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Primitive Ideal Space of Ultragraph C^*-algebras

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Abstract. In this paper, we describe the primitive ideal space of the C^*-algebra $C^*(\mathcal{G})$ associated to the ultragraph $\mathcal{G}$. We investigate the structure of the closed ideals of the quotient ultragraph C^*-algebra $C^*(\mathcal{G}/(H,S))$ which contain no nonzero set projections and then we characterize all non gauge-invariant primitive ideals. Our results generalize the Hong and Szymański's description of the primitive ideal space of a graph C^*-algebra by a simpler method.

1. Introduction

Let E be a countable directed graph and $C^*(E)$ be the associated C^*-algebra [2, 7, 10]. Using a special representation of $C^*(E)$, Hong and Szymański characterized the primitive ideal space of $C^*(E)$ and it's hull-kernel topology [8]. They constructed a family of irreducible representations of $C^*(E)$ associated to each maximal tail containing a cycle without exits. Theorem 2.10 of [8] shows that every non gauge-invariant primitive ideal of $C^*(E)$ is the kernel of such representation. Furthermore, there is a completely different approach for the primitive ideal space of graph C^*-algebras [4] (see also [3, 12]).

The motivation of the definition of ultragraph C^*-algebras [13] is to unify the theory of graph C^*-algebras and Exel-Laca algebras [5, 6]. The structure of ultragraph C^*-algebras is more complicated, because in ultragraphs the range of each edge is allowed to be a nonempty set of vertices rather than a single vertex. Any graph C^*-algebra can be considered as an ultragraph C^*-algebra and the C^*-algebras of ultragraphs with no singular vertices are precisely the Exel-Laca algebras.

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Furthermore, the class of ultragraph C^*-algebras are strictly lager than this class of directed graphs as well as the class of Exel-Laca algebras.

The aim of this paper is twofold. First, obtaining the Hong and Szymański's description of the primitive ideal space of $C^*(E)$ by a quite different and simpler method. Secondly, characterizing all primitive ideals of the ultragraph C^*-algebra $C^*(\mathcal{G})$ which are not invariant under the gauge action. The gauge-invariant primitive ideals of $C^*(\mathcal{G})$ are characterized in [11].

We start by recalling the definition of the quotient ultragraph $\mathcal{G}/(H,S)$ and its C^*-algebra $C^*(\mathcal{G}/(H,S))$ from [11]. To achieve our main result, we investigate the structure of the closed ideals of $C^*(\mathcal{G}/(H,S))$ which contain no nonzero projections. In particular, we show that if $\mathcal{G}/(H,S)$ contains a unique loop α without exits and I is a closed ideal of $C^*(\mathcal{G}/(H,S))$ containing no nonzero set projections, then I is contained in I_{α^0} (the closed ideal of $C^*(\mathcal{G}/(H,S))$ generated by the projections associated to the vertices of α). We then use this fact to show that there is a bijection between the downward directed sets containing a loop without exits and the set of all non gauge-invariant primitive ideals of $C^*(\mathcal{G})$.

2. Preliminaries

We begin by reviewing some background material on ultragraph, quotient ultragraph and their C^*-algebras. For more details see [11, 13].

An ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r_\mathcal{G}, s_\mathcal{G})$ consists of countable sets G^0 of vertices and $\mathcal{G}^1$ of edges, the source map $s_\mathcal{G} : \mathcal{G}^1 \to G^0$ and the range map $r_\mathcal{G} : \mathcal{G}^1 \to \mathcal{P}(G^0) \setminus \{\emptyset\}$, where $\mathcal{P}(G^0)$ is the collection of all subsets of G^0.

For a set X, a subcollection of $\mathcal{P}(X)$ is called an algebra if it is closed under the set operations $\cup$, $\cap$ and $\setminus$. If $\mathcal{G}$ is an ultragraph, we write $\mathcal{G}^0$ for the smallest algebra in $\mathcal{P}(G^0)$ containing $\big\{\{v\}, r_\mathcal{G}(e) : v \in G^0 \text{ and } e \in \mathcal{G}^1\big\}$.

Definition 2.1. Let $\mathcal{G}$ be an ultragraph. A subcollection $H \subseteq \mathcal{G}^0$ is hereditary if

(i) $\{s_\mathcal{G}(e)\} \in H$ implies $r_\mathcal{G}(e) \in H$ for all $e \in \mathcal{G}^1$,
(ii) $A \cup B \in H$ for all $A, B \in H$,
(iii) $A \in H$, $B \in \mathcal{G}^0$ and $B \subseteq A$, imply $B \in H$.

The hereditary subcollection $H \subseteq \mathcal{G}^0$ is saturated if for every $v \in G^0$ with $0 < \left|s_\mathcal{G}^{-1}(v)\right| < \infty$ we have

$$\big\{r_\mathcal{G}(e) : e \in \mathcal{G}^1 \text{ and } s_\mathcal{G}(e) = v\big\} \subseteq H \text{ implies } \{v\} \in H.$$

For a saturated hereditary subcollection $H \subseteq \mathcal{G}^0$, the breaking vertices of H is denoted by

$$B_H := \left\{ v \in G^0 : \left| s_{\mathcal{G}}^{-1}(v) \right| = \infty \text{ but } 0 < \left| s_{\mathcal{G}}^{-1}(v) \cap \{e : r_{\mathcal{G}}(e) \notin H\} \right| < \infty \right\}.$$

An admissible pair in $\mathcal{G}$ is a pair (H, S) of a saturated hereditary subcollection $H \subseteq \mathcal{G}^0$ and some $S \subseteq B_H$.

In order to define the quotient of ultragraphs we need to recall and introduce some notation. Let $\mathcal{G} = \left(G^0, \mathcal{G}^1, r_{\mathcal{G}}, s_{\mathcal{G}}\right)$ be an ultragraph and let (H, S) be an admissible pair in $\mathcal{G}$. Given $A \in \mathcal{P}(G^0)$, denote by $\overline{A} := A \cup \{w' : w \in A \cap (B_H \setminus S)\}$. Also, we write $\overline{\mathcal{G}^0}$ for the algebra in $\mathcal{P}(\overline{G^0})$ generated by the sets $\{v\}$, $\{w'\}$ and $\{\overline{r_{\mathcal{G}}(e)}\}$, where $v \in G^0$, $w \in B_H \setminus S$ and $e \in \mathcal{G}^1$.

Let $\sim$ be the relation on $\overline{\mathcal{G}^0}$ defined by $A \sim B$ if and only if there exists $V \in H$ such that $A \cup V = B \cup V$. Then, by [11, Lemma 3.5], $\sim$ is an equivalent relation on $\overline{\mathcal{G}^0}$ and the operations

$$[A] \cup [B] := [A \cup B], \qquad [A] \cap [B] := [A \cap B], \qquad [A] \setminus [B] := [A \setminus B],$$

are well-defined on the equivalent classes $\left\{[A] : A \in \overline{\mathcal{G}^0}\right\}$. One can see that $[A] = [B]$ if and only if both $A \setminus B$ and $B \setminus A$ belong to H.

Definition 2.2. Let $\mathcal{G} = \left(G^0, \mathcal{G}^1, r_{\mathcal{G}}, s_{\mathcal{G}}\right)$ be an ultragraph and let (H, S) be an admissible pair in $\mathcal{G}$. The quotient ultragraph of $\mathcal{G}$ by (H, S) is the quadruple $\mathcal{G}/(H, S) := \left(\Phi(G^0), \Phi(\mathcal{G}^1), r, s\right)$, where

$$\Phi(G^0) := \left\{[\{v\}] : v \in G^0 \setminus H\right\} \cup \left\{[\{w'\}] : w \in B_H \setminus S\right\},$$

$$\Phi(\mathcal{G}^1) := \left\{e \in \mathcal{G}^1 : r_{\mathcal{G}}(e) \notin H\right\},$$

and $s : \Phi(\mathcal{G}^1) \to \Phi(G^0)$ and $r : \Phi(\mathcal{G}^1) \to \{[A] : A \in \overline{\mathcal{G}^0}\}$ are the maps defined by $s(e) := [\{s_{\mathcal{G}}(e)\}]$ and $r(e) := [\overline{r_{\mathcal{G}}(e)}]$ for every $e \in \Phi(\mathcal{G}^1)$, respectively.

For the sake of simplicity, we will write $[v]$ instead of $[\{v\}]$ for every vertex $v \in G^0 \setminus H$. For $A, B \in \overline{\mathcal{G}^0}$, we write $[A] \subseteq [B]$ whenever $[A] \cap [B] = [A]$. The smallest algebra in $\{[A] : A \in \overline{\mathcal{G}^0}\}$ containing

$$\left\{[v], [w'] : v \in G^0 \setminus H, w \in B_H \setminus S\right\} \cup \left\{r(e) : e \in \Phi(\mathcal{G}^1)\right\},$$

is denoted by $\Phi(\mathcal{G}^0)$. It can be shown that $\Phi(\mathcal{G}^0) = \left\{[A] : A \in \overline{\mathcal{G}^0}\right\}$.

A vertex $[v] \in \Phi(G^0)$ is called a sink if $\left|s^{-1}\left([v]\right)\right| = \emptyset$ and is called an infinite emitter if $\left|s^{-1}\left([v]\right)\right| = \infty$. A singular vertex is a vertex that is either a sink or an infinite emitter. The set of singular vertices is denoted by $\Phi_{\text{sg}}(G^0)$.

Definition 2.3. Let $\mathcal{G}/(H,S)$ be a quotient ultragraph. A Cuntz-Krieger $\mathcal{G}/(H,S)$-family consists of projections $\left\{q_{[A]} : [A] \in \Phi(\mathcal{G}^0)\right\}$ and partial isometries $\left\{t_e : e \in \Phi(\mathcal{G}^1)\right\}$ with mutually orthogonal ranges such that

(1) $q_{[\emptyset]} = 0$, $q_{[A]}q_{[B]} = q_{[A]\cap[B]}$ and $q_{[A]\cup[B]} = q_{[A]} + q_{[B]} - q_{[A]\cap[B]}$;
(2) $t_e^* t_e = q_{r(e)}$;
(3) $t_e t_e^* \leq q_{s(e)}$;
(4) $q_{[v]} = \sum_{s(e)=[v]} t_e t_{e^*}$ whenever $0 < |s^{-1}([v])| < \infty$.

The C^*-algebra $C^*(\mathcal{G}/(H,S))$ is the universal C^*-algebra generated by a Cuntz-Krieger $\mathcal{G}/(H,S)$-family.

Due to the fact that in the quotient ultragraph $\mathcal{G}/(\emptyset,\emptyset)$, we have $[A] = \{A\}$, for every $A \in \mathcal{G}^0$. We can consider the ultragraph $\mathcal{G}$ as the quotient ultragraph $\mathcal{G}/(\emptyset,\emptyset)$. So, the definition of ultragraph C^*-algebras ([13, Defintion 2.7]) is an special case of Definition 2.3.

From now on we denote the universal Cuntz-Krieger $\mathcal{G}$-family and $\mathcal{G}/(H,S)$-family by $\{s,p\}$ and $\{t,q\}$, respectively. Also, we suppose that $C^*(\mathcal{G}) = C^*(s,p)$ and $C^*\left(\mathcal{G}/(H,S)\right) = C^*(t,q)$.

A path in the quotient ultragraph $\mathcal{G}/(H,S)$ is a sequence $\alpha := e_1 \cdots e_n$ of edges in $\Phi(\mathcal{G}^1)$ such that $s(e_{i+1}) \subseteq r(e_i)$ for $1 \leq i \leq n-1$. We say that the path α has length $|\alpha| := n$ and we consider the elements of $\Phi(\mathcal{G}^0)$ as the paths of length zero. We write $(\mathcal{G}/(H,S))^*$ for the set of finite paths. The maps r, s extend to $(\mathcal{G}/(H,S))^*$ in an obvious way.

By [11, Lemma 3.9], we have

$$C^*\left(\mathcal{G}/(H,S)\right) = \overline{\operatorname{span}}\left\{t_\alpha q_{[A]} t_\beta^* : [A] \in \Phi(\mathcal{G}^0) \text{ and } \alpha, \beta \in (\mathcal{G}/(H,S))^*\right\},$$

where $t_\alpha := t_{e_1} \cdots t_{e_n}$ if $\alpha = e_1 \cdots e_n$ and $t_\alpha := q_{[A]}$ if $\alpha = [A]$.

The universal property of $C^*\left(\mathcal{G}/(H,S)\right)$ gives the strongly continuous gauge action $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(\mathcal{G}/(H,S))$, which is characterized on generators by $\gamma_z(q_{[A]}) = q_{[A]}$ and $\gamma_z(t_e) = z t_e$ for every $A \in \Phi(\mathcal{G}^0)$, $e \in \Phi(\mathcal{G}^1)$ and $z \in \mathbb{T}$.

Definition 2.4. A loop in $\mathcal{G}/(H,S)$ is a path α with $|\alpha| \geq 1$ and $s(\alpha) \subseteq r(\alpha)$. A loop $\alpha = e_1 \cdots e_n$ has an exit if either $r(e_i) \neq s(e_{i+1})$ for some $1 \leq i \leq n$ or there exists an edge $f \in \Phi(\mathcal{G}^1)$ and an index i such that $s(f) \subseteq r(e_i)$ but $f \neq e_{i+1}$. The quotient ultragraph $\mathcal{G}/(H,S)$ satisfies Condition (L) if every loop in $\mathcal{G}/(H,S)$ has an exit.

Let H be a saturated hereditary subcollection of $\mathcal{G}^0$. We say that $\alpha = e_1 \cdots e_n$ is a loop in $\mathcal{G}^0 \setminus H$ if $r_\mathcal{G}(\alpha) \in \mathcal{G}^0 \setminus H$. Also, α has an exit in $\mathcal{G}^0 \setminus H$ if either $r_\mathcal{G}(e_i) \setminus s_\mathcal{G}(e_{i+1}) \in \mathcal{G}^0 \setminus H$ for some $1 \leq i \leq n$ or there exists an edge $f \in \mathcal{G}^1$ and an index i such that $r_\mathcal{G}(f) \in \mathcal{G}^0 \setminus H$ and $s_\mathcal{G}(f) \subseteq r_\mathcal{G}(e_i)$ but $f \neq e_{i+1}$. One can see that the quotient ultragraph

$\mathcal{G}/(H, B_H)$ satisfies Condition (L) if and only if every loop in $\mathcal{G}^0 \setminus H$ has an exit in $\mathcal{G}^0 \setminus H$.

Let (H, S) be an admissible pair in an ultragraph $\mathcal{G}$. For every $w \in B_H$, we define the projection

$$p_w^H := p_w - \sum_{s(e)=w,\ r_{\mathcal{G}}(e)\notin H} s_e s_e^*.$$

We denote by $I_{(H,S)}$ the (two-sided) ideal of $C^*(\mathcal{G})$ generated by the projections $\{p_A : A \in H\} \cup \{p_w^H : w \in S\}$. By [9, Theorem 6.12], the correspondence $(H, S) \mapsto I_{(H,S)}$ is a bijection from the set of all admissible pairs of $\mathcal{G}$ to the set of all gauge-invariant ideals of $C^*(\mathcal{G})$.

3. Closed Ideals of $C^*(\mathcal{G}/(H,S))$ Containing No Set Projections

The set of vertices in the loops without exits of $\mathcal{G}/(H, S)$ is denoted by $P_c(\mathcal{G}/(H, S))$.

Theorem 3.1. *Let $\mathcal{G}/(H, S)$ be a quotient ultragraph. If I is a closed ideal of $C^*(\mathcal{G}/(H, S))$ with $\{[A] \neq [\emptyset] : q_{[A]} \in I\} = \emptyset$, then $I \subseteq I_{P_c(\mathcal{G}/(H,S))}$.*

By [1, Theorem 5.4.4] we have Theorem 3.1 for row-finite directed graphs. To prove Theorem 3.1, we use the graph version of this theorem and the fact that a quotient ultragraph C^*-algebra is the direct limit of the certain C^*-algebras of finite graphs. We recall this direct limit from [11, Section 4] and then we prove Theorem 3.1.

Let $\mathcal{G}/(H, S)$ be a quotient ultragraph. For every finite subset F of $\Phi_{\mathrm{sg}}(G^0) \cup \Phi(\mathcal{G}^1)$ we construct a finite graph G_F as follows. Let

$$F^0 := F \cap \Phi_{\mathrm{sg}}(G^0), \qquad F^1 := F \cap \Phi\left(\mathcal{G}^1\right) = \{e_1, \ldots, e_n\}.$$

For every $\omega = (\omega_1, \ldots, \omega_n) \in \{0, 1\}^n \setminus \{0^n\}$, we define

$$r(\omega) := \bigcap_{\omega_i=1} r(e_i) \setminus \bigcup_{\omega_j=0} r(e_j), \qquad R(\omega) := r(\omega) \setminus \bigcup_{[v]\in F^0} [v],$$

which belong to $\Phi(\mathcal{G}^0)$. Set $\Gamma_0 := \{\omega \in \{0, 1\}^n \setminus \{0^n\}\}$ where vertices $[v_1], \ldots, [v_m]$ exist such that $R(\omega) = \bigcup_{i=1}^m [v_i]$ and $\emptyset \neq s^{-1}([v_i]) \subseteq F^1$ for $1 \leq i \leq m$, and

$$\Gamma_F := \{\omega \in \{0, 1\}^n \setminus \{0^n\} : R(\omega) \neq [\emptyset] \text{ and } \omega \notin \Gamma_0\}.$$

Define the finite graph $G_F = (G_F^0, G_F^1, r_F, s_F)$, where

$$\begin{aligned} G_F^0 &:= F^0 \cup F^1 \cup \Gamma_F, \\ G_F^1 &:= \left\{(e, f) \in F^1 \times F^1 : s(f) \subseteq r(e)\right\} \\ &\quad \cup \left\{(e, [v]) \in F^1 \times F^0 : [v] \subseteq r(e)\right\} \end{aligned}$$

$$\cup \left\{(e,\omega) \in F^1 \times \Gamma_F : \omega_i = 1 \text{ whenever } e = e_i\right\},$$

with $s_F(e,f) = s_F(e,[v]) = s_F(e,\omega) = e$ and $r_F(e,f) = f$, $r_F(e,[v]) = [v]$, $r_F(e,\omega) = \omega$.

Lemma 3.2. *Let $\mathcal{G}/(H,S)$ be a quotient ultragraph and let F be a finite subset of $\Phi_{\mathrm{sg}}(G^0) \cup \Phi(\mathcal{G}^1)$ containing $\{e_1, \dots, e_n\}$. Then $\alpha := e_1 \cdots e_n$ is a loop without exits in $\mathcal{G}/(H,S)$ if and only if $\widetilde{\alpha} := (e_1,e_2)\cdots(e_n,e_1)$ is a loop without exits in G_F.*

Proof. Since the elements of $F^0 \cup \Gamma_F$ are sinks in G_F, every loop in G_F is of the form $\widetilde{\beta}$ where β is a loop in $\mathcal{G}/(H,S)$. Suppose that $\widetilde{\alpha}$ has an exit in G_F. We distinguish three cases.

(i) If $(e_i,f) \in G_F^1$ is an exit for $\widetilde{\alpha}$, then $s(f) \subseteq r(e_i)$ and $(e_i,f) \neq (e_i,e_{i+1})$. Thus $f \neq e_{i+1}$ and hence f is an exit for α, which is impossible.

(ii) Since $r(e_{i+1}) = s(e_i)$ and $\left|s^{-1}(s(e_i))\right| = 1$ for every i, the elements of the form $(e_i,[v]) \in G_F^1$ can not be an exit for $\widetilde{\alpha}$.

(iii) Let $(e_i,\omega) \in G_F^1$ be an exit for $\widetilde{\alpha}$. Since $\omega_i = 1$,

$$\begin{aligned} r(\omega) &= \bigcap_{\omega_j=1} r(e_j) \setminus \bigcup_{\omega_j=0} r(e_j) \\ &\subseteq r(e_i) \\ &= s(e_{i+1}) \\ &= [s_{\mathcal{G}}(e_{i+1})]. \end{aligned}$$

As $\omega \in \Gamma_F$, we have $R(\omega) \neq [\emptyset]$ and hence $R(\omega) = s(e_{i+1}) = [s_{\mathcal{G}}(e_{i+1})]$. We note that $s^{-1}\left(s(e_{i+1})\right) = \{e_{i+1}\} \subseteq F^1$. Therefore $\omega \in \Gamma_0$, contradicts with $\omega \in \Gamma_F$.

Thus $\widetilde{\alpha}$ is a loop without exits in G_F. The converse follows from the argument of [11, Lemma 4.8]. $\square$

Let $C^*(\mathcal{G}/(H,S)) = C^*(t,q)$ and let F be a finite subset of $\Phi_{\mathrm{sg}}(G^0) \cup \Phi(\mathcal{G}^1)$. Then, by [11, Proposition 4.2] and [11, Corollary 4.3] the elements

$$\begin{aligned} &Q_e := t_e t_e^*, && T_{(e,[v])} := t_e Q_{[v]}, \\ &Q_\omega := q_{R(\omega)} \left(1 - \sum_{e\in F^1} t_e t_e^*\right), && T_{(e,f)} := t_e Q_f, \\ &Q_{[v]} := q_{[v]} \left(1 - \sum_{e\in F^1} t_e t_e^*\right), && T_{(e,\omega)} := t_e Q_\omega, \end{aligned}$$

form a Cuntz-Krieger G_F-family such that

$$\begin{aligned} C^*(G_F) &= C^*(T,Q) \\ &= C^*\left(t_e, q_{[v]} : [v] \in F^0, e \in F^1\right). \end{aligned}$$

Proof of Theorem 3.1. Let $\{F_n\}$ be an increasing sequence of finite subsets of $\Phi_{\rm sg}(G^0) \cup \Phi(\mathcal{G}^1)$ such that $\cup_{n=1}^{\infty} F_n = \Phi_{\rm sg}(G^0) \cup \Phi(\mathcal{G}^1)$. Then $C^*(G_{F_n}) \subseteq C^*(G_{F_{n+1}})$ and

$$\begin{aligned}\overline{\bigcup_n C^*(G_{F_n})} &= C^*\left(t_e, q_{[v]} : [v] \in \Phi_{\rm sg}(G^0), e \in \Phi(\mathcal{G}^1)\right) \\ &= C^*\left(\mathcal{G}/(H,S)\right).\end{aligned}$$

Thus we may write $C^*(\mathcal{G}/(H,S)) = \varinjlim C^*(G_{F_n})$. Now, let I be a closed ideal of $C^*(\mathcal{G}/(H,S))$ such that $\left\{[A] \neq [\emptyset] : q_{[A]} \in I\right\} = \emptyset$. Set $I_n := I \cap C^*(G_{F_n})$. We show that I_n is a closed ideal of $C^*(G_{F_n})$ containing no vertex projections, and then we conclude that there is no nonzero projection in I_n. Assume to the contrary that $Q_x \in I_n$ for some $x \in G^0_{F_n}$. If $x \in F_n^0$ or $x \in F_n^1$, then by multiplying Q_x with suitable members, it can be shown that I contains a set projection which is impossible. So let $x = \omega \in \Gamma_{F_n}$. Hence there exists a vertex $[v] \subseteq R(\omega)$ such that either $[v]$ is a sink or there is an edge $f \in \Phi(\mathcal{G}^1) \setminus F^1$ with $s(f) = [v]$. In the former case, we deduce that $q_{[v]}Q_x = q_{[v]} \in I$ and in the later case $t_f^* Q_x t_f = q_{r(f)} \in I$, which contradicts the hypothesis. Therefore I_n containing no vertex projections.

Suppose that I_n contains a projection $0 \neq p \in C^*(G_{F_n})$ and let J be the ideal of $C^*(G_{F_n})$ generated by p. It follows from [1, Corollary 5.3.7] that J is a gauge-invariant ideal and thus, by [2, Theorem 4.1], J is generated by a set of vertex projections, contradicting that I_n contains no vertex projections.

By [1, Proposition 5.4.3] we have $I_n \subseteq I_{P_c(G_{F_n})}$ for all n. From Lemma 3.2, we know that

$$P_c(G_F) = \left\{\widetilde{\alpha} : \alpha \in P_c(\mathcal{G}/(H,S)) \text{ and } \alpha^1 \subseteq G_F^1\right\}.$$

Thus $I_{P_c(G_{F_n})}$ is generated by the elements of the form $Q_e = t_e t_e^*$, where e is the edge of a loop without exits in $\mathcal{G}/(H,S)$. Hence for every $n \in \mathbb{N}$ we have $I_n \subseteq I_{P_c(G_{F_n})} \subseteq I_{P_c(\mathcal{G}/(H,S))}$. Since $I = \varinjlim(I \cap C^*(G_{F_n})) = \varinjlim I_n$, we deduce that $I \subseteq I_{P_c(\mathcal{G}/(H,S))}$, as desired. □

4. Primitive Ideals

In this section, we characterize all primitive ideals of the ultragraph C^*-algebra $C^*(\mathcal{G})$ which are not invariant under the gauge action.

We recall the definition of downward directed sets from [11, Definition 5.3]. Let $\mathcal{G}$ be an ultragraph. Define a relation on $\mathcal{G}^0$ by setting $A \geq B$ if either $B \subseteq A$ or there exists a path α of positive length such that $s_{\mathcal{G}}(\alpha) \in A$ and $B \subseteq r_{\mathcal{G}}(\alpha)$. A subcollection $M \subseteq \mathcal{G}^0$ is called downward directed if for every $A, B \in M$ there exists $\emptyset \neq C \in M$ such that $A, B \geq C$.

Let I be a closed ideal of $C^*(\mathcal{G})$. We denote $H_I := \{A \in \mathcal{G}^0 : p_A \in I\}$.

Lemma 4.1. *Let $\mathcal{G}$ be an ultragraph and I be a closed ideal of $C^*(\mathcal{G})$. If I is primitive, then $\mathcal{G}^0 \setminus H_I$ is downward directed.*

Proof. Let $A, B \in \mathcal{G}^0 \setminus H_I$. Denote by $\widetilde{x}$ the image of $x \in C^*(\mathcal{G})$ in $C^*(\mathcal{G})/I$. Since $A, B \notin H_I$, the projections $\widetilde{p}_A$ and $\widetilde{p}_B$ are nonzero. Thus the ideals

$$J_1 := \big(C^*(\mathcal{G})/I\big)\widetilde{p}_A\big(C^*(\mathcal{G})/I\big), \qquad J_2 := \big(C^*(\mathcal{G})/I\big)\widetilde{p}_B\big(C^*(\mathcal{G})/I\big),$$

are non-zero. Since $C^*(\mathcal{G})/I$ is a primitive C^*-algebra, it follows that

$$J_1 J_2 = \big(C^*(\mathcal{G})/I\big)\widetilde{p}_A\big(C^*(\mathcal{G})/I\big)\widetilde{p}_B\big(C^*(\mathcal{G})/I\big),$$

is also a nonzero ideal of $C^*(\mathcal{G})/I$. Thus, $\widetilde{p}_A\big(C^*(\mathcal{G})/I\big)\widetilde{p}_B \neq \{0\}$. We note that

$$C^*(\mathcal{G})/I = \overline{\operatorname{span}}\left\{\widetilde{s}_\alpha \widetilde{p}_C \widetilde{s}_\beta^* : C \in \mathcal{G}^0, \alpha, \beta \in \mathcal{G}^* \text{ and } r_{\mathcal{G}}(\alpha) \cap C \cap r_{\mathcal{G}}(\beta) \neq \emptyset\right\}.$$

Hence there exist $\alpha, \beta \in \mathcal{G}^*$ and $C \in \mathcal{G}^0$ such that $\widetilde{p}_A(\widetilde{s}_\alpha \widetilde{p}_C \widetilde{s}_\beta^*)\widetilde{p}_B \neq 0$, which implies that $p_A(s_\alpha p_C s_\beta^*)p_B \neq 0$. Thus $s_{\mathcal{G}}(\alpha) \in A$ and $s_{\mathcal{G}}(\beta) \in B$. If we set $D := r_{\mathcal{G}}(\alpha) \cap C \cap r_{\mathcal{G}}(\beta)$, then we deduce that $A, B \geq D$. Therefore $\mathcal{G}^0 \setminus H_I$ is downward directed. □

Lemma 4.2. *Let $\mathcal{G}/(H,S)$ contains a unique (up to permutation) loop α without exits and let $s(\alpha) = [v]$. If I is a nonzero primitive ideal of $C^*(\mathcal{G}/(H,S))$ with $\{[A] \neq [\emptyset] : q_{[A]} \in I\} = \emptyset$, then there exists $t \in \mathbb{T}$ such that I is generated by the element $tq_{[v]} - t_\alpha$.*

Proof. Note that every primitive ideal of $C(\mathbb{T})$ (which is maximal in the non-trivial closed ideals of $C(\mathbb{T})$) is of the form

$$N_t = \{f \in C(\mathbb{T}) : f(t) = 0\},$$

for some $t \in \mathbb{T}$. By Theorem 3.1, we have $I \subseteq I_{\alpha^0}$. From the proof of [11, Lemma 5.1], we know that I_{α^0} is Morita equivalent to $C(\mathbb{T})$ by the Morita correspondence $J \mapsto q_{[v]} J q_{[v]}$. Since the primeness is preserved by the Morita correspondence, there exists $t \in \mathbb{T}$ such that I maps to N_t. As the ideal N_t is generated by the function $f_t(z) = t - z$, we deduce that I is generated by $tq_{[v]} - t_\alpha$. □

Remark 4.3. Let $\mathcal{G}/(H,S)$ contain a loop α without exits and α' be a permutation of α. Denote $[v] = s(\alpha)$ and $[w] = s(\alpha')$. Let δ be a subpath of α and α' such that $\delta\alpha = \alpha'\delta$. If $t \in \mathbb{T}$, then the ideals of $C^*(\mathcal{G}/(H,S))$ generated by $tq_{[v]} - t_\alpha$ and $tq_{[w]} - t_{\alpha'}$ are equal, because $t_\delta\left(tq_{[v]} - t_\alpha\right)t_\delta^* = tq_{[w]} - t_{\alpha'}$.

Let H be a saturated hereditary subcollection of $\mathcal{G}^0$ and let $\mathcal{G}^0 \setminus H$ contain a unique (up to permutation) loop α without exits in $\mathcal{G}^0 \setminus H$. For $t \in \mathbb{T}$, the ideal of $C^*\big(\mathcal{G}/(H, B_H)\big)$ generated by

$$\left\{p_A, p_w^H : A \in H, w \in B_H\right\} \cup \left\{tp_{s_{\mathcal{G}}(\alpha)} - s_\alpha\right\},$$

is denoted by $I_{\langle H,B_H,t\rangle}$.

Theorem 4.4. *Let $\mathcal{G}$ be an ultragraph and I be a non gauge-invariant ideal of $C^*(\mathcal{G})$. Denote $H := H_I$. Then I is a primitive (prime) ideal if and only if $\mathcal{G}^0 \setminus H$ is downward directed, $\mathcal{G}^0 \setminus H$ contains a (unique) loop α without exits in $\mathcal{G}^0 \setminus H$ and there exists $t \in \mathbb{T}$ such that $I = I_{\langle H,B_H,t\rangle}$.*

Proof. Suppose that I is a primitive ideal of $C^*(\mathcal{G})$. By Lemma 4.1, $\mathcal{G}^0 \setminus H$ is downward directed. Denote $S := \{w \in B_H : p_w^H \in I\}$. If we write $\widetilde{I}$ for the image of I in the quotient $C^*(\mathcal{G})/I_{(H,S)}$, then by [11, Proposition 4.6], we have $C^*(\mathcal{G}/(H, S)) \cong C^*(\mathcal{G})/I_{(H,S)}$ and

$$\left\{[A] \neq [\emptyset] : [A] \in \Phi(\mathcal{G}^0) \text{ and } q_{[A]} \in \widetilde{I}\right\} = \emptyset.$$

If I is not gauge-invariant, then by [9, Theorem 6.12], $\widetilde{I} \neq \{0\}$. Hence by the Cuntz-Krieger uniqueness theorem for quotient ultragraphs [11, Theorem 4.9], $\mathcal{G}/(H, S)$ contains a loop α without exits. For the uniqueness, suppose β is a loop without exits in $\mathcal{G}/(H, S)$. Thus α and β are loops without exits in $\mathcal{G}^0 \setminus H$. By downward directed property of $\mathcal{G}^0 \setminus H$ there exists $\emptyset \neq C \in \mathcal{G}^0 \setminus H$ such that $s_{\mathcal{G}}(\alpha), s_{\mathcal{G}}(\beta) \geq C$. Since α, β have no exit in $\mathcal{G}^0 \setminus H$, we must have $s_{\mathcal{G}}(\alpha) \in \beta^0$ and $s_{\mathcal{G}}(\beta) \in \alpha^0$. The absence of exits implies that β is a permutation of α.

Now, we show that $S = B_H$. Since $\widetilde{I}$ is a primitive ideal of $C^*(\mathcal{G})/I_{(H,S)}$, we have that $C^*(\mathcal{G}/(H, S))/\widetilde{I}$ is a primitive C^*-algebra. Let $w \in B_H \setminus S$. Then $q_{[w']} + \widetilde{I}$ and $q_{s(\alpha)} + \widetilde{I}$ are nonzero projections in $C^*(\mathcal{G}/(H, S))/\widetilde{I}$. Similar to the proof of Lemma 4.1, let J_1 and J_2 be the ideals of $C^*(\mathcal{G}/(H, S))/\widetilde{I}$ generated by $q_{[w']} + \widetilde{I}$ and $q_{s(\alpha)} + \widetilde{I}$, respectively. Thus $J_1J_2 \neq 0$ and hence

$$\left(q_{[w']} + \widetilde{I}\right) \left(C^*(\mathcal{G}/(H, S))/\widetilde{I}\right) \left(q_{s(\alpha)} + \widetilde{I}\right) \neq \{0\}.$$

Consequently, there exist $\mu, \nu \in (\mathcal{G}/(H, S))^*$ and $[A] \in \Phi(\mathcal{G}^0)$ such that $q_{[w']}(t_\mu q_{[A]} t_\nu^*) q_{s(\alpha)} \neq 0$, which is impossible because $[w']$ is a sink and α has exit in $\mathcal{G}/(H, S)$.

From the proof of [11, Proposition 4.6], we know that $C^*(\mathcal{G}/(H, B_H)) = C^*(t_e, q_{[A]})$ where

$$\begin{aligned} q_{[A]} &:= p_A + I_{(H,B_H)} && \text{for } A \in \Phi(\mathcal{G}^0), \\ t_e &:= s_e + I_{(H,B_H)} && \text{for } e \in \Phi(\mathcal{G}^1). \end{aligned}$$

Since $\widetilde{I}$ is not gauge-invariant, by Lemma 4.2, there exists $t \in \mathbb{T}$ such that $\widetilde{I}$ is generated by $(tp_{s_{\mathcal{G}}(\alpha)} - s_\alpha) + I_{(H,B_H)}$. This implies that I is generated by

$$\left\{p_A, p_w^H, tp_{s_{\mathcal{G}}(\alpha)} - s_\alpha : A \in H, w \in B_H\right\},$$

and so $I = I_{\langle H,B_H,t\rangle}$.

For the converse, let $\mathcal{G}^0 \setminus H$ be downward directed. Thus the only loop (up to permutation) without exits in $\mathcal{G}/(H, B_H)$ is α. Hence $I = I_{\langle H,B_H,t\rangle}$ is well-defined. Let $\widetilde{x}$ be the image of $x \in C^*(\mathcal{G})$ in $C^*(\mathcal{G})/I_{(H,B_H)}$. As we pointed out in the proof of Lemma 4.2, since $\widetilde{I}$ is generated by

$$\begin{aligned} tq_{s(\alpha)} - t_\alpha &= t\widetilde{p}_{s_{\mathcal{G}}(\alpha)} - \widetilde{s}_\alpha \\ &= \left(tp_{s_{\mathcal{G}}(\alpha)} - s_\alpha\right) + I_{(H,B_H)}, \end{aligned}$$

we deduce that $\widetilde{I}$ is a non gauge-invariant primitive ideal of $C^*(\mathcal{G})/I_{(H,B_H)}$. Suppose that J_1, J_2 are two ideals in $C^*(\mathcal{G})$ and $J_1J_2 \subseteq I$. Then $\widetilde{J_1}\widetilde{J_2} = \widetilde{J_1J_2} \subseteq \widetilde{I}$, and so either $\widetilde{J_1} \subseteq \widetilde{I}$ or $\widetilde{J_2} \subseteq \widetilde{I}$. Since $I_{(H,B_H)} \subset I$, we have either $J_1 \subseteq I$ or $J_2 \subseteq I$. Consequently, I is a non gauge-invariant primitive ideal. □

Let α be a loop in $\mathcal{G}$. We say that every vertex on α lies on exactly one loop, if for every loop β (distinct from α) and every $e \in \beta^1$ we have $r_{\mathcal{G}}(e) \cap \alpha^0 = \emptyset$. We denote such loops (up to permutation) by $C_\kappa(\mathcal{G})$.

Let $\mathcal{G}^0 \setminus H$ contain a loop α without exits in $\mathcal{G}^0 \setminus H$. Then $\alpha \in C_\kappa(\mathcal{G})$. Define

$$M(\alpha^0) := \left\{\emptyset \neq A \in \mathcal{G}^0 : A \geq s_{\mathcal{G}}(\alpha)\right\}.$$

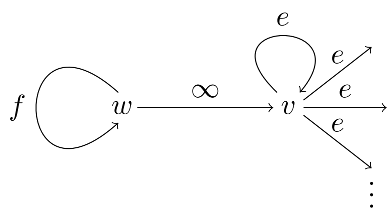
If $\mathcal{G}^0 \setminus H$ is downward directed, then it can be shown that $M(\alpha^0) = \mathcal{G}^0 \setminus H$. Conversely, if $\alpha \in C_\kappa(\mathcal{G})$ and if we set $H := \{A \in \mathcal{G}^0 : A \ngeq s_{\mathcal{G}}(\alpha)\}$, then H is a saturated hereditary subcollection of $\mathcal{G}^0$ and $M(\alpha^0) = \mathcal{G}^0 \setminus H$. Also, α is a loop without exits in $\mathcal{G}^0 \setminus H$. Since $\alpha \in C_\kappa(\mathcal{G})$, we deduce that $\mathcal{G}^0 \setminus H$ is downward directed. Therefore, we may conclude that every non gauge-invariant primitive ideal of $C^*(\mathcal{G})$ is exactly corresponding with a such loop $\alpha \in C_\kappa(\mathcal{G})$ and some $t \in \mathbb{T}$.

Corollary 4.5. *Let $\mathcal{G}$ be a ultragraph. If $\mathrm{Prim}_\tau(C^*(\mathcal{G}))$ is the set of non gauge-invariant primitive ideals of $C^*(\mathcal{G})$, then there exists a one-to-one corresponding between $C_\kappa(\mathcal{G}) \times \mathbb{T}$ and $\mathrm{Prim}_\tau(C^*(\mathcal{G}))$ as*

$$(\alpha, t) \longleftrightarrow I_{\langle H,B_H,t\rangle},$$

where $H := \mathcal{G}^0 \setminus M(\alpha^0)$.

Example 4.6. Let $\mathcal{G}$ be the following ultragraph.



There are two loops $\alpha_1 := e$ and $\alpha_2 := f$ in $\mathcal{G}$. Since $r(f) \cap \{v\} = r(e) \cap \{w\} = \emptyset$, we have $\alpha_1, \alpha_2 \in C_\kappa(\mathcal{G})$. The bijection of Corollary 4.5 identifies $\{\alpha_1, \alpha_2\} \times \mathbb{T}$ with $\mathrm{Prim}_\tau(C^*(\mathcal{G}))$.

Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$. We observe that

$$\mathcal{G}^0 = \overline{\{\emptyset, \{v\}, \{w\}, r(e), \{v, w\}, r(e) \cup \{w\}, r(e) \setminus \{v\}, \{w\} \cup r(e) \setminus \{v\}\}}.$$

We now see that $M(\alpha_1^0) = \mathcal{G}^0 \setminus \overline{\{r(e) \setminus \{v\}\}}$ and $M(\alpha_2^0) = \{\{w\}\}$. Set $H_2 := \overline{\{r(e) \setminus \{v\}\}}$ and $H_1 := \mathcal{G}^0 \setminus \{\{w\}\}$. Then $B_{H_1} = \{w\}$ and $B_{H_2} = \emptyset$. Consequently,

$$\mathrm{Prim}_\tau(C^*(\mathcal{G})) = \left\{ I_{\langle H_1, B_{H_1}, t\rangle}, I_{\langle H_2, B_{H_2}, t\rangle} : t \in \mathbb{T} \right\}.$$

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