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A Proximal Point Algorithm for Finding a Common Zero of a Finite Family of Maximal Monotone Operators

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Abstract. In this paper, we consider a proximal point algorithm for finding a common zero of a finite family of maximal monotone operators in real Hilbert spaces. Also, we give a necessary and sufficient condition for the common zero set of finite operators to be nonempty, and by showing that in this case, this iterative sequence converges strongly to the metric projection of some point onto the set of common zeros of operators.

1. Introduction

Let K_1 and K_2 be nonempty, closed and convex subsets of a real Hilbert space H with nonempty intersection. In 1933, von Neumann showed that the following problem

$$\text{find an } x \in H \text{ such that } x \in K_1 \cap K_2, \tag{1.1}$$

can be solved by means of an iterative process. In 1965, Bregman [7] showed that the sequence (x_n) generated from the method of alternating projections, converges weakly to a point in $K_1 \cap K_2$. For more information to these methods, see for example [2, 7, 9, 10, 12, 14, 15, 17], and the references therin. Recently, the authores consider the method of resolvents for solving the problem (1.1). The general one was given in [5] and [6]. See also [1, 4], and the references therin. In 2012, Boikanyo and Morosanu [3] considered the following proximal point algorithm(PPA)

$$\begin{cases} x_{2n+1} = \alpha_n u + \delta_n x_{2n} + \gamma_n J^A_{\beta_n}(x_{2n}) + e_n, & n \geq 0, \\ x_{2n} = \lambda_n u + \rho_n x_{2n-1} + \delta_n J^B_{\mu_n}(x_{2n-1}) + e'_n, & n \geq 1, \end{cases}$$

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for some $u, x_0 \in H$, where (e_n) and (e'_n) are sequences of computational errors, A and B are maximal monotone operators, and $\alpha_n, \delta_n, \gamma_n \in (0,1)$ and $\beta_n, \mu_n \in (0,\infty)$. Here $J_\beta^A := (I+\beta A)^{-1}, \beta > 0$ (the resolvent operator of A). They proved under minimal assumptions on the sequences of parameteres definded (x_n), that the sequence (x_n) converges strongly to a point in $F = A^{-1}(0) \cap B^{-1}(0)$ that is nearest to u. They assumed that the set of common zeros of A and B is nonempty.

In this paper, we give a necessary and sufficient condition for the set of common zeros of a finite family of maximal monotone operators is nonempty, and by showing that in this case, this iterative sequence converges strongly to the metric projection of some point onto the set of common zeros of operators.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|.\|$. We recall that a map $T : H \to H$ is called nonexpansive if for every $x, y \in H$, we have $\|Tx - Ty\| \leq \|x - y\|$. The map T is called firmly nonexpansive if for every $x, y \in H$, we have

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I-T)x - (I-T)y\|^2.$$

Obviousely, every firmly nonexpansive mapping is nonexpansive. For more information on firmly nonexpansive mappings, see for example [8]. An operator $A : D(A) \subset H \rightrightarrows H$ is said to be monotone if its graph is a monotone subset of $H \times H$, that is,

$$\langle y_2 - y_1, x_2 - x_1 \rangle \geq 0,$$

for all $x_1, x_2 \in D(A)$ and $y_1 \in A(x_1)$ and $y_2 \in A(x_2)$. A is maximal monotone if A is monotone and the graph of A is not properly contained in the graph of any other monotone operator. Note that if A is maximal monotone, then so is its inverse A^{-1}. For a maximal monotone operator A, and for every $t > 0$, the operator $J_t : H \to H$ defined by $J_t(x) := (I + tA)^{-1}(x)$ is well-defined, single-valued and nonexpansive on H. It is called the resolvent of A. It is known that the Yosida approximation of A, an operator defined by $A_\beta = \beta^{-1}(I - J_\beta^A)$, is maximal monotone and Lipschitzian with constant $\frac{1}{\beta}$ for every $\beta > 0$. We denote weak convergence in H by $\xrightarrow{w}$ and strong convergence by $\to$. The weak ω-limit set of a sequence (x_n) will be denoted by $\omega_w((x_n))$, that is, $\omega_w((x_n)) = \{x \in H : x_{n_k} \xrightarrow{w} x \text{ for some subsequence } (x_{n_k}) \text{ of } (x_n)\}$.
For the main results of this paper, we need the following useful lemmas.

Lemma 2.1 ([3])**.** *For all $x, y \in H$, we have*

$$\|x + y\|^2 \leq \|y\|^2 + 2\langle x, x + y \rangle. \tag{2.1}$$

Lemma 2.2 ([13]). *Any maximal monotone operator $A : D(A) \subset H \rightrightarrows H$ satisfies the demiclosedness principle. In other words, given any two sequences x_n and y_n satisfying x_n converges strongly to x and y_n converges weakly to y with $(x_n, y_n) \in G(A)$, then $(x, y) \in G(A)$.*

Lemma 2.3 ([16]). *For any $x \in H$ and $\mu \geq \beta > 0$,*

$$\|x - J_\beta^A(x)\| \leq 2\|x - J_\mu^A(x)\|, \tag{2.2}$$

where $A : D(A) \subset H \rightrightarrows H$ is a maximal monotone operator.

Lemma 2.4 ([11]). *Let (S_n) be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence (S_{n_j}) of (S_n) such that $S_{n_j} < S_{n_j+1}$ for all $j \geq 0$. Define an integer sequence $(\tau(n))_{n \geq n_0}$ as*

$$\tau(n) = \max\left\{n_0 \leq k \leq n : S_k < S_{k+1}\right\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq n_0$,

$$\max\{S_{\tau(n)}, S_n\} \leq S_{\tau(n)+1}. \tag{2.3}$$

In section 3, we consider the sequence generated by (3.5) and give a necessary and sufficient condition for the common zero set of maximal monotone operatores $A_1, A_2, \ldots, A_n$ to be nonempty, and we show that in this case, the sequence V_n converges strongly to the metric projection of u onto $F = \cap_{i=1}^n A_i^{-1}(0)$. These results significantly improve upon the results of Boikanyo and Morosanu [3], who assumed that the common zero set of A and B is nonempty.

3. Main Results

The proof of our main result is based on the following useful lemma.

Lemma 3.1. *Let (S_n) be a sequence of non-negative real numbers satisfying*

$$\begin{aligned} S_{n+1} \leq & \left[(1-\alpha_n^1)(1-\alpha_n^2)\ldots(1-\alpha_n^k)\right] S_n \\ & + \left[\alpha_n^1 b_n^1 + \alpha_n^2 b_n^2 + \ldots + \alpha_n^k b_n^k\right] + d_n, \quad n \geq 0, \end{aligned} \tag{3.1}$$

where, for every $j = 1, 2, \ldots, k$, sequences $(\alpha_n^j), (b_n^j)$ and (d_n) satisfy the conditions:

(i) *$\alpha_n^j \in (0, 1)$, and for some $1 \leq l \leq k$, $\prod_{n=1}^{\infty}(1 - \alpha_n^l) = 0$, or equivalently $\sum_{n=1}^{\infty} \alpha_n^l = \infty$,*

(ii) *for every $j = 1, 2, \ldots k, \limsup b_n^j \leq 0$,*

(iii) $d_n \geq 0$ $(n \geq 0)$, $\sum\limits_{n=1}^{\infty} d_n < \infty$.

Then $\lim\limits_{n\to\infty} S_n = 0$.

Proof. For any $\epsilon > 0$, let N be an integer sufficiently large enough so that for every $j = 1, 2, \ldots, k$,

$$b_n^j < \frac{\epsilon}{k+1}, \qquad \sum_{n=N}^{\infty} d_n < \frac{\epsilon}{k+1}, \quad n \geq N. \tag{3.2}$$

By using (3.1) and by induction, we obtain, for $n > N$, that

$$\begin{aligned} S_{n+1} &\leq \left[\prod_{i=N}^{n} (1-\alpha_i^1)(1-\alpha_i^2)\ldots(1-\alpha_i^k)\right] S_N \\ &+ \frac{\epsilon}{k+1}\left[1 - \prod_{i=N}^{n}(1-\alpha_i^1)\right] + \cdots + \frac{\epsilon}{k+1}\left[1 - \prod_{i=N}^{n}(1-\alpha_i^k)\right] \\ &+ \sum_{i=N}^{n} d_i. \end{aligned} \tag{3.3}$$

Then,

$$S_{n+1} \leq \left[\prod_{i=N}^{n}(1-\alpha_i^1)\right] S_N + \epsilon. \tag{3.4}$$

Now the result follows immediately by letting $n \to \infty$ and $\epsilon \to 0$. □

The proximal point algorithm for a finite family of maximal monotone operators $\{A_i\}_{i=1}^k$ in real Hilbert spaces H, is the iterative sequence generated by

$$\begin{cases} V_{kn+k-1} = \alpha_n^k u + \delta_n^k V_{kn+k-2} + \gamma_n^k J_{\beta_n^k}^{A_k} V_{kn+k-2} \\ \vdots \\ V_{kn+1} = \alpha_n^2 u + \delta_n^2 V_{kn} + \gamma_n^2 J_{\beta_n^2}^{A_2} V_{kn} \\ V_{kn} = \alpha_n^1 u + \delta_n^1 V_{kn-1} + \gamma_n^1 J_{\beta_n^1}^{A_1} V_{kn-1}, \end{cases} \tag{3.5}$$

where for every $i = 1, 2, \ldots, k$; $\alpha_n^i, \delta_n^i, \gamma_n^i \in (0,1)$ with $\alpha_n^i + \delta_n^i + \gamma_n^i = 1$ and $\beta_n^i \in (0, \infty)$. The inexact version of (3.5) can be formulated as follow

$$
(3.6)\qquad \begin{cases} x_{kn+k-1} = \alpha_n^k u + \delta_n^k x_{kn+k-2} + \gamma_n^k J_{\beta_n^k}^{A_k} x_{kn+k-2} + e_n^k \\ \vdots \\ x_{kn+1} = \alpha_n^2 u + \delta_n^2 x_{kn} + \gamma_n^2 J_{\beta_n^2}^{A_2} x_{kn} + e_n^2 \\ x_{kn} = \alpha_n^1 u + \delta_n^1 x_{kn-1} + \gamma_n^1 J_{\beta_n^1}^{A_1} x_{kn-1} + e_n^1, \end{cases}
$$

where $\left(e_n^1\right), \left(e_n^2\right), \dots, \left(e_n^k\right)$ are error sequences.

Theorem 3.2. *Let $A_i : D(A_i) \subset H \to H (i = 1, \dots, k)$ be maximal monotone operators with $\bigcap_{i=1}^k A_i^{-1}(0) = F \neq \emptyset$. For fixed vectors $V_0, u \in H$, let (V_n) be the sequence generated by (3.5), where for every $j = 1, 2, \dots, k$; $\alpha_n^j, \delta_n^j, \gamma_n^j \in (0,1)$ with $\alpha_n^j + \delta_n^j + \gamma_n^j = 1$ and $\beta_n^j \in (0, \infty)$. Assume that*

(i) $\lim_{n\to\infty} \alpha_n^j = 0$,

(ii) *for some* $1 \leq l \leq k$, $\sum_{n=1}^{\infty} \alpha_n^l = \infty$,

(iii) $\beta_n^j \geq \beta^j$ *for* $\beta^j > 0$,

(iv) $\gamma_n^j \geq \gamma^j$ *for* $\gamma^j > 0$.

Then (V_n) converges strongly to $P_F u$.

Proof. Assume that $F \neq \emptyset$ and $p \in F$. From (3.5), and the fact that the resolvent operator is nonexpansive, for all $n \geq 0$ we have:

$$
\begin{aligned}
(3.7)\qquad \|V_{kn} - p\| &= \|V_{kn} - J_{\beta_n^1}^{A^1} p\| \\
&\leq \alpha_n^1 \|u - p\| + \delta_n^1 \|V_{kn-1} - p\| + \gamma_n^1 \|V_{kn-1} - p\| \\
&= \alpha_n^1 \|u - p\| + (1 - \alpha_n^1) \|V_{kn-1} - p\|.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
(3.8)\qquad \|V_{kn+1} - p\| &\leq \alpha_n^2 \|u - p\| + (1 - \alpha_n^2) \|V_{kn} - p\| \\
&\vdots \\
\|V_{kn+k-1} - p\| &\leq \alpha_n^{k-1} \|u - p\| + (1 - \alpha_n^{k-1}) \|V_{kn+k-2} - p\|.
\end{aligned}
$$

For all $n \geq 0$, from (3.7) and (3.8)

(3.9)

$$
\begin{aligned}
\|V_{k(n+1)-1} - p\| &\leq \alpha_n^{k-1} \|u - p\| \\
&\quad + (1 - \alpha_n^{k-1}) \left[\alpha_n^{k-2} \|u - p\| + (1 - \alpha_n^{k-2}) \|V_{k(n+1)-3} - p\|\right] \\
&= \left[1 - (1 - \alpha_n^{k-1})(1 - \alpha_n^{k-2})\right] \|u - p\|
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big[(1-\alpha_n^{k-1})(1-\alpha_n^{k-2})\Big] \|V_{k(n+)-3} - p\| \\
&\vdots \\
&\leq \Big[1-(1-\alpha_n^{k-1})(1-\alpha_n^{k-2})\dots(1-\alpha_n^{1})\Big] \|u-p\| \\
&\quad + \Big[(1-\alpha_n^{k-1})(1-\alpha_n^{k-2})\dots(1-\alpha_n^{1})\Big] \|V_{kn-1} - p\| \\
&\leq \Big[1-(1-\alpha_n^{k-1})(1-\alpha_n^{k-2})\dots(1-\alpha_n^{1})\Big] \|u-p\| \\
&\quad + \Big[(1-\alpha_n^{k-1})(1-\alpha_n^{k-2})\dots(1-\alpha_n^{1})\Big] \\
&\quad \Big(\Big[1-(1-\alpha_{n-1}^{k-1})(1-\alpha_{n-1}^{k-2})\dots(1-\alpha_{n-1}^{1})\Big] \|u-p\| \\
&\quad + \Big[(1-\alpha_{n-1}^{k-1})(1-\alpha_{n-1}^{k-2})\dots(1-\alpha_{n-1}^{1})\Big] \|V_{k(n-1)-1} - p\|\Big) \\
&= \left[1-\prod_{m=n-1}^{n}\prod_{j=1}^{k}(1-\alpha_m^{j})\right] \|u-p\| \\
&\quad + \left[\prod_{m=n-1}^{n}\prod_{j=1}^{k}(1-\alpha_m^{j})\right] \|V_{k(n-1)-1} - p\| \\
&\leq \\
&\vdots \\
&\leq \left[1-\prod_{m=1}^{n}\prod_{j=1}^{k}(1-\alpha_m^{j})\right] \|u-p\| \\
&\quad + \left[\prod_{m=1}^{n}\prod_{j=1}^{k}(1-\alpha_m^{j})\right] \|V_{k-1} - p\|.
\end{aligned}
$$

Thus the subsequence $(V_{k(n+1)-1})$ of (V_n) is bounded. Similarly, by using (3.7) and (3.8), the subsequences $(V_{k(n+1)-2}), \dots, (V_{kn})$, are also bounded. Hence the sequence (V_n) is bounded.
By using the fact that the resolvent operator is firmly nonexpansive, for every $n \geq 0$ we have,

(3.10)

$$
\begin{aligned}
\|J_{\beta_n^k}^{A^k} V_{k(n+1)-2} - p\|^2 &= \|J_{\beta_n^k}^{A^k} V_{k(n+1)-2} - J_{\beta_n^k}^{A^k} p\|^2 \\
&\leq \|V_{k(n+1)-2} - p\|^2 - \|V_{k(n+1)-2} - J_{\beta_n^k}^{A^k} V_{k(n+1)-2}\|^2.
\end{aligned}
$$

Also

$$2\left\langle V_{k(n+1)-2}-p, J_{\beta_n^k}^{A^k}V_{k(n+1)-2}-p\right\rangle \tag{3.11}$$
$$= \|V_{k(n+1)-2}-p\|^2 + \|J_{\beta_n^k}^{A^k}V_{k(n+1)-2}-p\|^2 - \|V_{k(n+1)-2}-J_{\beta_n^k}^{A^k}V_{k(n+1)-2}\|^2$$
$$\leq 2\left[\|V_{k(n+1)-2}-p\|^2 - \|V_{k(n+1)-2}-J_{\beta_n^k}^{A^k}V_{k(n+1)-2}\|^2\right],$$

where the above inequality follows from (3.10). Again by using the firmly nonexpansiveness property of the resolvent operator, we see that

$$\|\delta_n^k(V_{k(n+1)-2}-p)+\gamma_n^k(J_{\beta_n^k}^{A^k}V_{k(n+1)-2}-p)\|^2 \tag{3.12}$$
$$= (\delta_n^k)^2\|V_{k(n+1)-2}-p\|^2 + (\gamma_n^k)^2\|J_{\beta_n^k}^{A^k}V_{k(n+1)-2}-p\|^2 + 2\gamma_n^k\delta_n^k\left\langle V_{k(n+1)-2}-p, J_{\beta_n^k}^{A^k}V_{k(n+1)-2}-p\right\rangle$$
$$\leq (\delta_n^k)^2\|V_{k(n+1)-2}-p\|^2 + (\gamma_n^k)^2\|V_{k(n+1)-2}-p\|^2 - (\gamma_n^k)^2\|V_{k(n+1)-2}-J_{\beta_n^k}^{A^k}V_{k(n+1)-2}\|^2 + 2\gamma_n^k\delta_n^k\|V_{k(n+1)-2}-p\|^2 - 2\gamma_n^k\delta_n^k\|V_{k(n+1)-2}-J_{\beta_n^k}^{A^k}V_{k(n+1)-2}\|^2$$
$$= (1-\alpha_n^k)^2\|V_{k(n+1)-2}-p\|^2 - \gamma_n^k(\gamma_n^k+2\delta_n^k)\|V_{k(n+1)-2}-J_{\beta_n^k}^{A^k}V_{k(n+1)-2}\|^2$$

where the above inequality follows from (3.10) and (3.11). From (3.5) and Lemma 2.1, we have

$$\|V_{k(n+1)-1}-p\|^2 \tag{3.13}$$
$$= \|\alpha_n^k(u-p)+\delta_n^k(V_{k(n+1)-2}-p)+\gamma_n^k(J_{\beta_n^k}^{A^k}V_{k(n+1)-2}-p)\|^2$$
$$\leq \|\delta_n^k(V_{k(n+1)-2}-p)+\gamma_n^k(J_{\beta_n^k}^{A^k}V_{k(n+1)-2}-p)\|^2 + 2\alpha_n^k\left\langle u-p, V_{k(n+1)-1}-p\right\rangle$$
$$\leq (1-\alpha_n^k)\|V_{k(n+1)-2}-p\|^2 - \epsilon_1\|V_{k(n+1)-2}-J_{\beta_n^k}^{A^k}V_{k(n+1)-2}\|^2 + 2\alpha_n^k\left\langle u-p, V_{k(n+1)-1}-p\right\rangle,$$

where the second inequality follows from (3.12) and ϵ_1 is a positive number with $\gamma_n^k(\gamma_n^k+2\delta_n^k)\geq\epsilon_1$. Similarly, we get

$$\|V_{k(n+1)-1}-p\|^2 \leq (1-\alpha_n^k)\Big[(1-\alpha_n^{k-1})\|V_{k(n+1)-3}-p\|^2 - \sqrt{\epsilon_2}\|V_{k(n+1)-3}-J_{\beta_n^{k-1}}^{A^{k-1}}V_{k(n+1)-3}\|^2 \tag{3.14}$$

$$
\begin{aligned}
&+ 2\alpha_n^{k-1}\left\langle u-p, V_{k(n+1)-2}-p\right\rangle \Big] \\
&- \epsilon_1\|V_{k(n+1)-2} - J_{\beta_n^k}^{A^k} V_{k(n+1)-2}\|^2 \\
&+ 2\alpha_n^{k}\left\langle u-p, V_{k(n+1)-1}-p\right\rangle \\
\le\ & (1-\alpha_n^k)(1-\alpha_n^{k-1})\|V_{k(n+1)-3}-p\|^2 \\
&- \epsilon_1\|V_{k(n+1)-2} - J_{\beta_n^k}^{A^k} V_{k(n+1)-2}\|^2 \\
&- \epsilon_2\|V_{k(n+1)-3} - J_{\beta_n^{k-1}}^{A^{k-1}} V_{k(n+1)-3}\|^2 \\
&+ 2\alpha_n^{k}\left\langle u-p, V_{k(n+1)-1}-p\right\rangle \\
&+ 2\alpha_n^{k-1}\left\langle u-p, V_{k(n+1)-2}-p\right\rangle \\
&\ \vdots \\
\le\ & (1-\alpha_n^k)\dots(1-\alpha_n^1)\|V_{kn-1}-p\|^2 \\
&- \epsilon_1\|V_{k(n+1)-2} - J_{\beta_n^k}^{A^k} V_{k(n+1)-2}\|^2 - \cdots \\
&- \epsilon_k\|V_{kn-1} - J_{\beta_n^1}^{A^1} V_{kn-1}\|^2 \\
&+ 2\alpha_n^{k}\left\langle u-p, V_{k(n+1)-1}-p\right\rangle + \cdots \\
&+ 2\alpha_n^{1}\left\langle u-p, V_{kn}-p\right\rangle .
\end{aligned}
$$

Setting $S_n = \|V_{kn-1}-p\|^2$, then for some positive constant M we have

(3.15)

$$
\begin{aligned}
S_{n+1} - S_n + \epsilon_1\|V_{k(n+1)-2} - J_{\beta_n^k}^{A^k} V_{k(n+1)-2}\|^2 + \cdots + \epsilon_k\|V_{kn-1} - J_{\beta_n^1}^{A^1} V_{kn-1}\|^2 \\
\le (\alpha_n^1 + \ldots + \alpha_n^k)M.
\end{aligned}
$$

Now, we show that $S_n \to 0$ as $n \to \infty$. There exist two possible cases for the sequence (S_n).

Case(I): (S_n) is eventually decreasing (i.e., there exists $N \ge 0$ such that (S_n) is decreasing for all $n \ge N$). In this case, (S_n) is convergent. Then by letting $n \to \infty$ in (3.15), we conclude that

$$
\begin{aligned}
&\lim_{n\to\infty} \|V_{k(n+1)-2} - J_{\beta_n^k}^{A^k} V_{k(n+1)-2}\| = 0 \qquad (3.16)\\
&\ \vdots \\
&\lim_{n\to\infty} \|V_{kn-1} - J_{\beta_n^1}^{A^1} V_{kn-1}\| = 0.
\end{aligned}
$$

Moreover, it follows from (3.5) that

$$
\begin{aligned}
\|V_{k(n+1)-1} - V_{k(n+1)-2}\| \le\ & \alpha_n^k\|u - V_{k(n+1)-2}\| \qquad (3.17)\\
&+ \gamma_n^k\|V_{k(n+1)-2} - J_{\beta_n^k}^{A^k} V_{k(n+1)-2}\|,
\end{aligned}
$$

and this shows that

$$\lim_{n\to\infty} \|V_{k(n+1)-1} - V_{k(n+1)-2}\| = 0. \tag{3.18}$$

Similarly

$$\begin{aligned} &\lim_{n\to\infty} \|V_{k(n+1)-2} - V_{k(n+1)-3}\| = 0 \\ &\quad\vdots \\ &\lim_{n\to\infty} \|V_{kn} - V_{kn-1}\| = 0, \end{aligned} \tag{3.19}$$

which implies that

$$\lim_{n\to\infty} \|V_{n+1} - V_n\| = 0. \tag{3.20}$$

On the other hand, since (3.16) holds and by using the Lemma 2.3,

$$\begin{aligned} \lim_{n\to\infty} \|A_{\beta^1}(V_{kn-1})\| &= \lim_{n\to\infty} \frac{1}{\beta^1}\|V_{kn-1} - J^{A^1}_{\beta^1} V_{kn-1}\| \\ &\leq \lim_{n\to\infty} \frac{1}{\beta^1}\|V_{kn-1} - J^{A^1}_{\beta^1_n} V_{kn-1}\| = 0. \end{aligned} \tag{3.21}$$

Since A_{β^1} is demiclosed, one can show that

$$\omega_w((V_{kn-1})) \subset (A^1_{\beta^1})^{-1}(0) = (A^1)^{-1}(0). \tag{3.22}$$

By a similar method,

$$\begin{aligned} &\omega_w((V_{kn})) \subset (A^2)^{-1}(0) \\ &\omega_w((V_{kn+1})) \subset (A^3)^{-1}(0) \\ &\quad\vdots \\ &\omega_w((V_{k(n+1)-2})) \subset (A^k)^{-1}(0). \end{aligned} \tag{3.23}$$

Moreover, from (3.20) we get,

$$\omega_w((V_n)) \subset F = \cap_{j=1}^k (A^j)^{-1}(0). \tag{3.24}$$

Therefore, there exists a subsequence (V_{n_l}) of (V_n) converging weakly to some $z \in F$ such that,

$$\begin{aligned} \limsup_{n\to\infty} \langle u - P_F u, V_n - P_F u\rangle &= \lim_{l\to\infty} \langle u - P_F u, V_{n_l} - P_F u\rangle \\ &= \langle u - P_F u, z - P_F u\rangle \\ &\leq 0. \end{aligned} \tag{3.25}$$

Now, replacing p by $P_F u$ in (3.14) gives,

$$\begin{aligned} \|V_{k(n+1)-1} - P_F u\|^2 \leq\ & (1-\alpha_n^k)\ldots(1-\alpha_n^1)\|V_{kn-1} - P_F u\|^2 \\ & + 2\alpha_n^k \langle u - P_F u, V_{k(n+1)-1} - P_F u\rangle \end{aligned} \tag{3.26}$$

$$+\cdots+2\alpha_n^1\left\langle u-P_Fu, V_{kn}-P_Fu\right\rangle.$$

By using above inequality and Lemma 3.1, we get

$$\lim_{n\to\infty}\|V_{k(n+1)-1}-P_Fu\|=0. \tag{3.27}$$

From (3.27), (3.7) and (3.8) we have,

$$\begin{aligned}\lim_{n\to\infty}\|V_{kn}-P_Fu\|&=0\\ &\vdots\\ \lim_{n\to\infty}\|V_{k(n+1)-2}-P_Fu\|&=0.\end{aligned} \tag{3.28}$$

Thus

$$\lim_{n\to\infty}\|V_n-P_Fu\|=0. \tag{3.29}$$

Case(II): (S_n) is not eventually decreasing, that is, there is a subsequence (S_{n_j}) of (S_n) such that $S_{n_j}<S_{n_j+1}$ for all $j\geq 0$. Define an integer sequence $(\tau(n))_{n\geq n_0}$ as in Lemma 2.4. Since $S_{\tau(n)}\leq S_{\tau(n)+1}$ for all $n\geq n_0$, it follows from (3.15) that

$$\begin{aligned}&\lim_{n\to\infty}\|V_{k(\tau(n)+1)-2}-J^{A^k}_{\beta^k_{\tau(n)}}V_{k(\tau(n)+1)-2}\|=0\\ &\vdots\\ &\lim_{n\to\infty}\|V_{k\tau(n)-1}-J^{A^1}_{\beta^1_{\tau(n)}}V_{k\tau(n)-1}\|=0.\end{aligned} \tag{3.30}$$

As in Case(I), one can conclude that

$$\begin{aligned}&\omega_w((V_{k\tau(n)-1}))\subset(A^1)^{-1}(0)\\ &\omega_w((V_{k\tau(n)}))\subset(A^2)^{-1}(0)\\ &\vdots\\ &\omega_w((V_{k(\tau(n)+1)-2}))\subset(A^k)^{-1}(0).\end{aligned} \tag{3.31}$$

It follows from (3.5) that

$$\begin{aligned}\|V_{k(\tau(n)+1)-1}-V_{k(\tau(n)+1)-2}\|\leq{}&\alpha^k_{\tau(n)}\|u-V_{k(\tau(n)+1)-2}\|\\ &+\gamma^k_{\tau(n)}\|V_{k(\tau(n)+1)-2}-J^{A^k}_{\beta^k_{\tau(n)}}V_{k(\tau(n)+1)-2}\|,\end{aligned} \tag{3.32}$$

and from (3.30) we get,

$$\lim_{n\to\infty}\|V_{k(\tau(n)+1)-1}-V_{k(\tau(n)+1)-2}\|=0. \tag{3.33}$$

Similarly

$$\lim_{n\to\infty}\|V_{k(\tau(n)+1)}-V_{k(\tau(n))}\|=0 \tag{3.34}$$

$$\vdots$$
$$\lim_{n\to\infty} \|V_{k(\tau(n)+1)-2} - V_{k(\tau(n)+1)-3}\| = 0.$$

Thus

$$\begin{aligned} &\omega_w((V_{k\tau(n)-1})) \subset F \\ &\omega_w((V_{k\tau(n)})) \subset F \\ &\quad\vdots \\ &\omega_w((V_{k(\tau(n)+1)-2})) \subset F. \end{aligned} \tag{3.35}$$

Therefore, there exists a subsequence $(V_{k\tau(n)_j})$ of $(V_{k\tau(n)})$ converging weakly to some $z \in F$ such that,

$$\begin{aligned} \limsup_{n\to\infty} \left\langle u - P_F u, V_{k\tau(n)} - P_F u \right\rangle &= \lim_{j\to\infty} \left\langle u - P_F u, V_{k\tau(n)_j} - P_F u \right\rangle \\ &= \langle u - P_F u, z - P_F u \rangle \\ &\le 0. \end{aligned} \tag{3.36}$$

Similarly,

$$\begin{aligned} &\limsup_{n\to\infty} \left\langle u - P_F u, V_{k\tau(n)+1} - P_F u \right\rangle \le 0 \\ &\quad\vdots \\ &\limsup_{n\to\infty} \left\langle u - P_F u, V_{k(\tau(n)+1)-1} - P_F u \right\rangle \le 0. \end{aligned} \tag{3.37}$$

Now, replacing p by $P_F u$ in (3.14) and n by $\tau(n)$ gives,

$$\begin{aligned} S_{\tau(n)+1} \le (1 - \alpha^k_{\tau(n)}) \dots (1 - \alpha^1_{\tau(n)}) S_{\tau(n)} \\ + 2\alpha^k_{\tau(n)} \left\langle u - P_F u, V_{k(\tau(n)+1)-1} - P_F u \right\rangle \\ + \dots + 2\alpha^1_{\tau(n)} \left\langle u - P_F u, V_{k\tau(n)} - P_F u \right\rangle. \end{aligned} \tag{3.38}$$

Thus

$$\begin{aligned} &[1 - (1 - \alpha^k_{\tau(n)}) \dots (1 - \alpha^1_{\tau(n)})] S_{\tau(n)+1} \\ &\le 2\alpha^k_{\tau(n)} \left\langle u - P_F u, V_{k(\tau(n)+1)-1} - P_F u \right\rangle \\ &\quad + \dots + 2\alpha^1_{\tau(n)} \left\langle u - P_F u, V_{k\tau(n)} - P_F u \right\rangle. \end{aligned} \tag{3.39}$$

Therefore,

$$S_{\tau(n)+1} \le \frac{2\alpha^k_{\tau(n)}}{\left[1 - (1 - \alpha^k_{\tau(n)}) \dots (1 - \alpha^1_{\tau(n)})\right]} \left\langle u - P_F u, V_{k(\tau(n)+1)-1} - P_F u \right\rangle$$

$$+\cdots+\frac{2\alpha^1_{\tau(n)}}{\left[1-(1-\alpha^k_{\tau(n)})\dots(1-\alpha^1_{\tau(n)})\right]}\left\langle u-P_Fu, V_{k\tau(n)}-P_Fu\right\rangle,$$

This shows that, $\lim_{n\to\infty} S_{\tau(n)+1}=0$. Now by using the Lemma 2.4 we conclude that $\lim_{n\to\infty} S_n=0$. From (3.7) and (3.8), we get $\lim_{n\to\infty} V_n = P_Fu$ and this completes the proof. □

Theorem 3.3. *Let $A_i : D(A_i) \subset H \to H (i = 1,\dots,k)$ be maximal monotone operators with $\bigcap_{i=1}^k A_i^{-1}(0) = F \neq \emptyset$. For fixed vectors $x_0, u \in H$, let (x_n) be the sequence generated by (3.6), where for every $j = 1, 2, \dots, k$; $\alpha^j_n, \delta^j_n, \gamma^j_n \in (0, 1)$ with $\alpha^j_n + \delta^j_n + \gamma^j_n = 1$, $\beta^j_n \in (0,\infty)$ and $\sum_{n=1}^{\infty} \|e^j_n\| < \infty$. Assume that*

(i) $\lim_{n\to\infty} \alpha^j_n = 0$,

(ii) *for some* $1 \le l \le k$, $\sum_{n=1}^{\infty} \alpha^l_n = \infty$,

(iii) $\beta^j_n \ge \beta^j$ *for* $\beta^j > 0$,

(iv) $\gamma^j_n \ge \gamma^j$ *for* $\gamma^j > 0$.

Then (x_n) converges strongly to P_Fu.

Proof. Taking Theorem (3.2) into account, it is enough to prove that $\lim_{n\to\infty} \|x_n - V_n\| = 0$. Since the resolvent of A_k is nonexpansive, we derive from (3.5) and (3.6) that

$$\begin{aligned}\|x_{kn+k-1}-V_{kn+k-1}\| &\le \delta^k_n\|x_{kn+k-2}-V_{kn+k-2}\| \\ &\quad + \gamma^k_n\|J^{A_k}_{\beta^k_n}x_{kn+k-2}-J^{A_k}_{\beta^k_n}V_{kn+k-2}\|+e^k_n \\ &\le (1-\alpha^k_n)\|x_{kn+k-2}-V_{kn+k-2}\|+e^k_n. \end{aligned} \tag{3.40}$$

Similarly

(3.41)

$$\|x_{kn+k-2}-V_{kn+k-2}\| \le (1-\alpha^{k-1}_n)\|x_{kn+k-3}-V_{kn+k-3}\|+e^{k-1}_n,$$

$$\vdots$$

$$\|x_{kn}-V_{kn}\| \le (1-\alpha^1_n)\|x_{kn-1}-V_{kn-1}\|+e^1_n.$$

The above inequalities imply that

$$\begin{aligned}&\|x_{kn+k-1}-V_{kn+k-1}\| \\ &\qquad \le (1-\alpha^1_n)(1-\alpha^2_n)\dots(1-\alpha^k_n)\|x_{kn-1}-V_{kn-1}\|+\sum_{j=1}^k\|e^j_n\|.\end{aligned}$$

By letting $n \to \infty$ and by using Lemma(3.1), then

$$\lim_{n\to\infty} \|x_{kn+k-1} - V_{kn+k-1}\| = 0. \tag{3.42}$$

From (3.41) and (3.42) we have

$$\lim_{n\to\infty} \|x_{kn+k-2} - V_{kn+k-2}\| = 0, \tag{3.43}$$

$$\vdots$$

$$\lim_{n\to\infty} \|x_{kn} - V_{kn}\| = 0.$$

This completes the proof. □

In the following theorem, we give a necessary and sufficient condition for the common zero set of $A_1, A_2, \dots, A_k$ to be nonempty.

Theorem 3.4. *Let $A_i : D(A_i) \subset H \to H (i = 1, \dots, k)$ be maximal monotone operators. For fixed vectors $x_0, u \in H$, let (x_n) be the sequence generated by (3.6), where for every $j = 1, 2, \dots, k$; $\alpha_n^j, \delta_n^j, \gamma_n^j \in (0, 1)$ with $\alpha_n^j + \delta_n^j + \gamma_n^j = 1$ and $\sum_{n=1}^{\infty} \|e_n^j\| < \infty$. Assume that*

(i) $\lim_{n\to\infty} \alpha_n^j = 0$,

(ii) *for some* $1 \le l \le k$, $\sum_{n=1}^{\infty} \alpha_n^l = \infty$,

(iii) $\beta_n^j \to \infty$,

(iv) $\gamma_n^j \to 1$.

Then the following statement holds:

if (x_n) is weakly convergent then $\bigcap_{i=1}^k A_i^{-1}(0) = F \neq \emptyset$,

Proof. Let (x_n) be weakly converges to some $z \in H$. Therefore subsequences $(x_{kn}), (x_{kn+1}), \dots, (x_{k(n+1)-1})$, of (x_n) are weakly converges to z. It follows from (3.5) that

$$J_{\beta_n^1}^{A^1} x_{k(n)-1} = \frac{x_{kn} - \alpha_n^1 u - \delta_n^1 x_{kn-1} - e_n^1}{\gamma_n^1}. \tag{3.44}$$

Let $J_{\beta_n^1}^{A^1} x_{k(n)-1} = z_n$. Then z_n is weakly converges to z and

$$\lim_{m\to\infty} \frac{x_{k(n)-1} - z_n}{\beta_n^1} = 0.$$

On the other hand from the definition of resolvent operator we have,

$$\frac{x_{k(n)-1} - z_n}{\beta_n^1} \in A_1(z_n). \tag{3.45}$$

The demiclosedness property of the operator A_1 implies that, $z \in (A_1)^{-1}(0)$. Similarly, for subsequences $x_{k(n)+1}, \dots, x_{k(n+1)-1}$, of x_n, we get $z \in$

$(A_2)^{-1}(0), \ldots, z \in (A_k)^{-1}(0)$ and this completes the proof. □

4. Conclusions

In this paper, we give a necessary and sufficient condition for the set of common zeros of a finite family of maximal monotone operators is nonempty, and showed the strong convergence of the scheme to a zero of the operator in this case. As a future direction for research, since numerous other algorithms have been developed and their convergence studied by many authors, it might be interesting to investigate the possibility of implementing the ideas and methods developed in this paper to these other Algorithm. In particular, in this connection, we can mention the recent work of N. Nimit, A. P. Farajzadeh and N. Petrots [15].

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