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# Generalized Weighted Composition Operators From Logarithmic Bloch Type Spaces to $n$'th Weighted Type Spaces

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Abstract. Let $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions on the open unit disc $\mathbb{D}$. For a weight $\mu$ and a nonnegative integer $n$, the $n$'th weighted type space $\mathcal{W}_\mu^{(n)}$ is the space of all $f \in \mathcal{H}(\mathbb{D})$ such that $\sup_{z\in\mathbb{D}} \mu(z) \left|f^{(n)}(z)\right| < \infty$. Endowed with the norm

$$\|f\|_{\mathcal{W}_\mu^{(n)}} = \sum_{j=0}^{n-1} \left|f^{(j)}(0)\right| + \sup_{z\in\mathbb{D}} \mu(z) \left|f^{(n)}(z)\right|,$$

the $n$'th weighted type space is a Banach space. In this paper, we characterize the boundedness of generalized weighted composition operators $\mathcal{D}_{\varphi,u}^m$ from logarithmic Bloch type spaces $\mathcal{B}_{\log^\beta}^\alpha$ to $n$'th weighted type spaces $\mathcal{W}_\mu^{(n)}$, where $u$ and $\varphi$ are analytic functions on $\mathbb{D}$ and $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. We also provide an estimation for the essential norm of these operators.

## 1. Introduction

Let $\mathbb{D}$ denote the open unit disc in the complex plane $\mathbb{C}$ and $\mathcal{H}(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$. For a weight function $\nu : \mathbb{D} \to \mathbb{R}_+$, a continuous strictly positive and bounded function, the weighted Banach space of analytic functions, denoted by $\mathcal{H}_\nu^\infty$, is the space of all functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_\nu = \sup_{z\in\mathbb{D}} \nu(z) \left|f(z)\right| < \infty,$$

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and $\mathcal{H}_\nu^0$ is the subspace of $\mathcal{H}_\nu^\infty$ consisting of all functions $f$ for which $\lim_{|z|\to 1^-} \nu(z)\,|f(z)| = 0$. Endowed with the norm $\|\cdot\|_\nu$, $\mathcal{H}_\nu^\infty$ and $\mathcal{H}_\nu^0$ are Banach spaces. For a weight $\nu$, the associated weight $\tilde{\nu}$ is defined by

$$\tilde{\nu}(z) := \left(\sup\left\{|f(z)| : f \in \mathcal{H}_\nu^\infty, \|f\|_\nu \le 1\right\}\right)^{-1}, \quad z \in \mathbb{D}.$$

It is known that $\nu(z) \le \tilde{\nu}(z)$ for every $z \in \mathbb{D}$. Moreover, $\mathcal{H}_{\tilde{\nu}}^\infty = \mathcal{H}_\nu^\infty$ and $\|f\|_{\tilde{\nu}} = \|f\|_\nu$, for all $f \in \mathcal{H}_\nu^\infty$. A weight $\nu$ is called essential if for some positive constant $c$, $\nu(z) \ge c\tilde{\nu}(z)$ for every $z \in \mathbb{D}$. See [2, 3] for more results for the associated weights. The weight $\nu$ is called radial if $\nu(|z|) = \nu(z)$ for all $z \in \mathbb{D}$.

For an arbitrary weight $\nu$, the weighted Bloch space $\mathcal{B}_\nu$ is the space of all functions $f \in \mathcal{H}(\mathbb{D})$ such that $f' \in \mathcal{H}_\nu^\infty$. By $\mathcal{B}_\nu^0$ we mean the subspace of $\mathcal{B}_\nu$ consisting of functions $f$ for which $f' \in \mathcal{H}_\nu^0$. Endowed with the norm $\|f\|_{\mathcal{B}_\nu} := |f(0)| + \|f'\|_\nu$, the weighted Bloch space $\mathcal{B}_\nu$ is a Banach space and $\mathcal{B}_\nu^0$ is a closed subspace of $\mathcal{B}_\nu$. For the standard weight $\nu_\alpha(z) = \left(1 - |z|^2\right)^\alpha$ $(0 < \alpha < \infty)$, we denote the weighted Bloch space $\mathcal{B}_{\nu_\alpha}$, so-called $\alpha$-Bloch spaces, by $\mathcal{B}_\alpha$ (See [9]).

Consider the weight

$$\nu_{\alpha,\beta}(z) := \left(1 - |z|^2\right)^\alpha \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1 - |z|^2}\right)^\beta, \quad z \in \mathbb{D},$$

where, $\alpha > 0$ and $\beta \ge 0$. The logarithmic Bloch type space, denoted by $\mathcal{B}_{\log^\beta}^\alpha$, is the weighted Bloch space $\mathcal{B}_{\nu_{\alpha,\beta}}$ and was first introduced by Stević in [12]. The little logarithmic Bloch type space $\mathcal{B}_{\nu_{\alpha,\beta}}^0$ is denoted by $\mathcal{B}_{\log^\beta}^{\alpha,0}$. For $\beta = 0$, the space $\mathcal{B}_{\log^\beta}^\alpha$ coincides with $\alpha$-Bloch spaces $\mathcal{B}_\alpha$ and specifically, for every $\alpha > 0$ and $\beta \ge 0$, $\mathcal{B}_{\log^\beta}^\alpha \subseteq \mathcal{B}_\alpha$. For $\alpha = \beta = 1$, $\mathcal{B}_{\log^\beta}^\alpha$ becomes the logarithmic Bloch space $\mathcal{B}_{\log}$, which appeared in [1, 4, 16]. Let

$$\mu_{\alpha,\beta}(z) := (1 - |z|)^\alpha \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1 - |z|}\right)^\beta, \quad z \in \mathbb{D}.$$

Since the function $h(x) = x^\alpha \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{x}\right)^\beta$ is increasing on $(0, 1]$, we obtain

$$\mu_{\alpha,\beta}(z) \le \nu_{\alpha,\beta}(z) \le 2^\alpha \mu_{\alpha,\beta}(z), \quad z \in \mathbb{D}. \tag{1.1}$$

Therefore, the Bloch spaces $\mathcal{B}_{\mu_{\alpha,\beta}}$ and $\mathcal{B}_{\log^\beta}^\alpha$ are the same and their norms are equivalent. Furthermore, $\nu_{\alpha,\beta}$ is a non-increasing essential weight tending to zero at the boundary of $\mathbb{D}$ [11].

For an arbitrary weight $\mu$ and a nonnegative integer $n$, the $n$'th weighted type space $\mathcal{W}_\mu^{(n)}$ consists of all $f \in \mathcal{H}(\mathbb{D})$ such that $f^{(n)} \in \mathcal{H}_\mu^\infty$.

For $n = 0$ the space becomes $\mathcal{H}_\mu^\infty$, for $n = 1$ the weighted Bloch space $\mathcal{B}_\mu$ and for $n = 2$ the weighted Zygmund space $\mathcal{Z}_\mu$. The $n$'th weighted type space $\mathcal{W}_\mu^{(n)}$ with the norm

$$\|f\|_{\mathcal{W}_\mu^{(n)}} = \sum_{j=0}^{n-1} \left|f^{(j)}(0)\right| + \left\|f^{(n)}\right\|_\mu,$$

is a Banach space.

Let $u \in \mathcal{H}(\mathbb{D})$ and $\varphi$ be a non-constant analytic self-map on $\mathbb{D}$. The weighted composition operator $uC_\varphi$ induced by $u$ and $\varphi$, is defined by

$$(uC_\varphi f)(z) = u(z) \cdot f(\varphi(z)), \quad f \in \mathcal{H}(\mathbb{D}).$$

In the case $u \equiv 1$, we have the composition operator $C_\varphi$ and if $\varphi$ is the identity map, we have the multiplication operator $M_u : f \mapsto u \cdot f$.

Let $m$ be a nonnegative integer. The generalized weighted composition operator $\mathcal{D}_{\varphi,u}^m$, is defined as following

$$\left(\mathcal{D}_{\varphi,u}^m f\right)(z) = u(z) \cdot f^{(m)}(\varphi(z)), \quad f \in \mathcal{H}(\mathbb{D}), z \in \mathbb{D}.$$

The generalized weighted composition operator was first introduced by Zhu in [19] and it includes many known operators. In the case $m = 0$, we get weighted composition operator $uC_\varphi$ which has been studied by several authors, see [5–8] and references therein. The boundedness and compactness of $\mathcal{D}_{\varphi,u}^m$ between Bers type spaces and $\alpha$-Bloch spaces have been studied by Zhu in [18]. Stević and Sharma in [14] characterized the boundedness and compactness of $\mathcal{D}_{\varphi,u}^m$ from $\alpha$- Bloch spaces to weighted BMOA spaces. Qu, et. al. in [10] characterized the boundedness and compactness of $\mathcal{D}_{\varphi,u}^m$ from logarithmic Bloch space $\mathcal{B}_{\log}$ to weighted Zygmund spaces. Ramos-Fernández in [11] has characterized bounded weighted composition operators from logarithmic Bloch type spaces to weighted Bloch spaces and has obtained an essential norm estimate for these operators. In this paper, we provide a characterization for the boundedness and an estimation for the essential norm of $\mathcal{D}_{\varphi,u}^m$ from logarithmic Bloch type spaces $\mathcal{B}_{\log^\beta}^\alpha$ to $n$th weighted type space $\mathcal{W}_\mu^{(n)}$. Our results are generalization of [10, 11]. In Section 2, we state some lemmas to obtain our main results and we characterize the boundedness of $\mathcal{D}_{\varphi,u}^m : \mathcal{B}_{\log^\beta}^\alpha \to \mathcal{W}_\mu^{(n)}$. Section 3 is devoted to estimates for the essential norms of this operator.

To provide our main results we need the test functions defined as follows. Suppose $\alpha > 0$, $\beta \geq 0$ and let $m$ be a positive integer such that

$\alpha > m-1$. For a fixed $w \in \mathbb{D}$, define

$$K_w^{m,\alpha}(z) = \frac{\left(1-|\varphi(w)|^2\right)^m}{\left(1-\overline{\varphi(w)}z\right)^\alpha}, \qquad f_w = \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(w)|}\right)^{-\beta} K_w^{m,\alpha}.$$

Then

$$f_w' = \alpha\overline{\varphi(w)} \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(w)|}\right)^{-\beta} K_w^{m,\alpha+1},$$

and

$$\begin{aligned}\sup_{z\in\mathbb{D}} \nu_{\alpha-m+1,\beta}(z)\left|f_w'(z)\right| &\leq \alpha 2^m \sup_{z\in\mathbb{D}} \frac{\nu_{\alpha-m+1,\beta}(z)}{(1-|\varphi(w)z|)^{\alpha-m+1}\left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(w)z|}\right)^\beta} \\ &\quad\times \left(\frac{\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(w)z|}}{\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(w)|}}\right)^\beta \\ &\leq \alpha 2^m \sup_{z\in\mathbb{D}} \frac{\nu_{\alpha-m+1,\beta}(z)}{H\left(|\varphi(w)z|\right)},\end{aligned}$$

where

$$H(z) = (1-z)^{\alpha-m+1}\left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-z}\right)^\beta.$$

By [11, Lemma 2.1] we obtain

$$\sup_{z\in\mathbb{D}} \nu_{\alpha-m+1,\beta}(z)\left|f_w'(z)\right| \leq c(\alpha,m) < \infty,$$

where $c(\alpha,m)$ is a positive constant depending on $\alpha$ and $m$. It follows that $f_w \in \mathcal{B}_{\log^\beta}^{\alpha-m+1}$ and $\sup_{w\in\mathbb{D}} \|f_w\|_{\mathcal{B}_{\log^\beta}^{\alpha-m+1}} < \infty$. Since

$$\lim_{x\to 0} x^{\alpha-m+1}\left(\ln \frac{e^{\frac{\beta}{\alpha}}}{x}\right)^\beta = 0,$$

an easy computation gives $f_w \in \mathcal{B}_{\log^\beta}^{\alpha-m+1,0}$.

We shall use the following results to provide our main theorems.

**Theorem 1.1** ([8, Theorem 2.1])**.** *Let $\nu$ and $\omega$ be weights. Then the weighted composition operator $uC_\varphi$ maps $\mathcal{H}_\nu^\infty$ into $\mathcal{H}_\omega^\infty$ if and only if*

$$\|uC_\varphi\|_{\mathcal{H}_\nu^\infty\to\mathcal{H}_\omega^\infty} = \sup_{z\in\mathbb{D}} \frac{\omega(z)}{\tilde{\nu}(\varphi(z))}|u(z)| < \infty.$$

*Moreover,* $\|uC_\varphi\|_{e,\mathcal{H}_\nu^\infty\to\mathcal{H}_\omega^\infty} = \lim_{|\varphi(z)|\to 1}\sup \frac{\omega(z)}{\tilde{\nu}(\varphi(z))}|u(z)|$.

**Lemma 1.2** ([13, Lemma 4])**.** *Assume $n \in \mathbb{N}_0$, $g, u \in \mathcal{H}(\mathbb{D})$ and $\varphi$ is an analytic self-map on $\mathbb{D}$. Then*

$$(u(z)g(\varphi(z)))^{(n)} = \sum_{k=0}^{n} g^{(k)}(\varphi(z)) \sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z), \dots, \varphi^{(l-k+1)}(z)\right),$$

*where, $B_{l,k}(x_1, \dots, x_{l-k+1})$ is the Bell polynomial.*

Recall that the essential norm $\|T\|_{e;X\to Y}$ of a bounded linear operator $T : X \to Y$ is defined as the distance from $T$ to $\mathcal{K}(X,Y)$, the space of compact operators from $X$ into $Y$. The norm of a bounded operator $T : X \to Y$ is denoted by $\|T\|_{X\to Y}$. The notation $A \lesssim B$ means that for some constant $c$, $A \leq cB$ and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

## 2. Boundedness

In this section, we provide the necessary and sufficient conditions for $\mathcal{D}^m_{\varphi,u} : \mathcal{B}^\alpha_{\log^\beta} \to \mathcal{W}^{(n)}_\mu$ to be bounded, where $m \in \mathbb{N}_0$, $\alpha > 0$, $\beta \geq 0$ such that $\alpha + m > 1$ and $\mu$ is an arbitrary weight. Before stating our main results, we need some preliminary lemmas as follows. The following lemma is used frequently in this paper.

**Lemma 2.1** ([12, Lemma 2 (a)])**.** *Assume $\alpha > 1$ and $\beta \geq 0$. Then*

$$\int_0^x \frac{dt}{(1-t)^\alpha \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-t}\right)^\beta} \approx \frac{1}{(1-x)^{\alpha-1}\left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-x}\right)^\beta},$$

*as $x \to 1^-$.*

The following result may be proved in the same way as [10, Lemma 1].

**Lemma 2.2.** *Let $\alpha > 0$, $\beta \geq 0$ and $n \in \mathbb{N}_0$ such that $\alpha + n > 1$. There exists a constant $c$ such that*

$$\left|f^{(n)}(z)\right| \leq \frac{c\,\|f\|_{\mathcal{B}^\alpha_{\log^\beta}}}{\left(1-|z|^2\right)^{\alpha+n-1}\left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|z|^2}\right)^\beta}, \quad z \in \mathbb{D}.$$

In the next lemma, which is a generalization of [17, Proposition 7], we show that $\mathcal{H}^\infty_{\nu_{\alpha,\beta}} = \mathcal{B}^{\alpha+1}_{\log^\beta}$ and their norms are equivalent.

**Lemma 2.3.** *Suppose $\alpha > 1$ and $\beta \geq 0$. Then $f$ is in $\mathcal{B}^\alpha_{\log^\beta}$ if and only if $f$ is in $\mathcal{H}^\infty_{\nu_{\alpha-1,\beta}}$. Specifically, $\|f\|_{\mathcal{B}^\alpha_{\log^\beta}} \approx \|f\|_{\nu_{\alpha-1,\beta}}$.*

*Proof.* Let $\alpha > 1$ and $f \in \mathcal{B}^{\alpha}_{\log^{\beta}}$. Using (1.1) and Lemma 2.1 we have

$$
\begin{aligned}
|f(z) - f(0)| &\lesssim \|f\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}} \int_0^1 \frac{|z|\, dt}{(1-|z|\,t)^{\alpha} \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|z|t}\right)^{\beta}} \\
&\lesssim \frac{\|f\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}}}{(1-|z|)^{\alpha-1} \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|z|}\right)^{\beta}} \\
&\lesssim \frac{\|f\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}}}{\nu_{\alpha-1,\beta}(z)}.
\end{aligned}
$$

Thus $f \in \mathcal{H}^{\infty}_{\nu_{\alpha-1,\beta}}$ and $\|f\|_{\nu_{\alpha-1,\beta}} \lesssim \|f\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}}$. Now let $f \in \mathcal{H}^{\infty}_{\nu_{\alpha-1,\beta}}$. To show that $f \in \mathcal{B}^{\alpha}_{\log^{\beta}}$, we apply the arguments given in the proof of [10, Lemma 1]. Fixing $z \in \mathbb{D}$, let $r = \frac{1+|z|}{2}$. Then $r < 1$ and $\frac{z}{r} \in \mathbb{D}$. The Cauchy formula yields

$$
\begin{aligned}
\left|f'(z)\right| &= \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{f\left(re^{i\theta}\right)}{\left(re^{i\theta} - z\right)^2} rie^{i\theta} d\theta \right| \\
&\lesssim \frac{r\,\|f\|_{\nu_{\alpha-1,\beta}}}{(1-r^2)^{\alpha-1}\left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-r^2}\right)^{\beta}} \int_0^{2\pi} \frac{d\theta}{2\pi \left|re^{i\theta} - z\right|^2} \\
&= \frac{r\,\|f\|_{\nu_{\alpha-1,\beta}}}{(1-r^2)^{\alpha-1}\left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-r^2}\right)^{\beta} \left(r^2 - |z|^2\right)} \\
&\lesssim \frac{\|f\|_{\nu_{\alpha-1,\beta}}}{(1-r^2)^{\alpha}\left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-r^2}\right)^{\beta}}.
\end{aligned}
$$

The last inequality holds since

$$
\begin{aligned}
\frac{r^2 - |z|^2}{r} &> r - |z| \\
&= \frac{1-|z|}{2} \\
&= 1 - r.
\end{aligned}
$$

Thus

$$\left|f'(z)\right| \lesssim \frac{\|f\|_{\nu_{\alpha-1,\beta}}}{(1-r^2)^\alpha \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-r^2}\right)^\beta}.$$

Since $\frac{1-|z|^2}{4} \leq 1-r^2 \leq 1-|z|^2$ and $h(x) = x^\alpha \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{x}\right)^\beta$ is increasing on $[0,1)$, we provide that

$$\left|f'(z)\right| \lesssim \frac{\|f\|_{\nu_{\alpha-1,\beta}}}{\left(1-|z|^2\right)^\alpha \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|z|^2}\right)^\beta}.$$

This shows that

$$|f(0)| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|z|^2}\right)^\beta \left|f'(z)\right| \lesssim \|f\|_{\nu_{\alpha-1,\beta}},$$

which completes the proof. □

In the following lemma we show that [17, Proposition 8] holds for the logarithmic Bloch type spaces.

**Lemma 2.4.** *Suppose $\alpha > 0$, $\beta \geq 0$ and $n \in \mathbb{N}_0$. A function $f \in \mathcal{H}(\mathbb{D})$ belongs to $\mathcal{B}^\alpha_{\log^\beta}$ if and only if $f^{(n)} \in \mathcal{B}^{\alpha+n}_{\log^\beta}$. Furthermore,*

$$\|f\|_{\mathcal{B}^\alpha_{\log^\beta}} \approx \sum_{j=0}^{n} \left|f^{(j)}(0)\right| + \sup_{z\in\mathbb{D}} \nu_{\alpha+n,\beta}(z) \left|f^{(n+1)}(z)\right|. \tag{2.1}$$

*Proof.* For $n = 0$ it is trivial, thus we assume that $n \in \mathbb{N}$. Let $f \in \mathcal{B}^\alpha_{\log^\beta}$. It follows from Lemma 2.2 that $f^{(n)} \in \mathcal{B}^{\alpha+n}_{\log^\beta}$ and $\left\|f^{(n)}\right\|_{\mathcal{B}^{\alpha+n}_{\log^\beta}} \lesssim \|f\|_{\mathcal{B}^\alpha_{\log^\beta}}$. For $k = 1, 2, \ldots, n$, by the Cauchy formula we have

$$\left|f^{(k)}(0)\right| \leq \frac{2^{k+2\alpha}(k-1)!}{3^\alpha \left(\ln \frac{4e^{\frac{\beta}{\alpha}}}{3}\right)^\beta} \left\|f'\right\|_{\nu_{\alpha,\beta}}.$$

Thus

$$\sum_{j=0}^{n} \left|f^{(j)}(0)\right| + \sup_{z\in\mathbb{D}} \nu_{\alpha+n,\beta}(z) \left|f^{(n+1)}(z)\right| \lesssim |f(0)| + \left\|f'\right\|_{\nu_{\alpha,\beta}} + \left\|f^{(n)}\right\|_{\mathcal{B}^{\alpha+n}_{\log^\beta}}$$
$$\lesssim \|f\|_{\mathcal{B}^\alpha_{\log^\beta}}.$$

Now let $f^{(n)} \in \mathcal{B}^{\alpha+n}_{\log^\beta}$. Then

$$\left|f^{(n)}(z) - f^{(n)}(0)\right| \le \int_0^1 |z| \left|f^{(n+1)}(tz)\right| dt \le \left\|f^{(n)}\right\|_{\mathcal{B}^{\alpha+n}_{\log^\beta}} \int_0^1 \frac{|z|\, dt}{\nu_{\alpha+n,\beta}(tz)}.$$

Noticing that $\alpha + n > 1$, then using Lemma 2.1, we get

$$\left|f^{(n)}(z)\right| \lesssim \left|f^{(n)}(0)\right| + \frac{\left\|f^{(n)}\right\|_{\mathcal{B}^{\alpha+n}_{\log^\beta}}}{\nu_{\alpha+n-1,\beta}(z)}. \tag{2.2}$$

By induction, we show that for every nonnegative integer $k$ with $n-k \ge 1$,

$$\left|f^{(n-k)}(z)\right| \lesssim \sum_{j=0}^{k} \left|f^{(n-j)}(0)\right| + \frac{\left\|f^{(n)}\right\|_{\mathcal{B}^{\alpha+n}_{\log^\beta}}}{\nu_{\alpha+n-k-1,\beta}(z)}. \tag{2.3}$$

The relation (2.2) implies that for $k = 0$, (2.3) is valid. Let $k = 1$ such that $n - k \ge 1$. By (2.2) we have

$$\begin{aligned}
\left|f^{(n-1)}(z) - f^{(n-1)}(0)\right| &\le \int_0^1 |z| \left|f^{(n)}(tz)\right| dt \\
&\lesssim \left|f^{(n)}(0)\right| |z| + \left\|f^{(n)}\right\|_{\mathcal{B}^{\alpha+n}_{\log^\beta}} \int_0^1 \frac{|z|\, dt}{\nu_{\alpha+n-1,\beta}(tz)} \\
&\lesssim \left|f^{(n)}(0)\right| + \frac{\left\|f^{(n)}\right\|_{\mathcal{B}^{\alpha+n}_{\log^\beta}}}{\nu_{\alpha+n-2,\beta}(z)},
\end{aligned}$$

where the latter holds by Lemma 2.1. Thus

$$\left|f^{(n-1)}(z)\right| \lesssim \left|f^{(n-1)}(0)\right| + \left|f^{(n)}(0)\right| + \frac{\left\|f^{(n)}\right\|_{\mathcal{B}^{\alpha+n}_{\log^\beta}}}{\nu_{\alpha+n-2,\beta}(z)}.$$

Assume (2.3) holds for every nonnegative integer $k$ with $n - k \ge 1$; we will prove it for $k + 1$ with $n - k - 1 \ge 1$. For $z \in \mathbb{D}$ we have

$$\begin{aligned}
\left|f^{(n-k-1)}(z) - f^{(n-k-1)}(0)\right| &\le \int_0^1 |z| \left|f^{(n-k)}(tz)\right| dt \\
&\lesssim \left(\sum_{j=0}^{k} \left|f^{(n-j)}(0)\right|\right) |z| \\
&\quad + \left\|f^{(n)}\right\|_{\mathcal{B}^{\alpha+n}_{\log^\beta}} \int_0^1 \frac{|z|\, dt}{\nu_{\alpha+n-k-1,\beta}(tz)} \\
&\lesssim \sum_{j=0}^{k} \left|f^{(n-j)}(0)\right| + \frac{\left\|f^{(n)}\right\|_{\mathcal{B}^{\alpha+n}_{\log^\beta}}}{\nu_{\alpha+n-k-2,\beta}(z)},
\end{aligned}$$

from which we get that

$$\left|f^{(n-k-1)}(z)\right| \lesssim \sum_{j=0}^{k+1}\left|f^{(n-j)}(0)\right| + \frac{\left\|f^{(n)}\right\|_{\mathcal{B}^{\alpha+n}_{\log^{\beta}}}}{\nu_{\alpha+n-k-2,\beta}(z)}.$$

Thus (2.3) is true. Taking $k = n-1$, the relation (2.3) yields

$$\left|f'(z)\right| \lesssim \sum_{j=1}^{n}\left|f^{(j)}(0)\right| + \frac{\left\|f^{(n)}\right\|_{\mathcal{B}^{\alpha+n}_{\log^{\beta}}}}{\nu_{\alpha,\beta}(z)},$$

which implies that

$$\nu_{\alpha,\beta}(z)\left|f'(z)\right| \lesssim \left\|\nu_{\alpha,\beta}\right\|_{\infty}\sum_{j=1}^{n}\left|f^{(j)}(0)\right| + \left\|f^{(n)}\right\|_{\mathcal{B}^{\alpha+n}_{\log^{\beta}}}.$$

Therefore,

$$\|f\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}} \lesssim \sum_{j=0}^{n}\left|f^{(j)}(0)\right| + \sup_{z\in\mathbb{D}}\nu_{\alpha+n,\beta}(z)\left|f^{(n+1)}(z)\right|,$$

from which we get the desired result. □

For convenience, hereafter we assume that

$$\Phi^{u,\varphi}_{k,n}(z) = \sum_{l=k}^{n} C^{n}_{l} u^{(n-l)}(z) B_{l,k}\left(\varphi'(z), \ldots, \varphi^{(l-k+1)}(z)\right),$$

where $k \in \{0, \ldots, n\}$. By Lemma 1.2, for every $f \in \mathcal{B}^{\alpha}_{\log^{\beta}}$,

$$\begin{aligned}\left(\mathcal{D}^{m}_{\varphi,u} f\right)^{(n)} &= \left(u \cdot f^{(m)} \circ \varphi\right)^{(n)}\\ &= \sum_{k=0}^{n}\Phi^{u,\varphi}_{k,n}\cdot f^{(m+k)}\circ\varphi.\end{aligned}$$

Hence

$$\mathcal{D}^{n}\left(\mathcal{D}^{m}_{\varphi,u}\right) = \sum_{k=0}^{n}\Phi^{u,\varphi}_{k,n}\cdot C_{\varphi}\mathcal{D}^{m+k}, \tag{2.4}$$

where $\mathcal{D}$ is the differentiation operator on $\mathcal{H}(\mathbb{D})$. Let $\alpha > 0$, $\beta \geq 0$ and $m \in \mathbb{N}_0$ such that $\alpha + m > 1$. For every $f \in \mathcal{B}^{\alpha}_{\log^{\beta}}$ with $\|f\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}} \leq 1$ we have

$$\left\|\mathcal{D}^{m}_{\varphi,u} f\right\|_{\mathcal{W}^{(n)}_{\mu}} = \sum_{j=0}^{n-1}\left|\left(\mathcal{D}^{m}_{\varphi,u} f\right)^{(j)}(0)\right| + \left\|\mathcal{D}^{n}\left(\mathcal{D}^{m}_{\varphi,u} f\right)\right\|_{\mu}. \tag{2.5}$$

Fixing $j \in \{1, \ldots, n-1\}$, by Lemma 2.2 we have

$$\begin{aligned}\left|\left(\mathcal{D}^m_{\varphi,u} f\right)^{(j)}(0)\right| &= \left|\sum_{k=0}^{j} \Phi^{u,\varphi}_{k,j}(0) f^{(m+k)}(\varphi(0))\right| \\ &\lesssim \sum_{k=0}^{j} \frac{\left|\Phi^{u,\varphi}_{k,j}(0)\right|}{\nu_{\alpha+m+k-1,\beta}(\varphi(0))}.\end{aligned}$$

For $j = 0$,

$$\begin{aligned}\left|\left(\mathcal{D}^m_{\varphi,u} f\right)(0)\right| &= \left|u(0) f^{(m)}(\varphi(0))\right| \\ &\lesssim \frac{|u(0)|}{\nu_{\alpha+m-1,\beta}(\varphi(0))}.\end{aligned}$$

Using (2.5) we provide that

$$\left\|\mathcal{D}^m_{\varphi,u} f\right\|_{\mathcal{W}^{(n)}_\mu} \lesssim c(\alpha, n, m) + \left\|\mathcal{D}^n\left(\mathcal{D}^m_{\varphi,u} f\right)\right\|_\mu, \tag{2.6}$$

where, $c(\alpha, n, m)$ is a positive constant depending on $\alpha, n$ and $m$. According to Lemma 2.4, for $k \in \{0, \ldots, n\}$, the operator $\mathcal{D}^{m+k} : \mathcal{B}^\alpha_{\log^\beta} \to \mathcal{B}^{\alpha+m+k}_{\log^\beta}$ is bounded. Therefore,

$$\begin{aligned}\left\|\Phi^{u,\varphi}_{k,n} \cdot C_\varphi \mathcal{D}^{m+k}\right\|_{\mathcal{B}^\alpha_{\log^\beta} \to \mathcal{H}^\infty_\mu} &\lesssim \left\|\Phi^{u,\varphi}_{k,n} \cdot C_\varphi\right\|_{\mathcal{B}^{\alpha+m+k}_{\log^\beta} \to \mathcal{H}^\infty_\mu} \\ &\approx \left\|\Phi^{u,\varphi}_{k,n} \cdot C_\varphi\right\|_{\mathcal{H}^\infty_{\nu_{\alpha+m+k-1,\beta}} \to \mathcal{H}^\infty_\mu},\end{aligned}$$

and by (2.4)

$$\left\|\mathcal{D}^n\left(\mathcal{D}^m_{\varphi,u}\right)\right\|_{\mathcal{B}^\alpha_{\log^\beta} \to \mathcal{H}^\infty_\mu} \lesssim \sum_{k=0}^{n} \left\|\Phi^{u,\varphi}_{k,n} \cdot C_\varphi\right\|_{\mathcal{H}^\infty_{\nu_{\alpha+m+k-1,\beta}} \to \mathcal{H}^\infty_\mu}. \tag{2.7}$$

By relations (2.6) and (2.7) we have

$$\left\|\mathcal{D}^m_{\varphi,u}\right\|_{\mathcal{B}^\alpha_{\log^\beta} \to \mathcal{W}^{(n)}_\mu} \lesssim c(\alpha, n, m) + \sum_{k=0}^{n} \left\|\Phi^{u,\varphi}_{k,n} \cdot C_\varphi\right\|_{\mathcal{H}^\infty_{\nu_{\alpha+m+k-1,\beta}} \to \mathcal{H}^\infty_\mu}. \tag{2.8}$$

The relation (2.8) gives a sufficient condition for $\mathcal{D}^m_{\varphi,u} : \mathcal{B}^\alpha_{\log^\beta} \to \mathcal{W}^{(n)}_\mu$ to be bounded.

In the next theorem we characterize the boundedness of $\mathcal{D}^m_{\varphi,u} : \mathcal{B}^\alpha_{\log^\beta} \to \mathcal{W}^{(n)}_\mu$.

**Theorem 2.5.** *Let $u, \varphi \in \mathcal{H}(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $\mu$ be an arbitrary weight. Suppose that $\alpha > 0$, $\beta \geq 0$ and $m \in \mathbb{N}_0$ such that $\alpha + m > 1$. Then the following statements are equivalent.*

(i) $\mathcal{D}^m_{\varphi,u}:\mathcal{B}^\alpha_{\log^\beta}\to\mathcal{W}^{(n)}_\mu$ *is bounded;*

(ii) *For every* $k\in\{0,\dots,n\}$, $\Phi^{u,\varphi}_{k,n}\cdot C_\varphi:\mathcal{H}^\infty_{\nu_{\alpha+m+k-1,\beta}}\to\mathcal{H}^\infty_\mu$ *is bounded;*

(iii) *For every* $k\in\{0,\dots,n\}$, $\sup_{z\in\mathbb{D}}\frac{\mu(z)\left|\Phi^{u,\varphi}_{k,n}(z)\right|}{\nu_{\alpha+m+k-1,\beta}(\varphi(z))}<\infty$.

*Proof.* By Theorem 1.1 and (2.8) the implications $(ii)\Leftrightarrow(iii)$ and $(ii)\Rightarrow(i)$ are valid.

Let $\mathcal{D}^m_{\varphi,u}:\mathcal{B}^\alpha_{\log^\beta}\to\mathcal{W}^{(n)}_\mu$ be bounded. We show that (iii) is necessary. For a fixed $z\in\mathbb{D}$ and constants $C_1,\dots,C_{n+1}$, define

$$g_z=\sum_{j=1}^{n+1}\frac{C_jK_z^{j+1,\alpha+j}}{\prod\limits_{l=0}^{m-1}(\alpha+j+l)}\left(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(z)|}\right)^{-\beta}.$$

As pointed out in Section 1, it follows that $g_z\in\mathcal{B}^{\alpha,0}_{\log^\beta}$ and

$$\sup_{z\in\mathbb{D}}\|g_z\|_{\mathcal{B}^{\alpha,0}_{\log^\beta}}<\infty.$$

Fix $z\in\mathbb{D}$ and $k\in\{0,\dots,n\}$. Applying the arguments of [13, Theorem 1], we may choose the constants $C_{1,k},\dots,C_{n+1,k}$ and the function

$$g_{z,k}=\sum_{j=1}^{n+1}\frac{C_{j,k}K_z^{j+1,\alpha+j}}{\prod\limits_{l=0}^{m-1}(\alpha+j+l)}\left(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(z)|}\right)^{-\beta},$$

satisfying

$$g_z^{(m+k)}(\varphi(z))=\frac{\overline{\varphi(z)}^{m+k}}{\nu_{\alpha+m+k-1,\beta}(\varphi(z))},$$

and

$$g_z^{(m+t)}(\varphi(z))=0,\quad t\in\{0,\dots,n\}\setminus\{k\}.$$

Therefore,

$$\begin{aligned}\frac{\mu(z)\,|\varphi(z)|^{m+k}\left|\Phi^{u,\varphi}_{k,n}(z)\right|}{\nu_{\alpha+m+k-1,\beta}(\varphi(z))}&\le\sup_{z\in\mathbb{D}}\left\|\mathcal{D}^m_{\varphi,u}(g_{z,k})\right\|_{\mathcal{W}^{(n)}_\mu}\\&\lesssim\left\|\mathcal{D}^m_{\varphi,u}\right\|_{\mathcal{B}^\alpha_{\log^\beta}\to\mathcal{W}^{(n)}_\mu},\end{aligned}$$

from which we get

$$\sup_{|\varphi(z)|>\frac{1}{2}}\frac{\mu(z)\left|\Phi^{u,\varphi}_{k,n}(z)\right|}{\nu_{\alpha+m+k-1,\beta}(\varphi(z))}\lesssim\left\|\mathcal{D}^m_{\varphi,u}\right\|_{\mathcal{B}^\alpha_{\log^\beta}\to\mathcal{W}^{(n)}_\mu}. \tag{2.9}$$

As in the proof of [13, Theorem 1] for $k \in \{0, \dots, n\}$ one shows that

$$\sup_{z\in\mathbb{D}} \mu(z) \left|\Phi_{k,n}^{u,\varphi}(z)\right| \lesssim \left\|\mathcal{D}_{\varphi,u}^m\right\|_{\mathcal{B}^\alpha_{\log^\beta}\to\mathcal{W}_\mu^{(n)}}.$$

Thus

(2.10)

$$\sup_{|\varphi(z)|\le\frac{1}{2}} \frac{\mu(z)\left|\Phi_{k,n}^{u,\varphi}(z)\right|}{\nu_{\alpha+m+k-1,\beta}(\varphi(z))} \lesssim \sup_{z\in\mathbb{D}} \mu(z)\left|\Phi_{k,n}^{u,\varphi}(z)\right| \lesssim \left\|\mathcal{D}_{\varphi,u}^m\right\|_{\mathcal{B}^\alpha_{\log^\beta}\to\mathcal{W}_\mu^{(n)}}.$$

Addition of (2.9) and (2.10) yields

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)\left|\Phi_{k,n}^{u,\varphi}(z)\right|}{\nu_{\alpha+m+k-1,\beta}(\varphi(z))} \lesssim \left\|\mathcal{D}_{\varphi,u}^m\right\|_{\mathcal{B}^\alpha_{\log^\beta}\to\mathcal{W}_\mu^{(n)}},$$

which completes the proof. □

## 3. Essential Norm

In this section we estimate the essential norm of generalized weighted composition operators from logarithmic Bloch type spaces to the $n$'th weighted type spaces.

Let $\mathcal{D}_{\varphi,u}^m : \mathcal{B}^\alpha_{\log^\beta} \to \mathcal{W}_\mu^{(n)}$ be bounded. Define $\Lambda : \mathcal{B}^\alpha_{\log^\beta} \to \mathcal{W}_\mu^{(n)}$ by

$$\Lambda(f) = \sum_{l=0}^{n-1} \left(\mathcal{D}_{\varphi,u}^m f\right)^{(l)}(0)\frac{z^l}{l!}, \quad f \in \mathcal{B}^\alpha_{\log^\beta}.$$

Clearly, $\Lambda \in \mathcal{K}\left(\mathcal{B}^\alpha_{\log^\beta}, \mathcal{W}_\mu^{(n)}\right)$ and $(\Lambda f)^{(j)}(0) = \left(\mathcal{D}_{\varphi,u}^m f\right)^{(j)}(0)$. For $K \in \mathcal{K}\left(\mathcal{B}^\alpha_{\log^\beta}, \mathcal{W}_\mu^{(n)}\right)$ and $f \in \mathcal{B}^\alpha_{\log^\beta}$,

$$\begin{aligned}\left\|\left(\mathcal{D}_{\varphi,u}^m - \Lambda - K\right) f\right\|_{\mathcal{W}_\mu^{(n)}} &= \sum_{j=0}^{n-1} \left|\left(\mathcal{D}_{\varphi,u}^m f - \Lambda f\right)^{(j)}(0) - (Kf)^{(j)}(0)\right| \\ &\quad + \left\|\left((\mathcal{D}_{\varphi,u}^m - K)f\right)^{(n)}\right\|_\mu \\ &= \sum_{j=0}^{n-1} \left|(Kf)^{(j)}(0)\right| + \left\|\left(\mathcal{D}^n\left(\mathcal{D}_{\varphi,u}^m - K\right)\right) f\right\|_\mu.\end{aligned}$$

Using (2.4) we obtain the following

$$\left\|\mathcal{D}_{\varphi,u}^m - \Lambda\right\|_{e;\mathcal{B}^\alpha_{\log^\beta}\to\mathcal{W}_\mu^{(n)}} = \inf_{K\in\mathcal{K}\left(\mathcal{B}^\alpha_{\log^\beta},\mathcal{W}_\mu^{(n)}\right)} \left\|\mathcal{D}^n\left(\mathcal{D}_{\varphi,u}^m - K\right)\right\|_{\mathcal{B}^\alpha_{\log^\beta}\to\mathcal{H}_\mu^\infty}$$

$$= \inf_{K\in\mathcal{K}\left(\mathcal{B}^{\alpha}_{\log^{\beta}},\mathcal{W}^{(n)}_{\mu}\right)} \left\| \sum_{k=0}^{n} \Phi^{u,\varphi}_{k,n} \cdot C_{\varphi}\mathcal{D}^{m+k} - \mathcal{D}^{n}K \right\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}\to\mathcal{H}^{\infty}_{\mu}}$$

$$= \inf_{K\in\mathcal{K}\left(\mathcal{B}^{\alpha}_{\log^{\beta}},\mathcal{H}^{\infty}_{\mu}\right)} \left\| \sum_{k=0}^{n} \Phi^{u,\varphi}_{k,n} \cdot C_{\varphi}\mathcal{D}^{m+k} - K \right\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}\to\mathcal{H}^{\infty}_{\mu}}.$$

Accordingly,

$$\begin{aligned}(3.1)\qquad \left\|\mathcal{D}^{m}_{\varphi,u}\right\|_{e;\mathcal{B}^{\alpha}_{\log^{\beta}}\to\mathcal{W}^{(n)}_{\mu}} &= \left\|\mathcal{D}^{m}_{\varphi,u} - \Lambda\right\|_{e;\mathcal{B}^{\alpha}_{\log^{\beta}}\to\mathcal{W}^{(n)}_{\mu}} \\ &= \left\| \sum_{k=0}^{n} \Phi^{u,\varphi}_{k,n} \cdot C_{\varphi}\mathcal{D}^{m+k} \right\|_{e;\mathcal{B}^{\alpha}_{\log^{\beta}}\to\mathcal{H}^{\infty}_{\mu}} \\ &\le \sum_{k=0}^{n} \left\| \Phi^{u,\varphi}_{k,n} \cdot C_{\varphi}\mathcal{D}^{m+k} \right\|_{e;\mathcal{B}^{\alpha}_{\log^{\beta}}\to\mathcal{H}^{\infty}_{\mu}} \\ &\lesssim \sum_{k=0}^{n} \left\| \Phi^{u,\varphi}_{k,n} \cdot C_{\varphi} \right\|_{e;\mathcal{H}^{\infty}_{\nu_{\alpha+m+k-1,\beta}}\to\mathcal{H}^{\infty}_{\mu}}.\end{aligned}$$

The latter holds because $\mathcal{D}^{m+k} : \mathcal{B}^{\alpha}_{\log^{\beta}} \to \mathcal{H}^{\infty}_{\nu_{\alpha+m+k-1,\beta}}$ is bounded. The relation (3.1) gives an upper bound for the essential norm of $\mathcal{D}^{m}_{\varphi,u} : \mathcal{B}^{\alpha}_{\log^{\beta}} \to \mathcal{W}^{(n)}_{\mu}$. In the preceding theorem, we show that

$$\left\|\mathcal{D}^{m}_{\varphi,u}\right\|_{e;\mathcal{B}^{\alpha}_{\log^{\beta}}\to\mathcal{W}^{(n)}_{\mu}} \approx \sum_{k=0}^{n} \left\| \Phi^{u,\varphi}_{k,n} \cdot C_{\varphi} \right\|_{e;\mathcal{H}^{\infty}_{\nu_{\alpha+m+k-1,\beta}}\to\mathcal{H}^{\infty}_{\mu}}.$$

**Theorem 3.1.** *Let $u, \varphi \in \mathcal{H}(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $\mu$ be an arbitrary weight. Suppose that $\alpha > 0$, $\beta \ge 0$, $m \in \mathbb{N}_0$ such that $\alpha + m > 1$ and $\mathcal{D}^{m}_{\varphi,u} : \mathcal{B}^{\alpha}_{\log^{\beta}} \to \mathcal{W}^{(n)}_{\mu}$ is bounded. Then*

$$\left\|\mathcal{D}^{m}_{\varphi,u}\right\|_{e;\mathcal{B}^{\alpha}_{\log^{\beta}}\to\mathcal{W}^{(n)}_{\mu}} \approx \sum_{k=0}^{n} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|\Phi^{u,\varphi}_{k,n}(z)\right|}{\nu_{\alpha+m+k-1,\beta}(\varphi(z))}.$$

*Proof.* Since $\mathcal{D}^{m}_{\varphi,u} : \mathcal{B}^{\alpha}_{\log^{\beta}} \to \mathcal{W}^{(n)}_{\mu}$ is bounded, Theorem 1.1 and (3.1) imply that

$$\left\|\mathcal{D}^{m}_{\varphi,u}\right\|_{e;\mathcal{B}^{\alpha}_{\log^{\beta}}\to\mathcal{W}^{(n)}_{\mu}} \lesssim \sum_{k=0}^{n} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|\Phi^{u,\varphi}_{k,n}(z)\right|}{\nu_{\alpha+m+k-1,\beta}(\varphi(z))}.$$

Now, let $\{z_l\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_l)| > \frac{1}{2}$ and $|\varphi(z_l)| \to 1$. Fixing $k \in \{0, \dots, n\}$ and let $h_{l,k} = g_{z_l,k}$, defined in the proof of Theorem

2.5. Then, $\{h_{l,k}\}$ is a bounded sequences in $\mathcal{B}^{\alpha,0}_{\log^\beta}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$. Let $M=\sup\limits_{l}\|h_{l,k}\|_{\mathcal{B}^{\alpha}_{\log^\beta}}$ and $K\in\mathcal{K}(\mathcal{B}^{\alpha}_{\log^\beta},\mathcal{W}^{(n)}_\mu)$. It follows from [15, Lemma 2.10] that

$$\lim_{l\to\infty}\|Kh_{l,k}\|_{\mathcal{W}^{(n)}_\mu}=0.$$

Thus

$$\begin{aligned}M\left\|\mathcal{D}^m_{\varphi,u}-K\right\|_{\mathcal{B}^{\alpha}_{\log^\beta}\to\mathcal{W}^{(n)}_\mu}&\geq\limsup_{l\to\infty}\left\|\left(\mathcal{D}^m_{\varphi,u}-K\right)h_{l,k}\right\|_{\mathcal{W}^{(n)}_\mu}\\&\geq\limsup_{l\to\infty}\left\|\mathcal{D}^m_{\varphi,u}h_{l,k}\right\|_{\mathcal{W}^{(n)}_\mu}-\limsup_{l\to\infty}\|Kh_{l,k}\|_{\mathcal{W}^{(n)}_\mu}\\&\geq\limsup_{l\to\infty}\mu\left(z_l\right)\left|\sum_{j=0}^{n}h^{(m+j)}_{l,k}\left(\varphi(z_l)\right)\Phi^{u,\varphi}_{j,n}(z_l)\right|\\&=\limsup_{l\to\infty}\frac{\mu(z_l)\left|\varphi(z_l)\right|^{m+k}\left|\Phi^{u,\varphi}_{k,n}(z_l)\right|}{\nu_{\alpha+m+k-1,\beta}\left(\varphi(z_l)\right)}\\&=\limsup_{l\to\infty}\frac{\mu(z_l)\left|\Phi^{u,\varphi}_{k,n}(z_l)\right|}{\nu_{\alpha+m+k-1,\beta}\left(\varphi(z_l)\right)}.\end{aligned}$$

Therefore

$$\left\|\mathcal{D}^m_{\varphi,u}\right\|_{e;\mathcal{B}^{\alpha}_{\log^\beta}\to\mathcal{W}^{(n)}_\mu}\gtrsim\limsup_{|\varphi(z)|\to1}\frac{\mu(z)\left|\Phi^{u,\varphi}_{k,n}(z)\right|}{\nu_{\alpha+m+k-1,\beta}(\varphi(z))},$$

which completes the proof. □

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