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Some Results about the Contractions and the Pendant Pairs of a Submodular System

Saeid Hanifehnezhad[1] and Ardeshir Dolati[2*]

Abstract. Submodularity is an important property of set functions with deep theoretical results and various applications. Submodular systems appear in many applicable area, for example machine learning, economics, computer vision, social science, game theory and combinatorial optimization. Nowadays submodular functions optimization has been attracted by many researchers. Pendant pairs of a symmetric submodular system play essential role in finding a minimizer of this system. In this paper, we investigate some relations between pendant pairs of a submodular system and pendant pairs of its contractions. For a symmetric submodular system (V, f) we construct a suitable sequence of $|V|-1$ pendant pairs of its contractions. By using this sequence, we present some properties of the system and its contractions. Finally, we prove some results about the minimizers of a posimodular function.

1. Introduction

Submodular functions are widely applied in various fields such as combinatorics [18], economics [19], image segmentation [8], machine learning [9] and game theory [14]. For more details and general background about submodular functions see [2]. Many combinatorial optimization problems can be formulated as submodular function optimization. Therefore, submodular functions play important roles in many efficiently solvable

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* Corresponding author.

combinatorial optimization problems. Some well known examples of submodular functions are the rank function of a matroid, cut capacity function and entropy function. One of the most important problems in combinatorial optimization is minimizing a submodular function $f : 2^V \mapsto \mathbb{R}$. Its importance is similar to minimizing convex function in continuous optimization. The first weakly polynomial time algorithm [4] and strongly polynomial time algorithm [5] have been developed by Grötschel, Lovász, and Schrijver. These algorithms are based on ellipsoid method. However, these results are ultimately undesirable, since ellipsoid method is not very practical, and also does not give much combinatorial intuition. Then the minimizing submodular function problem shifted to "Is there a combinatorial (non-ellipsoid method) polynomial time algorithm to minimize a submodular function?". Scherijver [17] and Iwata, Fujishige and Fleischer [7] independently gave combinatorial algorithms to this problem. To the best of our knowledge, the best running time of a combinatorial algorithm is $O\left(|V|^5 \tau + |V|^6\right)$ by Orlin [15], where τ is the time for function evaluation. Currently, the fastest algorithm to minimize submodular functions is due to Lee, Sidford, and Wong [10] with running time $O\left(|V|^3 \log^2 |V| \tau + |V|^4 \log^{O(1)} |V|\right)$. Their algorithm is an improved variant of the ellipsoid method. However, the mentioned algorithms do not work well for large scale instances [1].

To find a minimizer of special cases of submodular function such as symmetric submodular functions and posimodular and submodular functions there exist faster and simpler algorithms that run in $O\left(|V|^3 \tau\right)$ time [12, 13, 16]. All of these algorithms are based on maximum adjacency orderings which is a sequence of all elements of the ground set. Maximum adjacency ordering can be constructed in a greedy way and the last two elements of it give a pendant pair [16]. Using pendant pairs of a symmetric submodular function and pendant pairs of its contractions one can find a minimizer of the function. Therefore, pendant pairs play a key role in minimizing submodular functions. Goemans and Soto [3] recently developed an algorithm to find a minimizer of a PP-admissible system. A system is said to be a PP-admissible system if any of its contractions has a pendant pair. The PP-admissible systems are extensions of symmetric submodular systems.

In this paper, we investigate some relations between pendant pairs of a symmetric submodular system and pendant pairs of its contractions. Also we state a property for posimodular functions.

The rest of the paper is organized as follows. Section 2 provides necessary notations and definitions. In Section 3 we discuss about pendant pairs of a submodular system and of its contractions and also we state some results. Finally, we present our conclusion in Section 4.

2. Preliminaries

Let V be a nonempty finite set and a and b be two distinct elements of it. We say that a set $X \in 2^V$ *separates* two elements a, b if $|X \cap \{a, b\}| = 1$. By $C(a, b)$ we denote all subsets of V that separate a and b. It is called that two subsets $X, Y \in 2^V$ *intersect* each other if $X \cap Y \neq \emptyset$, $X \nsubseteq Y$ and $Y \nsubseteq X$. A family $\Gamma \subseteq 2^V$ is called a *laminar family* if no two subsets in Γ intersect each other. A family $\mathcal{D} \subseteq 2^V$ is called a *lattice family* if

$$X, Y \in \mathcal{D} \quad \Rightarrow \quad X \cup Y, X \cap Y \in \mathcal{D}.$$

A subfamily $\mathcal{L}$ of the lattice family $\mathcal{D}$ is called a *parity family* if

$$X, Y \in \mathcal{D} \backslash \mathcal{L} \quad \Rightarrow \quad (X \cup Y \in \mathcal{L} \quad \Leftrightarrow \quad X \cap Y \in \mathcal{L}).$$

A real valued function f defined on subsets of V is called a set function.

For a given lattice $\mathcal{D}$, a set function $f : \mathcal{D} \mapsto \mathbb{R}$ is called a *submodular function* if

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y), \quad \forall X, Y \in \mathcal{D}.$$

A pair (V, f) is a system if f is a real valued function defined on 2^V. A system (V, f) is called a submodular system if f is a submodular function on 2^V.There is an equivalent definition of submodularity that is sometimes useful for proofs. A set function $f : 2^V \mapsto \mathbb{R}$ is said to be submodular if for all $X \subseteq Y \subset V$ and $j \in V \backslash Y$, we have [11]

$$f(Y \cup \{j\}) - f(Y) \leq f(X \cup \{j\}) - f(X).$$

A set function $f : 2^V \mapsto \mathbb{R}$ is called a posimodular function if

$$f(X) + f(Y) \geq f(X \backslash Y) + f(Y \backslash X),$$

for every pair of sets $X, Y \in 2^V$. A system (V, f) is called a posimodular system if f is a posimodular function on 2^V.

A set function f is called symmetric if for all $X \in 2^V$

$$f(X) = f(V \backslash X).$$

If f is a symmetric submodular function then f is a posimodular function, but the converse is not generally true [13].

For every $x, y \in V$, an ordered pair (x, y) is called a pendant pair of (V, f) if $\{y\}$ has a minimum value among all subsets of V separating x and y, that is

$$f(\{y\}) = \min \{f(X) \,|\, X \subset V, X \in C(x, y)\}.$$

The element y is called the leaf of the pendant pair (x, y).

For a given system (V, f), the system (V', f') obtained by identifying two elements $x, y \in V$ into a new single element t is defined by $V' = (V \setminus \{x, y\}) \cup \{t\}$ and

$$f'(X) = \begin{cases} f(X), & \text{if } t \notin X, \\ f((X \setminus \{t\}) \cup \{x, y\}), & \text{if } t \in .X \end{cases}$$

Consider a system (V, f) with $n = |V| \geq 2$. An ordering $\lambda = (v_1, v_2, \ldots, v_n)$ of all the elements of V is called a maximum adjacency ordering (MA-ordering) of (V, f) if it satisfies

$$f(V_{i-1} + v_i) - f(v_i) \leq f(V_{i-1} + v_j) - f(v_j), \quad 1 \leq i \leq j \leq n,$$

where $V_0 = \emptyset$ and $V_i = \{v_1, v_2, \ldots, v_i\} \, (1 \leq i \leq n-1)$.

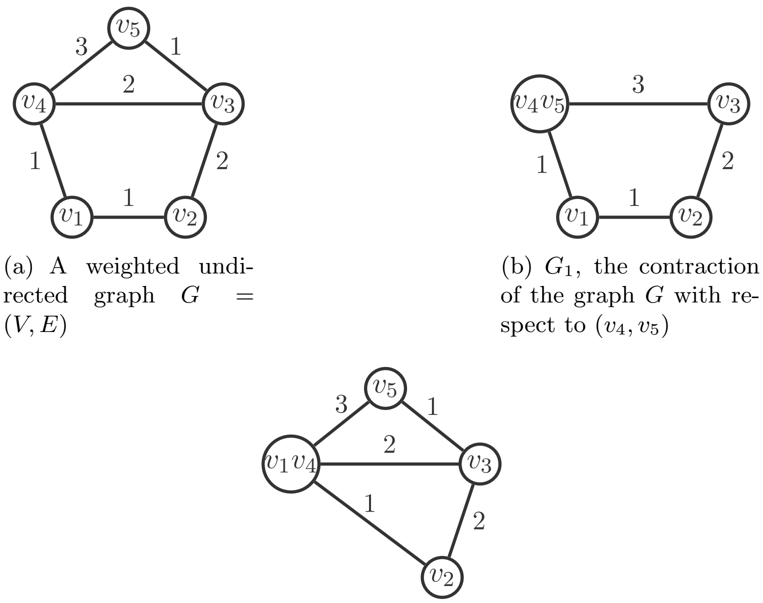
If $\lambda = (v_1, v_2, \ldots, v_n)$ is an MA-ordering of a symmetric submodular system (V, f), then (v_{n-1}, v_n) is a pendant pair of it [16]. Queyranne's algorithm, by repeatedly finding a pendant pair of a symmetric submodular system and contracting the system with respect to that pair computes a minimizer of the system [16].

Let $G = (V, E, w)$ be a weighted undirected graph with node set V, edge set $E \subseteq V \times V$ and weight function $w : E \mapsto \mathbb{R}^+$. A *cut* of the graph G is a set of all edges that connects X and $V \backslash X$, for some subset X of V. Therefore, every $X \subseteq V$ determines a cut. In other words, for a nonempty proper subset X of V, the induced cut on X is the set of all edges that connects X to $V \backslash X$. For a subset $X \subseteq V$, by $\delta(X)$ we mean the set of all edges connecting X to $V \backslash X$. The capacity of $\delta(X)$, called the cut function of the graph G, is defined as follows.

$$f(X) = \sum_{e \in \delta(X)} w(e).$$

The cut function $f : 2^V \mapsto \mathbb{R}^+$ of the graph G is a symmetric submodular function [16]. Therefore, (V, f) is a symmetric submodular system. A minimum cut of the graph G is a cut with minimum capacity. For example $\delta(\{v_1\})$ is a minimum cut of the graph G, depicted in Figure 1, with the capacity $f(\{v_1\}) = 2$. For a pair of distinct vertices s and t of V, an $st-$cut is a set of edges whose removal disconnects all paths between s and t. Thus, each $X \in C(s, t)$ determines an $st-$cut.

To obtain an MA-ordering of the vertices of the graph $G = (V, E)$, we choose an arbitrary singleton subset A of V. Then, A grows until it equals to V, by adding a new vertex outside of A that is most tightly connected with it. This one by one adding impose an MA-ordering of elements of V. For example $\mu = (v_2, v_3, v_4, v_5, v_1)$ is an MA-ordering of the graph $G = (V, E)$ depicted in Figure 1 and therefore (v_5, v_1) is a pendant pair of G.



(a) A weighted undirected graph $G = (V, E)$

(b) G_1, the contraction of the graph G with respect to (v_4, v_5)

(c) G_2, the contraction of the graph G with respect to (v_4, v_1)

FIGURE 1. A weighted undirected graph $G = (V, E)$ and its contractions

3. Contractions of a System

Let $\mathcal{S} = (V, f)$ be a symmetric submodular system. Suppose that $\mathcal{S} = (V, f)$ is denoted by $\mathcal{S}^0 = \left(V^0, f^0\right)$; initially and $\mathcal{S}^k = \left(V^k, f^k\right)$ is a contraction of $\mathcal{S}^{k-1}$ with respect to a pendant pair of $\mathcal{S}^{k-1}$. Then we say $\mathcal{S}^k$ is a contraction of $\mathcal{S}$ of rank k. Every pendant pair of $\mathcal{S}^k$ is also called a pendant pair of $\mathcal{S}$ of rank k. For more convenience, $\mathcal{S}$ is called to be a contraction of rank zero of itself and its pendant pairs are called the pendant pair of rank zero. By $\mathcal{R}(\mathcal{S})$ we mean the set of all the penant pairs of its contractions of any rank.

For a given symmetric submodular system $\mathcal{S} = (V, f)$, suppose that $\Delta = \left((u_0, v_0), \ldots, \left(u_{|V|-2}, v_{|V|-2}\right)\right)$ is a $(|V| - 1)$ −tuple of distinct pendant pairs in $\mathcal{R}(\mathcal{S})$ with no two elements of the same rank. If there is a sequence $\left\{\mathcal{S}^i\right\}_{i=0}^{|V|-1}$ of the contractions of $\mathcal{S}$ such that each (u_i, v_i) is a pendant pair of $\mathcal{S}^i$, (for $i = 0, 1, \ldots, |V| - 2$) and $\mathcal{S}^i$ is a contraction of $\mathcal{S}^{i-1}$ with respect to the pair (u_{i-1}, v_{i-1}) (for $i = 1, \ldots, |V| - 1$) then we call Δ a proper ordered $(|V| - 1)-$ tuple of pendant pairs in $\mathcal{R}(\mathcal{S})$. Every proper ordered $(|V| - 1)$ −tuple of pendant pairs in $\mathcal{R}(\mathcal{S})$, can be constructed by repeatedly choosing a pendant pair of the system $\mathcal{S}^i$ (for

$i = 0, 1, \ldots, |V| - 2)$ and contracting $\mathcal{S}^i$ with respect to that pair to obtain $\mathcal{S}^{i+1}$. The $|V| - 1$ pendant pairs that have been chosen in the above procedure, with their ranks, consist an ordered proper $(|V| - 1)$-tuple of pendant pair in $\mathcal{R}(\mathcal{S})$. It is easily shown that $\left|V^{|V|-1}\right| = 1$. See Figure 2.

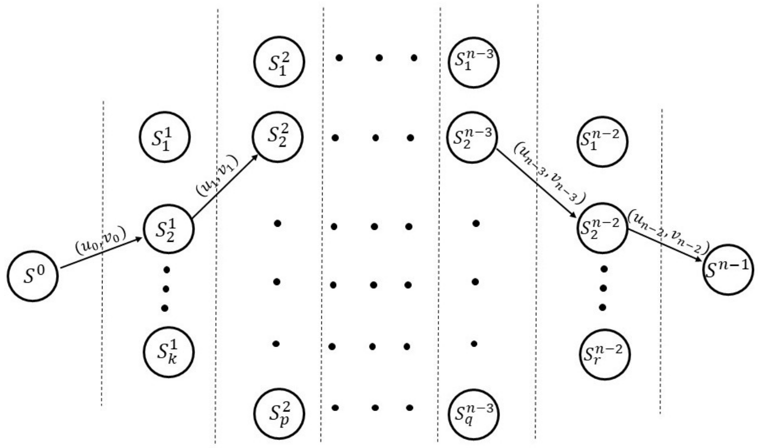


FIGURE 2. A system $\mathcal{S}_j^i$ is connected to a system $\mathcal{S}_{j'}^{i+1}$ if $\mathcal{S}_{j'}^{i+1}$ is a contraction of $\mathcal{S}_j^i$ with respect to a pendant pairof $\mathcal{S}_j^i$.

Qeyranne [16] showed that every symmetric submodular system has a pendant pair. In addition, he proved that every contraction of a symmetric submodular system with respect to a pendant pair of the system, is a symmetric submodular system. Then by considering these results and also Figure 2, we have the following theorem.

Theorem 3.1. *If $\mathcal{S} = (V, f)$ is a symmetric submodular system with $|V| = n$, then there exists at least one directed path from $\mathcal{S}^0$ to $\mathcal{S}^{n-1}$.*

Let $\mathcal{S} = (V, f)$ be a symmetric submodular system and Δ be a proper ordered $(|V| - 1)-$ tuple of pendant pairs in $\mathcal{R}(\mathcal{S})$. We denote by $\Pi(\Delta)$, the family of all subsets of V which their contractions are appeared as a leaf in pendant pairs of Δ. In other words, $\Pi(\Delta) = \{\psi(v_i) \,|\, (u_i, v_i) \in \Delta\}_{i=0}^{|V|-2}$, where $\psi(x)$ denotes the set of all elements of V which are unified into the element x. As an example, one can observe that $\Delta = ((v_4, v_1), (v_2, v_1v_4), (v_5, v_1v_2v_4), (v_3, v_1v_2v_3v_4))$ is a proper ordered 4-tuple of pendant pairs for the symmetric submodular system induced by cut function of the weighted graph G presented in Figure 1 (a). Also, $\Pi(\Delta) = \{\psi(v_1), \psi(v_1v_4), \psi(v_1v_2v_4), \psi(v_1v_2v_3v_4)\} =$

$\{\{v_1\}, \{v_1, v_4\},$
$\{v_1, v_2, v_4\}, \{v_1, v_2, v_3, v_4\}\}.$

It is observed that $\Pi(\Delta)$ is a nested laminar family. In the following theorem we state about this property.

Theorem 3.2. *Let $\mathcal{S} = (V, f)$ be a symmetric submodular system and Δ be a proper $(|V|-1)-$tuple of pendant pairs for $\mathcal{S}$. Then $\Pi(\Delta)$ is a laminar family.*

Proof. Suppose that X and Y are two arbitrary elements of $\Pi(\Delta)$. Therefore, we have two cases:

Case 1: $X \cap Y = \emptyset$. There is nothing to prove in this case.

Case 2: $X \cap Y \neq \emptyset$. To show that X and Y do not intersect each other, it suffices to state that either $X \subseteq Y$ or $Y \subseteq X$. We assume that there are pendant pairs (u_i, v_i) and (u_j, v_j) in $\Pi(\Delta)$ such that $X = \psi(v_i)$ and $Y = \psi(v_j)$. Suppose, without loss of generality, that $i < j$. In this case for obtaining the contraction of rank $i+1$ with respect to $\Pi(\Delta)$ the sets $\psi(u_i)$ and $\psi(v_i)$ are unified in an element. In all contractions of rank $k > i$ with respect to $\Pi(\Delta)$ we can find an element w such that $X \subseteq \psi(w)$ such that for each other element $z \neq w$ we have $\psi(z) \cap X = \emptyset$. Since Y is an element of a contraction whose rank is greater than i and $X \cap Y \neq \emptyset$, therefore $X \subseteq Y$ and this completes the proof.

□

Let $\mathcal{S}$ be a symmetric submodular system. By $\mathcal{F}(\mathcal{S}^k)$ and $\mathcal{P}(\mathcal{S})$, we denote the set of all contractions of rank k of $\mathcal{S}$ and all pendant pairs of $\mathcal{S}$, respectively.

Theorem 3.3. *Let $\mathcal{S} = (V, f)$ with $|V| > 2$, be a symmetric submodular system. Then, $|\mathcal{F}(\mathcal{S}^k)| \geq 2$, for every $k \in \{1, 2, 3, \ldots, |V|-2\}$.*

Proof. Since for every $\mathcal{S}^k, (0 \leq k \leq |V|-3)$ we can find at least two pendant pairs. We know that if $\mu = (v_1, v_2, \ldots, v_{|V|-k})$ is an MA-ordering of $\mathcal{S}^k$, then $(v_{|V|-k-1}, v_{|V|-k})$ is a pendant pair of $\mathcal{S}^k$. Now, we can obtain another pendant pair for $\mathcal{S}^k$ using MA-ordering by starting from $v_{|V|-k}$. By contracting each of these two distinct pendant pairs we obtain two distinct contractions of $\mathcal{S}$ with the same rank. Therefore, $|\mathcal{F}(\mathcal{S}^k)| \geq 2$, for every $k \in \{1, 2, 3, \ldots, |V|-2\}$ and the proof is completed. □

For a given symmetric submodular system $\mathcal{S} = (V, f)$, Hanifehnezhad and Dolati [6] proved that two pairs (x, y) and (y, x) simultaneously are pendant pairs of $\mathcal{S}$ if and only if $f(x) = f(y)$. The following theorem discusses about the number of the pendant pairs of contractions of rank k of a symmetric submodular system.

Theorem 3.4. *We can find a symmetric submodular system* $\mathcal{S} = (V, f)$, *a contraction of rank* $k \in \{1, 2, \ldots, |V| - 2\}$ *of it and two systems* $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{F}\left(\mathcal{S}^k\right)$ *such that* $|\mathcal{P}(\mathcal{H}_1)| \neq |\mathcal{P}(\mathcal{H}_2)|$.

Proof. Consider the graph G depicted in Figure 1 (a). It can be proved that (v_4, v_5) and (v_4, v_1) are two pendant pairs of G. Also, we can show that in the graph G_1 in Figure 1 (b) we have

$$\mathcal{P}(G_1) = \{(v_2, v_1), (v_3, v_1), (v_3, v_2), (v_3, v_4v_5), (v_4v_5, v_1), (v_4v_5, v_2)\},$$

and in the graph G_2 in Figure 1 (c) we have

$$\mathcal{P}(G_2) = \{(v_1v_4, v_2), (v_1v_4, v_5), (v_3, v_2), (v_3, v_5), (v_5, v_2)\}.$$

Therefore, $|\mathcal{P}(G_1)| = 6$ and $|\mathcal{P}(G_2)| = 5$. The proof is completed. □

The following theorem states an important property of posimodular functions.

Theorem 3.5. *Let* $f : 2^V \mapsto \mathbb{R}$ *be an arbitrary posimodular function and* $\mathcal{M} = \left\{X | X \in 2^V,\ f(X) = \min\left\{f(Y) | Y \in 2^V\right\}\right\}$ *be the minimizers family of it. If* $X, Y \in \mathcal{M}$, *then* $X \backslash Y, Y \backslash X \in \mathcal{M}$.

Proof. Let X and Y be two elements of $\mathcal{M}$. According to the posimodularity we have

$$f(X) + f(Y) \geq f(X \backslash Y) + f(Y \backslash X).$$

Thus,

$$(f(X) - f(X \backslash Y)) + (f(Y) - f(Y \backslash X)) \geq 0.$$

Since $X, Y \in \mathcal{M}$, we have $f(X) - f(X \backslash Y) \leq 0$ and $f(Y) - f(Y \backslash X) \leq 0$. It follows that $f(X) = f(Y) = f(X \backslash Y) = f(Y \backslash X)$. It means that $X \backslash Y$ and $Y \backslash X$ are two minimzers of f. In other words $X \backslash Y, Y \backslash X \in \mathcal{M}$. □

4. Conclusion

Submodularity is an important property of set functions with deep theoretical results and various applications. Submodular functions appear in a wide variety of applications such as computer science, discrete mathematics, economics and game theory. Minimizing submodular functions has been applied in many applicable areas. Pendant pairs of a symmetric submodular system play essential role in finding a minimizer of this system. In this paper, we investigated some relations between pendant pairs of a submodular system and pendant pairs of its contractions. For a submodular system (V, f) we constructed a suitable sequence of $|V| - 1$ pendant pairs of its contractions. By using this sequence, we studied some properties of the system and its contractions. Finally, we presented some results about the minimizers of a posimodular function.

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[1] Department of Mathematics, Shahed University, Tehran, Iran.
E-mail address: `Saeid.Hanifehnezhad@gmail.com`

[2] Department of Computer Science, Shahed University, Tehran, Iran.
E-mail address: `dolati@shahed.ac.ir`