a Banach algebra and X be a Banach A-bimodule with the left and right module actions $\pi_\ell : A \times X \to X$ and $\pi_r : X \times A \to X$, respectively. In this paper, we study the topological centers of the left module action $\pi_{\ell_n} : A \times X^{(n)} \to X^{(n)}$ and the right module action $\pi_{r_n} : X^{(n)} \times A \to X^{(n)}$, which inherit from the module actions π_ℓ and π_r, and also the topological centers of their adjoints, from the factorization property point of view, and then, we investigate conditions under which these bilinear maps are Arens regular or strongly Arens irregular.

1. Introduction

Let $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ be a bounded bilinear map on normed spaces. According to [1] and [2], we have two natural extensions f^{***} and f^{t***t} from $\mathcal{X}^{**} \times \mathcal{Y}^{**}$ to $\mathcal{Z}^{**}$, where the adjoint $f^* : \mathcal{Z}^* \times \mathcal{X} \to \mathcal{Y}^*$ of f is defined by

$$\langle f^*(\nu, x), y\rangle = \langle \nu, f(x,y)\rangle\,, \quad (x \in \mathcal{X}, y \in \mathcal{Y} \text{ and } \nu \in \mathcal{Z}^*)\,.$$

Also we define $f^{**} = (f^*)^*$ and $f^{***} = (f^{**})^*$.
It can be readily verified that f^{***} is the unique extension of f for which the maps

$$\cdot \mapsto f^{***}(\cdot, G), \quad \cdot \mapsto f^{***}(x, \cdot), \quad (x \in \mathcal{X}, G \in \mathcal{Y}^{**}),$$

are w^*-w^*-continuous.

Let $f^t : \mathcal{Y} \times \mathcal{X} \longrightarrow \mathcal{Z}$ be the flip map of f which is defined by $f^t(y,x) = f(x,y), (x \in \mathcal{X}, y \in \mathcal{Y})$. If we continue the latter process with f^t instead

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of f, we come to the bounded bilinear map $f^{t***t}: \mathcal{X}^{**} \times \mathcal{Y}^{**} \to \mathcal{Z}^{**}$, which is the unique extension of f such that the maps

$$\cdot \mapsto f^{t***t}(F, \cdot), \quad \cdot \mapsto f^{t***t}(\cdot, y), \quad (y \in \mathcal{Y}, F \in \mathcal{X}^{**}),$$

are w^*-w^*-continuous.

We define the left and right topological centers of f by

$$Z_\ell(f) = \{F \in \mathcal{X}^{**}; f^{***}(F,G) = f^{t***t}(F,G) \text{ for every } G \in \mathcal{Y}^{**}\},$$

and

$$Z_r(f) = \{G \in \mathcal{Y}^{**}; f^{***}(F,G) = f^{t***t}(F,G) \text{ for every } \ F \in \mathcal{X}^{**}\},$$

respectively. Clearly, $\mathcal{X} \subseteq Z_\ell(f), \mathcal{Y} \subseteq Z_r(f)$ and $Z_r(f) = Z_\ell\left(f^t\right)$.

A bounded bilinear map f is said Arens regular if $f^{***} = f^{t***t}$. This is equivalent to $Z_\ell(f) = \mathcal{X}^{**}$ as well as $Z_r(f) = \mathcal{Y}^{**}$. The map f is called left (right) strongly Arens irregular if $Z_\ell(f) = \mathcal{X}$ $(Z_r(f) = \mathcal{Y})$.

In the case where π is the multiplication of a Banach algebra $\mathcal{A}$, π^{***} and π^{t***t} are actually the first and second Arens products, respectively. In this case, we write $Z_\ell(\mathcal{A}^{**})$ and $Z_r(\mathcal{A}^{**})$ instead of $Z_\ell(\pi)$ and $Z_r(\pi)$, respectively. We also say that $\mathcal{A}$ is Arens regular, left strongly Arens irregular, and right strongly Arens irregular if the multiplication π of $\mathcal{A}$ has the corresponding property.

Let now $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule, and let $\pi_\ell: \mathcal{A} \times \mathcal{X} \to \mathcal{X}$ and $\pi_r: \mathcal{X} \times \mathcal{A} \to \mathcal{X}$ be the left and right module actions of $\mathcal{A}$ on $\mathcal{X}$. Then $\mathcal{X}^*$ is a Banach $\mathcal{A}$-bimodule with module actions π_r^{t*t} and π_ℓ^*. Similarly, for every positive integer n, $\mathcal{X}^{(n)}$ is a Banach $\mathcal{A}$-bimodule with the left module action $\pi_{\ell_n}: \mathcal{A} \times \mathcal{X}^{(n)} \to \mathcal{X}^{(n)}$ and the right module action $\pi_{r_n}: \mathcal{X}^{(n)} \times \mathcal{A} \to \mathcal{X}^{(n)}$ where $\pi_{\ell_n} = \pi_{r_{n-1}}^{t*t}$ and $\pi_{r_n} = \pi_{\ell_{n-1}}^*$. Also we put $\pi_{\ell_0} = \pi_\ell$ and $\pi_{r_0} = \pi_r$.

It is easy to see that

$$Z_\ell\left(\pi_{r_{n-2}}\right) \subseteq Z_\ell\left(\pi_{r_n}\right), Z_r\left(\pi_{\ell_{n-2}}\right) \subseteq Z_r\left(\pi_{\ell_n}\right),$$

and π_{r_n} and π_{ℓ_n} are extensions of $\pi_{r_{n-2}}$ and $\pi_{\ell_{n-2}}$, respectively.

We say that a bilinear mapping factors if it is onto. Many analysts have studied factorization of some module actions and their adjoints. For example in [6] the cases that the second adjoint of a module action factors and also factorization property of some module actions when A is an ideal in its second dual are studied. Also in [7] the factorization property of some approximately unital Banach modules and also the factorization property of the right module action when the left module action factors and vice versa are studied. In this paper, we will study the topological centers of some module actions or their adjoints from the factorization point of view.

2. Factorization Property and Arens Regularity

In this section, we will study the relation between topological centers and factorization property of module actions π_{ℓ_n} and π_{r_n} or their adjoints. The following lemma is useful in this study.

Lemma 2.1. *For each integer $n \geq 2$,*

(i) $\pi^{***}_{\ell_{n-2}}$ *and* $\pi^{t***t}_{\ell_{n-2}}$ *are extensions of* π_{ℓ_n}.
(ii) $\pi^{***}_{r_{n-2}}$ *and* $\pi^{t***t}_{r_{n-2}}$ *are extensions of* π_{r_n}.

Proof. We only prove part (i).
Let $\varphi \in \mathcal{X}^{(n)}$, and $\lambda \in \mathcal{X}^{(n-1)}$ and $a \in \mathcal{A}$. Consider a net (ν_α) in $\mathcal{X}^{(n-2)}$ such that it is w^*-convergent to φ. Then we have

$$\begin{aligned} \langle \pi_{\ell_n}(a, \varphi), \lambda \rangle &= \left\langle \varphi, \pi_{r_{n-1}}(\lambda, a) \right\rangle \\ &= \lim_\alpha \left\langle \pi_{r_{n-1}}(\lambda, a), \nu_\alpha \right\rangle \\ &= \lim_\alpha \left\langle \lambda, \pi_{\ell_{n-2}}\left(a, \nu_\alpha\right) \right\rangle \\ &= \left\langle \pi^{***}_{\ell_{n-2}}(a, \varphi), \lambda \right\rangle \\ &= \left\langle \pi^{t***t}_{\ell_{n-2}}(a, \varphi), \lambda \right\rangle . \end{aligned}$$

□

The following example gives some simple examples of bilinear maps which factor.

Example 2.2. (i) Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{X}$ be a left unital Banach $\mathcal{A}$-module with module action π_ℓ. Then π_{ℓ_n} and $\pi^*_{\ell_n}$ factor for each non negative integer n. For example if H is a Hilbert space and $\pi_\ell : B(H) \times K(H) \to K(H)$ is defined by $\pi_\ell(T, V) = T \circ V$, for each $T \in B(H)$ and $V \in K(H)$ or $\pi_\ell : \ell^\infty \times c_0 \to c_0$, which is defined by $\pi_\ell\left((a_\alpha), (b_\beta)\right) = (a_\alpha)(b_\beta)$ for each nets $(a_\alpha) \in \ell^\infty$ and $(b_\beta) \in c_0$, then π_{ℓ_n} and $\pi^*_{\ell_n}$ factor for each non negative integer n.

Similarly if $\mathcal{A}$ has a bounded approximate identity and $\mathcal{X}$ is left approximately unital Banach $\mathcal{A}$-module with module action π_ℓ, then it is easy to see that π^{***}_ℓ and π^{t***t}_ℓ factor.

(ii) Let $\mathcal{A}$ be a Banach algebra with non empty spectrum $\sigma(\mathcal{A})$ and $\mathcal{X}$ be a Banach space. For $f \in \sigma(\mathcal{A})$ define the left and right module actions $\pi_\ell : \mathcal{A} \times \mathcal{X} \to \mathcal{X}$ and $\pi_r : \mathcal{X} \times \mathcal{A} \to \mathcal{X}$ by $\pi_\ell(a, x) = \langle f, a \rangle x = \pi_r(x, a)$, for each $a \in \mathcal{A}, x \in \mathcal{X}$. Then one can verify that for every positive integer n,

$$\pi_{\ell_n}(a, \phi) = \langle f, a \rangle \phi = \pi_{r_n}(\phi, a), \quad \left(a \in \mathcal{A}, \phi \in \mathcal{X}^{(n)}\right).$$

Now consider $a_0 \in \mathcal{A}$ such that $\langle f, a_0\rangle = 1$. Then for each $\phi \in \mathcal{X}^{(n)}$ we have $\pi_{\ell_n}(a_0, \phi) = \phi = \pi_{r_n}(\phi, a_0)$. That is π_{ℓ_n} and π_{r_n} factor.

Theorem 2.3. *Consider an integer $n \geq 2$.*

(i) *Let π_{ℓ_n} factors. Then $F \in Z_\ell\left(\pi_{\ell_{n-2}}\right)$ if and only if $F\mathcal{A} \subseteq Z_\ell\left(\pi_{\ell_{n-2}}\right)$.*

(ii) *Let π_{r_n} factors. Then $F \in Z_r\left(\pi_{r_{n-2}}\right)$ if and only if $\mathcal{A}F \subseteq Z_r\left(\pi_{r_{n-2}}\right)$.*

(iii) *Let $\pi_{r_{n-1}}$ factors. Then $F \in Z_\ell\left(\pi_{\ell_{n-2}}\right)$ if and only if $\mathcal{A}F \subseteq Z_\ell\left(\pi_{\ell_{n-2}}\right)$.*

(iv) *Let $\pi_{\ell_{n-1}}$ factors. Then $F \in Z_r\left(\pi_{r_{n-2}}\right)$ if and only if $F\mathcal{A} \subseteq Z_r\left(\pi_{r_{n-2}}\right)$.*

Proof. (i) By Lemma 2.1 and since $\pi^{***}_{\ell_{n-2}}$ and $\pi^{t***t}_{\ell_{n-2}}$ are left module actions of $\mathcal{A}^{**}$ on $\mathcal{X}^{(n)}$, for each $F \in \mathcal{A}^{**}$, $a \in \mathcal{A}$ and $\psi \in \mathcal{X}^{(n)}$ we have

$$\begin{aligned}\pi^{***}_{\ell_{n-2}}\left(F, \pi_{\ell_n}(a,\psi)\right) &= \pi^{***}_{\ell_{n-2}}\left(F, \pi^{***}_{\ell_{n-2}}(a,\psi)\right)\\ &= \pi^{***}_{\ell_{n-2}}(Fa,\psi),\end{aligned}$$

and

$$\begin{aligned}\pi^{t***t}_{\ell_{n-2}}\left(F, \pi_{\ell_n}(a,\psi)\right) &= \pi^{t***t}_{\ell_{n-2}}\left(F, \pi^{t***t}_{\ell_{n-2}}(a,\psi)\right)\\ &= \pi^{t***t}_{\ell_{n-2}}(Fa,\psi).\end{aligned}$$

Now factorization of π_{ℓ_n} implies (i).

(ii) It is similar to (i).

(iii) If $\pi_{r_{n-1}}$ factors, then for each $F \in \mathcal{A}^{**}$, $\varphi \in \mathcal{X}^{(n)}$ and $\lambda \in \mathcal{X}^{(n-1)}$ that $\lambda = \pi_{r_{n-1}}(\eta, a)$ for some $\eta \in \mathcal{X}^{(n-1)}$ and $a \in \mathcal{A}$, we have

$$\begin{aligned}\left\langle \pi^{***}_{\ell_{n-2}}(F,\varphi), \lambda\right\rangle &= \left\langle \pi^{***}_{\ell_{n-2}}(F,\varphi), \pi_{r_{n-1}}(\eta,a)\right\rangle\\ &= \left\langle \pi^{***}_{\ell_{n-2}}(F,\varphi), \pi^{*}_{\ell_{n-2}}(\eta,a)\right\rangle\\ &= \left\langle \pi^{**}_{\ell_{n-2}}\left(\pi^{***}_{\ell_{n-2}}(F,\varphi), \eta\right), a\right\rangle\\ &= \left\langle \pi^{***}_{\ell_{n-2}}\left(a, \pi^{***}_{\ell_{n-2}}(F,\varphi)\right), \eta\right\rangle\\ &= \left\langle \pi^{***}_{\ell_{n-2}}(aF,\varphi), \eta\right\rangle.\end{aligned}$$

Similarly we can show that

$$\left\langle \pi^{t***t}_{\ell_{n-2}}(F,\varphi), \lambda\right\rangle = \left\langle \pi^{***}_{\ell_{n-2}}\left(a, \pi^{t***t}_{\ell_{n-2}}(F,\varphi)\right), \eta\right\rangle$$

$$= \left\langle \pi_{\ell_{n-2}}^{t***t}\left(a, \pi_{\ell_{n-2}}^{t***t}(F,\varphi)\right), \eta\right\rangle$$
$$= \left\langle \pi_{\ell_{n-2}}^{t***t}(aF,\varphi), \eta\right\rangle.$$

These prove (iii).

(iv) It is similar to (iii).

□

Note that the part (i) of the latter theorem says that if for a non negative integer n, $\pi_{\ell_{n+2}}$ factors, then $Z_\ell(\pi_{\ell_n}) = \{F \in \mathcal{A}^{**}; F\mathcal{A} \subseteq Z_\ell(\pi_{\ell_n})\}$ or in other word $Z_\ell(\pi_{\ell_n})\mathcal{A} \subseteq Z_\ell(\pi_{\ell_n})$. This says that in this case $Z_\ell(\pi_{\ell_n})$ is a right Banach $\mathcal{A}$-submodule of $\mathcal{A}^{**}$. In particular if $\mathcal{A}$ is a unital Banach algebra and $\mathcal{X}$ is a unital Banach $\mathcal{A}$-module then for each non negative integer n we have $Z_\ell(\pi_{\ell_n})\mathcal{A} = Z_\ell(\pi_{\ell_n}) = \mathcal{A}Z_\ell(\pi_{\ell_n})$. Also if π_{ℓ_n} is left strongly Arens irregular and $\pi_{\ell_{n+2}}$ factors, then $\{F \in \mathcal{A}^{**}; F\mathcal{A} \subseteq \mathcal{A}\} = \mathcal{A}$. Note that if $\mathcal{A}$ is Arens regular then it is easy to see that $Z_\ell(\pi_{\ell_n})$ is also a subalgebra of $\mathcal{A}^{**}$.

In the following, we give some examples for the part (i) of Theorem 2.3. Also, we have similar examples for other parts of this theorem.

Example 2.4. (i) If G is a locally compact group and π is the convolution of $M(G)$, then since $M(G)$ is unital, π_{ℓ_2} factors. Also we have $Z_\ell(\pi) = M(G)$ [8], and obviously $\mu \in M(G)$ if and only if $\mu\nu \in M(G)$, for $\nu \in M(G)$

(ii) For a locally compact group G, $L^\infty(G)$ with the pointwise product is Arens regular and π_{ℓ_2} factors. Also obviously $F \in L^\infty(G)^{**}$ if and only if $FL^\infty(G) \subseteq L^\infty(G)^{**}$.

(iii) Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be normed spaces and $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ be a bounded bilinear map. Consider the Banach algebra $\mathcal{A}_f = \mathcal{X} \oplus \mathcal{Y} \oplus \mathcal{Z}$ with the pointwise vector space operations, the multiplication $(x, y, z)(x', y', z') = (0, 0, f(x, y'))$ and the norm $\|(x, y, z)\| = \|x\| + \|y\| + \|z\|, (x, x' \in \mathcal{X}, y, y' \in \mathcal{Y}, z, z' \in \mathcal{Z})$. This algebra was defined for first time in [3]. Now let $\mathcal{W}$ be a left Banach $\mathcal{A}_f$-module with the left module action π_ℓ such that π_{ℓ_n} factors for some even integer n. Then since for each $(F, G, H) \in \mathcal{A}_f^{**}$ and $(x, y, z) \in \mathcal{A}_f$, we have $(F, G, H)(x, y, z) = (0, 0, f^{***}(F, y))$, therefore by Theorem 2.3, $(F, G, H) \in Z_\ell\left(\pi_{\ell_{n-2}}\right)$ if and only if $(0, 0, f^{***}(F, y)) \in Z_\ell\left(\pi_{\ell_{n-2}}\right)$. That is $Z_\ell\left(\pi_{\ell_{n-2}}\right) = B \oplus \mathcal{Y}^{**} \oplus \mathcal{Z}^{**}$, where

$$B = \{F \in \mathcal{X}^{**}; (0, 0, f^{***}(F, y)) \in Z_\ell\left(\pi_{\ell_{n-2}}\right), y \in \mathcal{Y}\}.$$

The parts (i) and (iv) of Theorem 2.3 also imply the following corollary.

Corollary 2.5. *For each integer $n \geq 2$,*

(i) *If* π_{ℓ_n} *factors and* $\mathcal{A}$ *is a left ideal of* $\mathcal{A}^{**}$*, then* $\pi_{\ell_{n-2}}$ *and* $\pi_{r_{n-1}}$ *are Arens regular.*
(ii) *If* π_{ℓ_n} *factors and* $\mathcal{A}$ *has a left approximate identity* (e_α) *with a* w^**-cluster point* E*, then* $E \in Z_\ell\left(\pi_{\ell_{n-2}}\right) \cap Z_r\left(\pi_{r_{n-1}}\right)$.

The part (i) of the latter corollary says that if for some even integer n, π_{ℓ_n} factors and $\mathcal{A}$ is a left ideal of $\mathcal{A}^{**}$, then π_ℓ and its adjoint are Arens regular and if for some odd integers n, π_{ℓ_n} factors and $\mathcal{A}$ is a left ideal of $\mathcal{A}^{**}$, then π_r and the adjoint of its flip map are Arens regular. In particular if π is the multiplication of $\mathcal{A}$ and for some integers $n \geq 2$, π_{ℓ_n} factors and $\mathcal{A}$ is a left ideal of $\mathcal{A}^{**}$, then $\mathcal{A}$ is Arens regular.

Note that similar corollaries can be given as above for cases (ii) and (iii).

Theorem 2.6. *Let* n *be a non zero integer.*
(i) *If* $\pi^*_{r_n}$ *factors, then* $Z_r(\pi_{r_n}) \subseteq Z_r\left(\mathcal{A}^{**}\right)$.
(ii) *If* $\pi^{t*}_{\ell_n}$ *factors, then* $Z_\ell\left(\pi_{\ell_n}\right) \subseteq Z_\ell\left(\mathcal{A}^{**}\right)$.

Proof. We only prove (i).

Let $F \in Z_r\left(\pi_{r_n}\right)$ and $f \in \mathcal{A}^*$. Since $\pi^*_{r_n}$ factors, there exist $\omega \in \mathcal{X}^{(n+1)}$ and $\phi \in \mathcal{X}^{(n)}$ such that $f = \pi^*_{r_n}(\omega, \phi)$. Also we can verify that for every $a \in \mathcal{A}, G \in \mathcal{A}^{**}, \upsilon \in \mathcal{X}^{(n+1)}$ and $\psi \in \mathcal{X}^{(n)}$, $\pi^{t*}\left(\pi^*_{r_n}(\upsilon, \psi), a\right) = \pi^*_{r_n}(\pi^{t*}_{r_n}(\upsilon, a), \psi)$ and $\pi^*_{r_n}\left(\upsilon, \pi_{r_n}(\psi, a)\right) = \pi^*\left(\pi^*_{r_n}(\upsilon, \psi), a\right)$, and so

$$\pi^{t**}\left(G, \pi^*_{r_n}(\upsilon, \psi)\right) = \pi^{t**}_{r_n}\left(\pi^{*t*}_{r_n}(G, \psi), \upsilon\right),$$

and

$$\pi^*_{r_n}\left(\pi^{**}_{r_n}(G, \upsilon), \psi\right) = \pi^{**}\left(G, \pi^*_{r_n}(\upsilon, \psi)\right).$$

Therefore, by above discussion, we have

$$\begin{aligned}
\left\langle \pi^{t***t}(G, F), f \right\rangle &= \left\langle \pi^{t***t}(G, F), \pi^*_{r_n}(\omega, \phi) \right\rangle \\
&= \left\langle F, \pi^{t**}\left(G, \pi^*_{r_n}(\omega, \phi)\right) \right\rangle \\
&= \left\langle F, \pi^{t**}_{r_n}\left(\pi^{*t*}_{r_n}(G, \phi), \omega\right) \right\rangle \\
&= \left\langle \pi^{t***t}_{r_n}\left(\pi^{*t*}_{r_n}(G, \phi), F\right), \omega \right\rangle \\
&= \left\langle \pi^{***}_{r_n}\left(\pi^{*t*}_{r_n}(G, \phi), F\right), \omega \right\rangle \\
&= \left\langle G, \pi^*_{r_n}\left(\pi^{**}_{r_n}(F, \omega), \phi\right) \right\rangle \\
&= \left\langle G, \pi^{**}\left(F, \pi^*_{r_n}(\omega, \phi)\right) \right\rangle \\
&= \left\langle \pi^{***}(G, F), \pi^*_{r_n}(\omega, \phi) \right\rangle \\
&= \left\langle \pi^{***}(G, F), f \right\rangle.
\end{aligned}$$

Therefore $\pi^{t***t}(G, F) = \pi^{***}(G, F)$, as required. □

Lemma 2.7. *Consider the Banach algebra* $\mathcal{A}$ *as an* $\mathcal{A}$*-module.*

(i) *If $\mathcal{A}$ has a bounded right approximate identity, then $\pi^*_{r_n}$ factors for every positive odd integer n.*

(ii) *If $\mathcal{A}$ has a bounded left approximate identity, then $\pi^*_{r_n}$ factors for every positive even integer n.*

(iii) *If $\mathcal{A}$ has a bounded right approximate identity, then $\pi^{t*}_{\ell_n}$ factors for every positive even integer n.*

(iv) *If $\mathcal{A}$ has a bounded left approximate identity, then $\pi^{t*}_{\ell_n}$ factors for every positive odd integer n.*

Proof. (i) Let (e_α) be a bounded right approximate identity in $\mathcal{A}$ with a cluster point $E \in \mathcal{A}^{**}$. We argue by induction and prove that $\pi^*_{r_{2k-1}}(E, f) = f$ for every $k \in \mathbb{N}$ and $f \in \mathcal{A}^*$.
Let $k = 1$, for each $a \in \mathcal{A}$ we have

$$\begin{aligned}\left\langle \pi^*_{r_1}(E, f), a\right\rangle &= \langle E, \pi_{r_1}(f, a)\rangle \\ &= \lim_\alpha \langle \pi_{r_1}(f, a), e_\alpha\rangle \\ &= \lim_\alpha \langle \pi^*(f, a), e_\alpha\rangle \\ &= \lim_\alpha \langle f, \pi(a, e_\alpha)\rangle \\ &= \langle f, a\rangle ,\end{aligned}$$

hence $\pi^*_{r_1}(E, f) = f$, and so $\pi^*_{r_1}$ factors.
Now we show that for every $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$, we have

$$\left\langle \pi^*_{r_{2k+1}}(E, f), a\right\rangle = \left\langle \pi^*_{r_{2k-1}}(E, f), a\right\rangle, \quad (k \geq 1),$$

which will clearly yield our claim. We have

$$\begin{aligned}\left\langle \pi^*_{r_{2k+1}}(E, f), a\right\rangle &= \left\langle E, \pi_{r_{2k+1}}(f, a)\right\rangle \\ &= \left\langle \pi^*_{\ell_{2k}}(f, a), E\right\rangle \\ &= \langle f, \pi_{\ell_{2k}}(a, E)\rangle \\ &= \left\langle \pi^{t*t}_{r_{2k-1}}(a, E), f\right\rangle \\ &= \left\langle E, \pi_{r_{2k-1}}(f, a)\right\rangle \\ &= \left\langle \pi^*_{r_{2k-1}}(E, f), a\right\rangle ,\end{aligned}$$

as required. Therefore for every odd integer $n \geq 1$, $\pi^*_{r_n}$ factors.

(ii) Suppose now that E is a cluster point in $\mathcal{A}^{**}$ of bounded left approximate identity (e_α) of $\mathcal{A}$. Then for $n = 2$ we have

$$\begin{aligned}\left\langle \pi^*_{r_2}(f, E), a\right\rangle &= \langle f, \pi_{r_2}(E, a)\rangle \\ &= \left\langle \pi^*_{\ell_1}(E, a), f\right\rangle \\ &= \langle E, \pi_{\ell_1}(a, f)\rangle\end{aligned}$$

$$= \lim_{\alpha} \left\langle \pi^{t*t}(a,f), e_\alpha \right\rangle$$
$$= \lim_{\alpha} \left\langle f, \pi\left(e_\alpha, a\right) \right\rangle$$
$$= \langle f, a \rangle \,,$$

and so $\pi^*_{r_2}$ factors. Also, we have for $k \geq 2$,

$$\begin{aligned}
\left\langle \pi^*_{r_{2k}}(f,E), a \right\rangle &= \left\langle f, \pi_{r_{2k}}(E,a) \right\rangle \\
&= \left\langle \pi^*_{\ell_{2k-1}}(E,a), f \right\rangle \\
&= \left\langle E, \pi_{\ell_{2k-1}}(a,f) \right\rangle \\
&= \left\langle \pi^{t*t}_{r_{2k-2}}(a,f), E \right\rangle \\
&= \left\langle f, \pi_{r_{2k-2}}(E,a) \right\rangle \\
&= \left\langle \pi^*_{r_{2k-2}}(f,E), a \right\rangle ,
\end{aligned}$$

Therefore $\pi^*_{r_n}$ factors for every even integer $n \geq 2$.

The proofs of (iii) and (iv) are similar. □

Theorem 2.8. *Let $\mathcal{A}$ be a Banach algebra with the product π and let $k \geq 1$.*

(i) *If $\mathcal{A}$ has a bounded right approximate identity, then $Z_r\left(\pi_{r_{2k-1}}\right) \subseteq Z_r\left(\mathcal{A}^{**}\right)$ and $Z_\ell\left(\pi_{\ell_{2k}}\right) \subseteq Z_\ell\left(\mathcal{A}^{**}\right)$.*

(ii) *If $\mathcal{A}$ has a bounded left approximate identity, then $Z_r\left(\pi_{r_{2k}}\right) \subseteq Z_r\left(\mathcal{A}^{**}\right)$ and $Z_\ell\left(\pi_{\ell_{2k-1}}\right) \subseteq Z_\ell\left(\mathcal{A}^{**}\right)$.*

Proof. Apply Theorem 2.6 and Lemma 2.7. □

Theorem 2.8 implies that if $\mathcal{A}$ has a bounded right (left) approximate identity and the right module action of $\mathcal{A}$ on $\mathcal{A}^{(n)}$ is Arens regular for some positive odd (even) integers n, then $\mathcal{A}$ is Arens regular. And if $\mathcal{A}$ has a bounded right (left) approximate identity and the left module action of $\mathcal{A}$ on $\mathcal{A}^{(n)}$ is Arens regular for some positive even (odd) integer n, then $\mathcal{A}$ is Arens regular. Also if $\mathcal{A}$ has a bounded approximate identity and it is right (left) strongly Arens irregular (for example, $L^1(G)$ [9] or $M(G)$ [8], for some locally compact groups G), then for every $n \geq 1$, the right (left) module action of $\mathcal{A}$ on $\mathcal{A}^{(n)}$ is right (left) strongly Arens irregular.

In the sequel we study some restricted factorizations. Let $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ be a bilinear mapping and $\mathcal{M}, \mathcal{N}$ and $\mathcal{W}$ be subspaces of $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$, respectively. We say that f factors $\mathcal{W}$ on $\mathcal{M} \times \mathcal{N}$ if for each $w \in \mathcal{W}$ there are $m \in \mathcal{M}$ and $n \in \mathcal{N}$ such that $f(m,n) = w$.

Theorem 2.9. *Consider an integer $n \geq 2$, a subspace B in $\mathcal{A}^{**}$ and a subspace V of $\mathcal{X}^{(n)}$.*

(i) *If* $\pi_{\ell_{n-2}}^{t***t}$ *factors* V *on* $B \times Z_r\left(\pi_{\ell_{n-2}}^{t*}\right)$ *and* $\pi_{\ell_{n-2}}^{***}\left(B, \mathcal{X}^{(n-2)}\right) \subseteq \mathcal{X}^{(n-2)}$*, then* $V \subseteq \mathcal{X}^{(n-2)}$.

(ii) *If* $\pi_{r_{n-2}}^{***}$ *factors* V *on* $Z_r\left(\pi_{r_{n-2}}^{*}\right) \times B$ *and* $\pi_{r_{n-2}}^{***}(\mathcal{X}^{(n-2)}, B) \subseteq \mathcal{X}^{(n-2)}$*, then* $V \subseteq \mathcal{X}^{(n-2)}$.

Proof. We only prove (i). Proof of the other part can be done similarly.

For each $\phi \in V$ there are $F \in B$ and $\psi \in Z_r\left(\pi_{\ell_{n-2}}^{t*}\right)$ such that $\phi = \pi_{\ell_{n-2}}^{t***t}(F, \psi)$ and so for every $\omega \in \mathcal{X}^{(n+1)}$,

$$\begin{aligned} \langle \omega, \phi \rangle &= \left\langle \omega, \pi_{\ell_{n-2}}^{t***t}(F, \psi) \right\rangle \\ &= \left\langle \pi_{\ell_{n-2}}^{t****}(\omega, \psi), F \right\rangle. \end{aligned}$$

Now if (φ_α) and (λ_β) are two nets in $\mathcal{X}^{(n-2)}$ and $\mathcal{X}^{(n-1)}$ respectively, such that $\varphi_\alpha \xrightarrow{w^*} \psi$ and $\lambda_\beta \xrightarrow{w^*} \omega$, then since for each $G \in \mathcal{A}^{**}, \varphi \in \mathcal{X}^{(n-2)}$ and $\lambda \in \mathcal{X}^{(n-1)}$,

$$\left\langle \pi_{\ell_{n-2}}^{t*}(\lambda, \varphi), G \right\rangle = \left\langle \lambda, \pi_{\ell_{n-2}}^{***}(G, \varphi) \right\rangle,$$

we have

$$\begin{aligned} \langle \omega, \phi \rangle &= \left\langle \pi_{\ell_{n-2}}^{t****}(\omega, \psi), F \right\rangle \\ &= \lim_\alpha \lim_\beta \left\langle F, \pi_{\ell_{n-2}}^{t*}\left(\lambda_\beta, \varphi_\alpha\right) \right\rangle \\ &= \lim_\alpha \lim_\beta \left\langle \lambda_\beta, \pi_{\ell_{n-2}}^{***}\left(F, \varphi_\alpha\right) \right\rangle \\ &= \lim_\alpha \left\langle \omega, \pi_{\ell_{n-2}}^{***}\left(F, \varphi_\alpha\right) \right\rangle. \end{aligned}$$

Therefore $\pi_{\ell_{n-2}}^{***}\left(F, \varphi_\alpha\right) \xrightarrow{w} \phi$ and so $\phi \in \overline{\mathcal{X}^{(n-2)}}^{w} = \mathcal{X}^{(n-2)}$. □

Note that if $\mathcal{A}$ has an approximate identity, $\mathcal{X}$ is left (right) approximately unital, E is a w^*-cluster point of the approximate identity and B in Theorem 2.9 is the subspace generated by E, then we have the following corollary, which is also a combination of [5] and Theorem 3.4 of [4].

Corollary 2.10. *If* $\mathcal{A}$ *has an approximate identity.*

(i) *If* $\mathcal{X}$ *is left approximately unital, then* π_ℓ^{t*} *is right strongly Arens irregular.*

(ii) *If* $\mathcal{X}$ *is right approximately unital, then* π_r^{*} *is right strongly Arens irregular.*

Corollary 2.11. *Let* V *be a subspace of* $\mathcal{A}^{**}$.

(i) *If $\mathcal{A}$ is a left ideal of $\mathcal{A}^{**}$ and $V \subseteq \mathcal{A}^{**} \Diamond Z_r\left(\pi^{t*}\right)$, then $V \subseteq \mathcal{A}$.*
(ii) *If $\mathcal{A}$ is a right ideal of $\mathcal{A}^{**}$ and $V \subseteq Z_r\left(\pi^{t*}\right) \Box \mathcal{A}^{**}$, then $V \subseteq \mathcal{A}$.*

In particular if in Corollary 2.11 we put $V = \mathcal{A}^{**}$, then $\mathcal{A}$ is reflexive and if $V = Z_r(\mathcal{A}^{**})$ $(V = Z_\ell(\mathcal{A}^{**}))$ then $\mathcal{A}$ is right (left) strongly Arens irregular. Consequently if $V = Z_r(\mathcal{A}^{**})$ $(V = Z_\ell(\mathcal{A}^{**}))$ satisfies the conditions of the corollary, then $\mathcal{A}$ is Arens regular if and only if it is reflexive.

In the following, we give some examples which satisfy in part (i) of Corollary 2.11. Also, we may give similar examples which satisfy in part (ii) of this corollary.

Example 2.12. (i) Let H be a Hilbert space and π be the product of $K(H)$, the algebra of compact operators on H. By Corollary 2.10, π^{t*} is right strongly Arens irregular. Now since $K(H)$ is an ideal of $B(H)$, $B(H) \Diamond Z_r\left(\pi^{t*}\right) = B(H)K(H) \subseteq K(H)$.

(ii) Let H be a Hilbert space. Consider the Banach algebra $A_\circ = K(H) \oplus K(H) \oplus K(H)$, with the product and norm as follows

$$(r,s,t)(u,v,w) = (0,0,r \circ v), \qquad \|(u,v,w)\| = \|u\| + \|v\| + \|w\|,$$

for each $r,s,t,u,v,w \in K(H)$.

Let F be the transpose of the product $\circ$, of $K(H)$. Then Corollary 2.10 implies that F^* is right strongly Arens irregular. If we denote the product of $A_\circ$ by π, then it is easy to see that for every $f,g,h \in K(H)^*$ and $u,v,w \in K(H)$,

$$\pi^{t*}((f,g,h),(u,v,w)) = (F^*(h,v),0,0)\,.$$

Therefore $Z_r\left(\pi^{t*}\right) = B(H) \oplus K(H) \oplus B(H)$. On the other hand $A_\circ$ is a left ideal of $A_\circ^{**}$ and we have for each $(T,R,S) \in A_\circ^{**}$ and $(U,u,V) \in Z_r\left(\pi^{t*}\right)$,

$$(T,R,S)\Diamond(U,u,V) = (0,0,T \circ u) \in A_\circ.$$

The following theorem implies that when $\mathcal{X}$ is not reflexive, then $\pi_{\ell_n}|_{\mathcal{A}\times Z_r\left(\pi^{t*}_{\ell_{n-2}}\right)}$ and also $\pi_{r_n}|_{Z_r\left(\pi^{*}_{r_{n-2}}\right)\times\mathcal{A}}$ do not factor.

Theorem 2.13. *Let $n \geq 2$ and V be a subspace of $\mathcal{X}^{(n)}$.*

(i) *If π_{ℓ_n} factors V on $\mathcal{A} \times Z_r\left(\pi^{t*}_{\ell_{n-2}}\right)$, then $V \subseteq \mathcal{X}^{(n-2)}$.*
(ii) *If π_{r_n} factors V on $Z_r\left(\pi^{*}_{r_{n-2}}\right) \times \mathcal{A}$, then $V \subseteq \mathcal{X}^{(n-2)}$.*

Proof. (i) For each $\Psi \in V$, there are $a \in \mathcal{A}$ and $\Phi \in Z_r\left(\pi^{t*}_{\ell_{n-2}}\right)$ such that $\Psi = \pi_{\ell_n}(a,\Phi)$. Now if the net $(\phi_\alpha) \subseteq \mathcal{X}^{(n-2)}$ is

w^*-convergent to Φ, then for each $\omega \in \mathcal{X}^{(n+1)}$ we have

$$\begin{aligned}
\lim_\alpha \left\langle \omega, \pi_{\ell_{n-2}}(a, \phi_\alpha) \right\rangle &= \lim_\alpha \lim_\beta \left\langle \lambda_\beta, \pi_{\ell_{n-2}}(a, \phi_\alpha) \right\rangle \\
&= \lim_\alpha \lim_\beta \left\langle \pi^{t*}_{\ell_{n-2}}(\lambda_\beta, \phi_\alpha), a \right\rangle \\
&= \left\langle \pi^{t*t***t}_{\ell_{n-2}}(\omega, \Phi), a \right\rangle \\
&= \left\langle \pi^{t****}_{\ell_{n-2}}(\omega, \Phi), a \right\rangle \\
&= \left\langle \omega, \pi^{t***t}_{\ell_{n-2}}(a, \Phi) \right\rangle \\
&= \langle \omega, \pi_{\ell_n}(a, \Phi) \rangle \\
&= \langle \omega, \Psi \rangle,
\end{aligned}$$

where $(\lambda_\beta) \subseteq \mathcal{X}^{(n-1)}$ is a w^*-convergent net to ω. Now since $\mathcal{X}^{(n-2)}$ is complete, $\Psi \in \overline{\mathcal{X}^{(n-2)}}^w = \mathcal{X}^{(n-2)}$.

(ii) It is similar.

□

Example 2.14. (i) Let G be a locally compact group and π be the product of $M(G)$ $\big($resp. $L^1(G)\big)$. Then by Corollary 2.10, π^{t*} is right strongly Arens irregular and so

$$\pi_{\ell_2}\left(M(G), Z_r\left(\pi^{t*}\right)\right) \subseteq M(G),$$

$$\left(\text{resp. } \pi_{\ell_2}\left(L^1(G), Z_r\left(\pi^{t*}\right)\right) \subseteq L^1(G)\right).$$

(ii) Let $\mathcal{X}$ be a normed space and "." be the scaler multiplication on $\mathcal{X}$. If π is the product of the Banach algebra $\mathcal{A}_. = \mathcal{X} \oplus \mathbb{C} \oplus \mathcal{X}$, then it is easy to see that π^{t*} is Arens regular and $\pi_{\ell_2}\left(\mathcal{A}_., Z_r\left(\pi^{t*}\right)\right) \subseteq \mathcal{A}_.$. For some other special examples, see Example 2.12, where a Banach algebra $\mathcal{A}$ is an ideal of $\mathcal{A}^{**}$.

Corollary 2.15. *Consider an integer $n \geq 2$.*

(i) *Let π_{ℓ_n} factors $Z_r\left(\pi^{t*}_{\ell_{n-2}}\right)$ on $\mathcal{A} \times Z_r\left(\pi^{t*}_{\ell_{n-2}}\right)$. Then $\pi^{t*}_{\ell_{n-2}}$ is right strongly Arens irregular.*

(ii) *Let π_{r_n} factors $Z_r\left(\pi^{*}_{r_{n-2}}\right)$ on $Z_r\left(\pi^{*}_{r_{n-2}}\right) \times \mathcal{A}$. Then $\pi^{*}_{r_{n-2}}$ is right strongly Arens irregular.*

(iii) *Let π_{ℓ_n} factors $Z_r\left(\pi_{\ell_{n-2}}\right)$ on $\mathcal{A} \times Z_r\left(\pi^{t*}_{\ell_{n-2}}\right)$. Then $\pi_{\ell_{n-2}}$ is right strongly Arens irregular.*

(iv) *Let π_{r_n} factors $Z_\ell\left(\pi_{r_{n-2}}\right)$ on $Z_r\left(\pi^{*}_{r_{n-2}}\right) \times \mathcal{A}$. Then $\pi^{*}_{r_{n-2}}$ is left strongly Arens irregular.*

(v) *If $\pi_{\ell_n}|_{\mathcal{A} \times Z_r\left(\pi^{t*}_{\ell_{n-2}}\right)}$ factors, then $\mathcal{X}$ is reflexive.*

(vi) *If* $\pi_{r_n}|_{Z_r\left(\pi^*_{r_{n-2}}\right)\times\mathcal{A}}$ *factors, then* $\mathcal{X}$ *is reflexive.*

Proposition 2.16. *Let* $n \geq 2$.

(i) *If for each* $\phi \in \mathcal{X}^{(n)}$ *there are* $\psi \in \mathcal{X}^{(n)}$ *and* $F, G \in \mathcal{A}^{**}$ *such that* $\phi = \pi^{t***t}_{\ell_{n-2}}(F, \psi)$ *and* $\psi = \pi^{t***t}_{\ell_{n-2}}(G, \phi)$*, then* $\pi^{t*}_{\ell_{n-2}}$ *is left strongly Arens irregular.*

(ii) *If for each* $\phi \in \mathcal{X}^{(n)}$ *there are* $\psi \in \mathcal{X}^{(n)}$ *and* $F, G \in \mathcal{A}^{**}$ *such that* $\phi = \pi^{***}_{r_{n-2}}(\psi, F)$ *and* $\psi = \pi_{r_{n-2}}(\phi, G)$*, then* $\pi^*_{r_{n-2}}$ *is left strongly Arens irregular.*

Proof. Let $\omega \in Z_\ell(\pi^{t*}_{\ell_{n-2}})$ and $\phi \in \mathcal{X}^{(n)}$, and F, G and ψ are as above. Then we have

$$\begin{aligned}\langle \omega, \phi \rangle &= \left\langle \omega, \pi^{t***t}_{\ell_{n-2}}(F, \psi) \right\rangle \\ &= \left\langle \pi^{t****}_{\ell_{n-2}}(\omega, \psi), F \right\rangle \\ &= \left\langle \pi^{t*t***t}_{\ell_{n-2}}(\omega, \psi), F \right\rangle \\ &= \left\langle \psi, \pi^{t*t**}_{\ell_{n-2}}(\omega, F) \right\rangle,\end{aligned}$$

therefore $\omega = \pi^{t**}_{\ell_{n-2}}\left(G, \pi^{t*t**}_{\ell_{n-2}}(\omega, F)\right) \in \chi^{(n-1)}$.

Proof of (ii) is similar. □

Note that the given conditions in part (i) (resp. part (ii)) of Proposition 2.16 are equivalent to "for each $\phi \in \mathcal{X}^{(n)}$ there is $F \in \mathcal{A}^{**}$ such that $\phi = \pi^{t***t}_{\ell_{n-2}}(F, \phi)$ (respectively $\phi = \pi^{***}_{r_{n-2}}(\phi, F)$)". So if $\mathcal{A}$ has a bounded approximate identity and $\mathcal{X}$ is approximately unital, then $\pi^{t*}_{\ell_n}$ and $\pi^*_{r_n}$ are left and right strongly Arens irregular. However by combination of the results of [4] and [5] we give this result for $n = 2$.

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