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Simple Construction of a Frame which is ϵ-nearly Parseval and ϵ-nearly Unit Norm

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Abstract. In this paper, we will provide a simple method for starting with a given finite frame for an n-dimensional Hilbert space $\mathcal{H}_n$ with nonzero elements and producing a frame which is ϵ-nearly Parseval and ϵ-nearly unit norm. Also, the concept of the ϵ-nearly equal frame operators for two given frames is presented. Moreover, we characterize all bounded invertible operators T on the finite or infinite dimensional Hilbert space $\mathcal{H}$ such that $\{f_k\}_{k=1}^{\infty}$ and $\{Tf_k\}_{k=1}^{\infty}$ are ϵ-nearly equal frame operators, where $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$. Finally, we introduce and characterize all operator dual Parseval frames of a given Parseval frame.

1. Introduction

Given a separable Hilbert space $\mathcal{H}$ with inner product $\langle .,. \rangle$, a sequence $\{f_k\}_{k=1}^{\infty}$ is called a frame for $\mathcal{H}$ if there exist constants $A > 0$, $B < \infty$ such that for all $f \in \mathcal{H}$,

$$A \|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B \|f\|^2, \tag{1.1}$$

where A, B are respectively the lower and upper frame bounds. The second inequality of the frame condition (1.1) is also known as the Bessel condition for $\{f_k\}_{k=1}^{\infty}$. $\{f_k\}_{k=1}^{\infty}$ is called a tight frame, if $A = B$. A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is called a frame sequence in $\mathcal{H}$, if it is a frame for $\overline{span}\,\{f_k\}_{k=1}^{\infty}$.

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The bounded linear operator T defined by

$$T : \ell_2(\mathbb{N}) \to \mathcal{H}, \qquad T\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k f_k,$$

is called the pre-frame operator or synthesis operator of $\{f_k\}_{k=1}^{\infty}$. Also the bounded linear operator S defined by

$$S : \mathcal{H} \to \mathcal{H}, \qquad Sf = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k,$$

is called the frame operator of $\{f_k\}_{k=1}^{\infty}$. A Riesz basis for $\mathcal{H}$ is a family of the form $\{Ve_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $V \in B(\mathcal{H})$ is an invertible operator. Every Riesz basis for $\mathcal{H}$ is a frame for $\mathcal{H}$. Two frames $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are dual frames for $\mathcal{H}$ if

$$\begin{aligned} f &= \sum_{k=1}^{\infty} \langle f, f_k \rangle g_k \\ &= \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \quad \forall f \in \mathcal{H}. \end{aligned}$$

The frame $\left\{\tilde{f}_k\right\}_{k=1}^{\infty}$ defined by $\tilde{f}_k = S^{-1} f_k$ is a dual frame of the frame $\{f_k\}_{k=1}^{\infty}$ that is called canonical dual frame of $\{f_k\}_{k=1}^{\infty}$. For more detailes concerning frames we refer to[2, 6, 12, 15].

A tight frame with frame bound 1 is called a Parseval frame. Parseval frames are useful in applications, as they provide decompostion

$$f = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, \quad \forall f \in \mathcal{H}.$$

It is well-known that $\left\{S^{\frac{-1}{2}} f_k\right\}_{k=1}^{\infty}$ is a Parseval frame for $\mathcal{H}$, if $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with frame operator S. However, in general the computation of the operator $S^{\frac{-1}{2}}$ is not very easy. A frame $\{f_k\}_{k=1}^{\infty}$ is a unit norm frame if $\|f_k\| = 1$ for all k. Parseval frames and unit norm frames are particularly useful. However, it is difficult to construct frames which possess both of these properties simultaneously, called unit norm Parseval frames. Unit norm Parseval frames are known to be exceptionally robust against additive noise and erasures [5, 10] for high redundancies.

In the next section, we present a simple procedure on which, when it is applied to a given frame for an n-dimensional Hilbert space $\mathcal{H}_n$ with nonzero elements, produces a frame which is almost Parseval and almost unit norm. A characterization of invertible operators $T \in B(\mathcal{H})$ which make $\{f_k\}_{k=1}^{\infty}$ and $\{Tf_k\}_{k=1}^{\infty}$ into ϵ-nearly equal frame operators, is provided in Section 3 . Section 4 is then devoted to the study of the operator dual Parseval frames.

2. ϵ-nearly Parseval Frames

Equal-norm Parseval frames have many applications in frame theory. Bodmann and Casazza presented a construction method which generates equal-norm Parseval frames of Parseval frames [3]. Also, they developed this method to generate an equal-norm Parseval frame which is close to given ϵ-nearly Parseval frame.

Definition 2.1. Given $0 < \epsilon < 1$, a frame (finite frame) $\{f_k\}_{k=1}^m$ for an n-dimensional Hilbert space $\mathcal{H}_n$ is ϵ-nearly Parseval frame, if its frame operator S satisfies

$$(1-\epsilon)\, I \leq S \leq (1+\epsilon)\, I.$$

Also the frame is ϵ-nearly unit norm, provided that

$$\left| \|f_k\|^2 - 1 \right| < \epsilon, \quad \forall\ k = 1, 2, \ldots, m.$$

Freeman and Speegle presented a method for construction ϵ-nearly Parseval frames in [9].

A simple construction of a family of ϵ-nearly Parseval frames is given in the next theorem.

Theorem 2.2. *Suppose $\{f_k\}_{k=1}^m$ is a frame for $\mathcal{H}_n$. Then there exist $\lambda > 0$ and $\epsilon \in (0,1)$ such that $\left\{\frac{1}{\sqrt{\lambda}} f_k\right\}_{k=1}^m$ is an ϵ-nearly Parseval frame.*

Proof. Suppose $\{e_j\}_{j=1}^n$ is an orthonormal basis for $\mathcal{H}_n$ consisting of eigenvectors of S, where S is the frame operator of $\{f_k\}_{k=1}^m$. Let $\{\lambda_j\}_{j=1}^n$ denote the corresponding eigenvalues. Note that for each j,

$$\lambda_j = \lambda_j \|e_j\| = \langle S e_j, e_j \rangle \geq 0.$$

Set $\alpha := \max\{\lambda_j\}_{j=1}^n$ and $\beta := \min\{\lambda_j\}_{j=1}^n$. Now $\beta \neq 0$. Indeed, if $\beta = 0$, then $S - \beta I = S$ is not invertible and this a contradiction by the invertibility of S.

For any $\epsilon \in \left(\frac{\alpha-\beta}{\alpha+\beta}, 1\right)$ and $\lambda \in \left(\frac{\alpha}{1+\epsilon}, \frac{\beta}{1-\epsilon}\right)$ we have

$$\left| \frac{\lambda_j}{\lambda} - 1 \right| < \epsilon, \quad \forall\ j = 1, 2, \ldots, n.$$

Now $\left\{\frac{1}{\sqrt{\lambda}} f_k\right\}_{k=1}^m$ is a weighted frame for $\mathcal{H}_n$ with frame operator $S_1 = \frac{1}{\lambda} S$ [1].

For all $f \in \mathcal{H}_n$ we have

$$\begin{aligned}
\|(I - S_1) f\|^2 &= \left\| \sum_{j=1}^{n} \langle f, e_j \rangle e_j - \sum_{j=1}^{n} \langle f, e_j \rangle S_1 e_j \right\|^2 \\
&= \left\| \sum_{j=1}^{n} \left(\frac{\lambda_j}{\lambda} - 1 \right) \langle f, e_j \rangle e_j \right\|^2 \\
&= \sum_{j=1}^{n} \left| \frac{\lambda_j}{\lambda} - 1 \right|^2 |\langle f, e_j \rangle|^2 \\
&< \epsilon^2 \|f\|^2 ,
\end{aligned}$$

and hence $\|I - S_1\| < \epsilon$.

Since $I - S_1$ is self adjoint, we have $|\langle (I - S_1) f, f \rangle| \leq \epsilon \|f\|^2$ and $\langle (I - S_1) f, f \rangle \in \mathbb{R}$, for all $f \in \mathcal{H}_n$. Thus,

$$-\epsilon \|f\|^2 \leq \langle (I - S_1) f, f \rangle \leq \epsilon \|f\|^2 , \quad \forall f \in \mathcal{H}_n.$$

This implies

$$(1 - \epsilon) I \leq S_1 \leq (1 + \epsilon) I.$$

Hence, $\left\{ \frac{1}{\sqrt{\lambda}} f_k \right\}_{k=1}^{m}$ is an ϵ-nearly Parseval frame. □

A simple procedure on which, when it is applied to a given frame for a n-dimensional Hilbert space $\mathcal{H}_n$ with nonzero elements, produces a frame which is almost Parseval and almost unit norm, is presented in the next theorem.

Theorem 2.3. *Suppose $\{f_k\}_{k=1}^{m}$ is a frame for $\mathcal{H}_n$ with nonzero elements. Then there exist $\lambda > 0$ and $\epsilon \in (0, 1)$ such that $\left\{ \frac{1}{\sqrt{\lambda}} f_k \right\}_{k=1}^{m}$ is an ϵ-nearly Parseval frame and ϵ-nearly unit norm.*

Proof. Let $\{\lambda_j\}_{j=1}^{n}$ and $\{e_j\}_{j=1}^{n}$ be as in the proof of Theorem 2.2. Set $\alpha := \max \{\lambda_j\}_{j=1}^{n} \cup \left\{ \|f_k\|^2 \right\}_{k=1}^{m}$ and $\beta := \min \{\lambda_j\}_{j=1}^{n} \cup \left\{ \|f_k\|^2 \right\}_{k=1}^{m}$. Now $\beta \neq 0$, since S is invertible and $f_k \neq 0$, for all $k = 1, 2, ..., m$. For any $\epsilon \in \left(\frac{\alpha - \beta}{\alpha + \beta}, 1 \right)$ and $\lambda \in \left(\frac{\alpha}{1+\epsilon}, \frac{\beta}{1-\epsilon} \right)$ we have

$$\left| \frac{\lambda_j}{\lambda} - 1 \right| < \epsilon, \quad \forall\ j = 1, 2, \ldots, n,$$

and

$$\left| \left\| \frac{f_k}{\sqrt{\lambda}} \right\|^2 - 1 \right| < \epsilon, \quad \forall\ k = 1, 2, \ldots, m.$$

Thus, $\left\{\frac{1}{\sqrt{\lambda}}f_k\right\}_{k=1}^{m}$ is an ϵ-nearly Parseval frame and ϵ-nearly unit norm. $\square$

3. ϵ-nearly Equal Frame Operators

A characterisation of operators which generate frames with a prescribed frame operator is given in [4]. In this section we introduce ϵ-nearly equal frame operators and characterise all operators $T \in B(\mathcal{H})$ shch that two frames $\{f_k\}_{k=1}^{\infty}$ and $\{Tf_k\}_{k=1}^{\infty}$ are ϵ-nearly equal frame operators. Also, we present a charactristion of ϵ-nearly Parseval frames by their frame operators.

Definition 3.1. Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be two frames for $\mathcal{H}$ with frame operators S_1 and S_2 , respectively and $0 < \epsilon < 1$. Then $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are called ϵ-nearly equal frame operators, if $\|S_1 - S_2\| \leq \epsilon$.

ϵ-nearly Parseval frames are precisely the frames which are ϵ-nearly equal frame operators with a Parseval frame.

Theorem 3.2. *Suppose $\{f_k\}_{k=1}^{\infty}$ is a Parseval frame for $\mathcal{H}$ and $\{g_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$. Then $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are ϵ-nearly equal frame operators if and only if $\{g_k\}_{k=1}^{\infty}$ is an ϵ-nearly Parseval frame for $\mathcal{H}$.*

Proof. If $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are ϵ-nearly equal frame operators, then $\|I - S\| \leq \epsilon$, where S is the frame operator of $\{g_k\}_{k=1}^{\infty}$. Thus, a similar argument as in the proof of Theorem 2.3 shows that

$$(1-\epsilon)\, I \leq S \leq (1+\epsilon)\, I.$$

This implies that $\{g_k\}_{k=1}^{\infty}$ is an ϵ-nearly Parseval frame for $\mathcal{H}$. Conversely if $\{g_k\}_{k=1}^{\infty}$ is an ϵ-nearly Parseval frame for $\mathcal{H}$, then

$$(1-\epsilon)\, I \leq S \leq (1+\epsilon)\, I,$$

and hence $\|I - S\| \leq \epsilon$. Thus, $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are ϵ-nearly equal frame operators. $\square$

Example 3.3. Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with frame operator S and $0 < \epsilon < 1$. There exists $N \in \mathbb{N}$ such that $\epsilon < N\|S\|$. Then $T : \mathcal{H} \to \mathcal{H}$, $Tf = \sqrt{1 - \frac{\epsilon}{N\|S\|}}f$ is an invertible bounded linear operator on $\mathcal{H}$. If S' is the frame operator of weighted frame $\{Tf_k\}_{k=1}^{\infty}$, then $S' = \left(1 - \frac{\epsilon}{N\|S\|}\right) S$ and hence

$$\begin{aligned}\left\|S - S'\right\| &= \left\|S - \left(1 - \frac{\epsilon}{N\,\|S\|}\right) S\right\| \\ &= \frac{\epsilon}{N} \\ &< \epsilon.\end{aligned}$$

Thus, $\{f_k\}_{k=1}^{\infty}$ and $\{Tf_k\}_{k=1}^{\infty}$ are ϵ-nearly equal frame operators but $S \neq S'$.

For a given frame $\{f_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ and $0 < \epsilon < 1$, can we classify all invertible operators $T \in B(\mathcal{H})$ so that $\{f_k\}_{k=1}^{\infty}$ and $\{Tf_k\}_{k=1}^{\infty}$ are ϵ-nearly equal frame operators?

Inspired by the pioneering work of Cahill, Casazza and Kutyniok in [4, Theorem 2.7], we will answer the above question in the following theorem.

Theorem 3.4. *Let $0 < \epsilon < 1$ and let $T \in B(\mathcal{H})$ be an invertible operator. Then two frames $\{f_k\}_{k=1}^{\infty}$ and $\{Tf_k\}_{k=1}^{\infty}$ are ϵ-nearly equal frame operators if and only if there exists an operator $A \in B(\mathcal{H})$ with $\|A\| \leq \epsilon$ and a unitary operator U on $\mathcal{H}$ such that $T = (S-A)^{\frac{1}{2}} US^{\frac{-1}{2}}$, where S is the frame operator of $\{f_k\}_{k=1}^{\infty}$.*

Proof. The frame operator of $\{Tf_k\}_{k=1}^{\infty}$ is TST^*. If $\{f_k\}_{k=1}^{\infty}$ and $\{Tf_k\}_{k=1}^{\infty}$ are ϵ-nearly equal frame operators, then

$$\begin{aligned}\|I-(I+TST^*-S)\| &= \|S-TST^*\| \\ &\leq \epsilon \\ &< 1.\end{aligned}$$

Thus, there exists an invertible operator $B \in B(\mathcal{H})$ such that $I+TST^*-S = B^{-1}$. Defining $A := I - B^{-1}$, we obtain that $\|A\| \leq \epsilon$ and $S - A = TST^*$.

Let $U := (S-A)^{\frac{-1}{2}} TS^{\frac{1}{2}}$. Thus, U is a unitary operator, since

$$\begin{aligned}UU^* &= (S-A)^{\frac{-1}{2}} TS^{\frac{1}{2}}S^{\frac{1}{2}}T^*(S-A)^{\frac{-1}{2}} \\ &= (S-A)^{\frac{-1}{2}}(S-A)(S-A)^{\frac{-1}{2}} \\ &= I.\end{aligned}$$

Also we have

$$\begin{aligned}(S-A)^{\frac{1}{2}} US^{\frac{-1}{2}} &= (S-A)^{\frac{1}{2}}(S-A)^{\frac{-1}{2}} TS^{\frac{1}{2}}S^{\frac{-1}{2}} \\ &= T.\end{aligned}$$

Conversely, if there exists an operator $A \in B(\mathcal{H})$ with $\|A\| \leq \epsilon$ and a unitary operator U on $\mathcal{H}$ such that $T = (S-A)^{\frac{1}{2}} US^{\frac{-1}{2}}$, then

$$\begin{aligned}TST^* &= (S-A)^{\frac{1}{2}} US^{\frac{-1}{2}}SS^{\frac{-1}{2}}U^*(S-A)^{\frac{1}{2}} \\ &= S-A.\end{aligned}$$

This implies that $\|S-TST^*\| = \|A\| \leq \epsilon$ and hence $\{f_k\}_{k=1}^{\infty}$ and $\{Tf_k\}_{k=1}^{\infty}$ are ϵ-nearly equal frame operators. $\square$

By using the frame bounds of $\{f_k\}_{k=1}^{\infty}$, we provide a sufficient condition for which $\{f_k\}_{k=1}^{\infty}$ and $\{Tf_k\}_{k=1}^{\infty}$ are ϵ-nearly equal frame operators.

Theorem 3.5. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with frame bounds A, B and $0 < \epsilon < B$. If $T \in B(\mathcal{H})$ is an invertible operator such that $\|T\|^2 \leq \frac{A}{B}$ and $\|T^{-1}\|^2 \leq \frac{A}{B-\epsilon}$, then $\{f_k\}_{k=1}^{\infty}$ and $\{Tf_k\}_{k=1}^{\infty}$ are ϵ-nearly equal frame operators.*

Proof. For any $f \in \mathcal{H}$ we have

$$A\|f\|^2 \leq \langle Sf, f\rangle \leq B\|f\|^2,$$

and

$$A\left\|T^{-1}\right\|^{-2}\|f\|^2 \leq \langle TST^*f, f\rangle \leq B\|T\|^2\|f\|^2.$$

Thus,

$$\begin{aligned}\|S - TST^*\| &= \sup_{\|f\|=1} |\langle (S - TST^*) f, f\rangle| \\ &\leq B - A\left\|T^{-1}\right\|^{-2} \\ &\leq \epsilon.\end{aligned}$$

This implies that $\{f_k\}_{k=1}^{\infty}$ and $\{Tf_k\}_{k=1}^{\infty}$ are ϵ-nearly equal frame operators. $\square$

Example 3.6. Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with frame bounds A, B such that $B < \frac{1+\sqrt{1+4A^2}}{2}$. Then $\{f_k\}_{k=1}^{\infty}$ and $\left\{\sqrt{\frac{A}{B}}f_k\right\}_{k=1}^{\infty}$ are ϵ-nearly equal frame operators for any $\epsilon \in [\frac{B^2-A^2}{B}, 1)$.

Indeed, the bounded linear operator T on $\mathcal{H}$ defined by $Tf = \sqrt{\frac{A}{B}}f$ satisfies $\|T\|^2 = \frac{A}{B}$ and $\left\|T^{-1}\right\|^2 = \frac{B}{A} \leq \frac{A}{B-\epsilon}$ for any $\epsilon \in [\frac{B^2-A^2}{B}, 1)$. Thus, $\{f_k\}_{k=1}^{\infty}$ and $\left\{\sqrt{\frac{A}{B}}f_k\right\}_{k=1}^{\infty}$ are ϵ-nearly equal frame operators by Theorem 3.5.

Using the frame operator of $\{f_k\}_{k=1}^{\infty}$, we provide a necessary or sufficient conditions for which $\{f_k\}_{k=1}^{\infty}$ and $\{Tf_k\}_{k=1}^{\infty}$ be ϵ-nearly equal frame operators.

Theorem 3.7. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with frame operator S. Let $T \in B(\mathcal{H})$ be an invertible operator and $0 < \epsilon < 1$.*

If $\|T\|^2 < 1 - \frac{\epsilon}{\|S\|}$, then $\{f_k\}_{k=1}^{\infty}$ and $\{Tf_k\}_{k=1}^{\infty}$ are not ϵ-nearly equal frame operators.

Proof. If $\{f_k\}_{k=1}^{\infty}$ and $\{Tf_k\}_{k=1}^{\infty}$ are ϵ-nearly equal frame operators, then

$$\epsilon \geq \|S - TST^*\|$$

$$\geq \|S\| - \|TST^*\|$$
$$\geq \left(1 - \|T\|^2\right) \|S\| .$$

Thus $\|T\|^2 \geq 1 - \frac{\epsilon}{\|S\|}$. This implies the result. □

4. Operator Dual Parseval Frames

The duals of frames have an essential role in reconstruction of vectors (or signals) in terms of the frame elements. From this point of view, dual frames have generalized:

S. Li and H. Ogawa defined and characterized pseudo duals of frames in [14]. They present the results of some studies on duals $\{g_k\}_{k=1}^{\infty}$ of a given nonexact frame in a separable Hilbert space $\mathcal{H}$ that may not be Bessel sequences. Oblique dual frames are another generalizations of dual frames [7, 13]. C. Heil, Y.Y. Koo and J.K. Lim divided dual frames to Type I duals and Type II duals in[11]. A Type I dual is a dual such that the range of its synthesis operator is contained in the range of the synthesis operator of original frame sequence, and a Type II dual is a dual such that the range of its analysis operator is contained in the range of the analysis operator of original frame sequence. They prove that all Type I and Type II duals are oblique duals, but the converse is not necessarily true. O. Christensen introduced the concept of approximately dual frames in [8].

In this section, operator dual Parseval frames of a frame in a separable Hilbert space $\mathcal{H}$ are introduced and characterized. By applying operator dual Parseval frames as well, (which also includes usual duals) we can achieve more reconstruction formulas to obtain signals.

Definition 4.1. Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$. The frame $\{g_k\}_{k=1}^{\infty}$ is an operator dual frame of $\{f_k\}_{k=1}^{\infty}$, if there exists an invertible operator $A \in B(\mathcal{H})$ with $\|A\| \leq 1$ such that

$$f = \sum_{k=1}^{\infty} \langle Af, g_k\rangle f_k, \quad \forall f \in \mathcal{H}. \tag{4.1}$$

If $\{g_k\}_{k=1}^{\infty}$ is also Parseval, then $\{g_k\}_{k=1}^{\infty}$ is called operator dual Parseval frame of $\{f_k\}_{k=1}^{\infty}$.

The operator A in (4.2) is unique, since for all $f \in \mathcal{H}$,

$$A^{-1} f = \sum_{k=1}^{\infty} \langle f, g_k\rangle f_k,$$

and hence $A^{-1} = TU^*$, where T and U are the pre-frame operators of $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$, respectively. All ordinary dual frames of a given frame $\{f_k\}_{k=1}^{\infty}$ are operator dual frames of $\{f_k\}_{k=1}^{\infty}$ with invertible operator $A = I$. Using the fact that $\|A\| = \|A^*\|$, it is easy to show that $\{g_k\}_{k=1}^{\infty}$ is an operator dual frame of $\{f_k\}_{k=1}^{\infty}$ if and only if $\{f_k\}_{k=1}^{\infty}$ is an operator dual frame of $\{g_k\}_{k=1}^{\infty}$. Indeed, if $\{f_k\}_{k=1}^{\infty}$ is an operator dual frame of $\{g_k\}_{k=1}^{\infty}$, then there exists an invertible operator $A \in B(\mathcal{H})$ with $\|A\| \leq 1$ such that

$$g = \sum_{k=1}^{\infty} \langle Ag, f_k \rangle g_k, \quad \forall g \in \mathcal{H}. \tag{4.2}$$

Now let $f \in \mathcal{H}$. Then there exists $g \in \mathcal{H}$, such that $f = Ag$ and $g = \sum_{k=1}^{\infty} \langle Ag, f_k \rangle g_k$. Therefore $f = Ag = \sum_{k=1}^{\infty} \langle f, f_k \rangle Ag_k$. Since $\{Ag_k\}_{k=1}^{\infty}$ is a Bessel sequence, by [6, Lemma 5.6.2] we have

$$\begin{aligned} f &= \sum_{k=1}^{\infty} \langle f, f_k \rangle Ag_k \\ &= \sum_{k=1}^{\infty} \langle f, Ag_k \rangle f_k \\ &= \sum_{k=1}^{\infty} \langle A^* f, g_k \rangle f_k, \end{aligned}$$

and hence $\{g_k\}_{k=1}^{\infty}$ is an operator dual frame of $\{f_k\}_{k=1}^{\infty}$. A similar argument shows that, $\{f_k\}_{k=1}^{\infty}$ is an operator dual frame of $\{g_k\}_{k=1}^{\infty}$ if $\{g_k\}_{k=1}^{\infty}$ is an operator dual frame of $\{f_k\}_{k=1}^{\infty}$.

We characterize all operator dual Parseval frames of a given Parseval frame in the next theorem.

Theorem 4.2. *Let $\{f_k\}_{k=1}^{\infty}$ be a Parseval frame for $\mathcal{H}$. Then all operator dual Parseval frames of $\{f_k\}_{k=1}^{\infty}$ have the form $\{Af_k\}_{k=1}^{\infty}$, where A is a unitary operator on $\mathcal{H}$.*

Proof. Let $\{f_k\}_{k=1}^{\infty}$ be a Parseval frame for $\mathcal{H}$ and A be a unitary operator on $\mathcal{H}$. Then $\|A\| \leq 1$ and $\{Af_k\}_{k=1}^{\infty}$ is a Parseval frame for $\mathcal{H}$ by [6, Lemma 5.3.3]. Also for all $f \in \mathcal{H}$ we have

$$\begin{aligned} \sum_{k=1}^{\infty} \langle Af, Af_k \rangle f_k &= \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k \\ &= f. \end{aligned}$$

Conversely, let $\{g_k\}_{k=1}^{\infty}$ be an operator dual Parseval frame of $\{f_k\}_{k=1}^{\infty}$. Then there exists an invertible operator $A \in B(\mathcal{H})$ with $\|A\| \leq 1$ such that

$$f = \sum_{k=1}^{\infty} \langle Af, g_k \rangle f_k, \quad \forall f \in \mathcal{H}.$$

Thus, $f = TU^*Af$, where T and U are the pre-frame operators of $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ respectively. Also for all $f \in \mathcal{H}$ we have

$$\begin{aligned}\|f\|^4 &= \left|\sum_{k=1}^{\infty} \langle Af, g_k\rangle \langle f_k, f\rangle\right|^2 \\ &\leq \sum_{k=1}^{\infty} |\langle Af, g_k\rangle|^2 \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \\ &= \|Af\|^2 \|f\|^2 .\end{aligned}$$

Therefore, $\|f\| \leq \|Af\|$. On the other hand $\|Af\| \leq \|A\| \|f\| \leq \|f\|$ and hence A is a surjective isometry. i.e. A is a unitary operator. Also

$$\begin{aligned}\|f\|^2 &= \|Af\|^2 \\ &= \|U^*Af\|^2 \\ &= \|T^*f + U^*Af - T^*f\|^2 \\ &= \langle T^*f + U^*Af - T^*f, T^*f + U^*Af - T^*f\rangle \\ &= \|T^*f\|^2 + \|(U^*A - T^*)\, f\|^2 + 2\mathrm{Re}\, \langle T^*f, (U^*A - T^*)\, f\rangle \\ &= \|f\|^2 + \|(U^*A - T^*)\, f\|^2 + 2\mathrm{Re}\, \langle f, (TU^*A - TT^*)\, f\rangle \\ &= \|f\|^2 + \|(U^*A - T^*)\, f\|^2 .\end{aligned}$$

Thus $U^*Af = T^*f$. This implies $\langle Af, g_k\rangle = \langle f, f_k\rangle$ for all $f \in \mathcal{H}$ and $k \in \mathbb{N}$. Hence $f_k = A^*g_k = A^{-1}g_k$, for all $k \in \mathbb{N}$. This implies $g_k = Af_k$, for all $k \in \mathbb{N}$. □

Corollary 4.3. *Let $\{f_k\}_{k=1}^{\infty}$ be a uniform Parseval frame for $\mathcal{H}$. Then all operator dual uniform Parseval frames of $\{f_k\}_{k=1}^{\infty}$ have the form $\{Af_k\}_{k=1}^{\infty}$, where A is a unitary operator on $\mathcal{H}$.*

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