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On Preserving Properties of Linear Maps on C^*-algebras

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ABSTRACT. Let A and B be two unital C^*-algebras and $\varphi : A \to B$ be a linear map. In this paper, we investigate the structure of linear maps between two C^*-algebras that preserve a certain property or relation. In particular, we show that if φ is unital, B is commutative and $V(\varphi(a)^*\varphi(b)) \subseteq V(a^*b)$ for all $a, b \in A$, then φ is a $*$-homomorphism. It is also shown that if $\varphi(|ab|) = |\varphi(a)\varphi(b)|$ for all $a, b \in A$, then φ is a unital $*$-homomorphism.

1. Introduction

In 1970, Kaplansky asked the following question:
Let $\varphi : A \longrightarrow B$ be a unital and invertibility preserving linear map between unital Banach algebras A and B. Is φ a Jordan homomorphism?[6]. The Kaplansky's question was originated by Gleason-Kahane-Zelazko Theorem which states that every invertibility preserving unital linear functional on a unital complex Banach algebra is multiplicative [18]. In this paper we explain and prove a Gleason-Kahane-Zelazko type Theorem and show that if A is a unital C^*-algebra and φ is a unital linear functional on A such that $V(\varphi(a)^*\varphi(b)) \subseteq V(a^*b)$ for all $a, b \in A$, then φ is a $*$-homomorphism.
Another kind of linear preserver problems is absolute value preserving linear maps and in this paper, we will characterize this kind of maps.

Let (X, d) and $(Y, d^{'})$ be metric spaces. A map $f : X \to Y$ is said to be a contraction if there exist $0 \leq k < 1$ such that

$$d^{'}(f(x_1), f(x_2)) \leq kd(x_1, x_2); \quad x_1, x_2 \in X.$$

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Let A be a complex unital normed algebra, and

$$D(A,1)=\left\{f\in A',f(1)=\| f \|=1\right\},$$

where A' is the dual space of A. The elements of $D(A,1)$ are called the normalized states on A. For $a\in A$ let,

$$V(a)=\{f(a):f\in D(A,1)\},\qquad v(a)=\sup\{|\lambda|:\lambda\ \in V(A)\}.$$

The sets $V(a)$ and $v(a)$ are called the numerical range and numerical radius of a respectively and the spectrum of a is denoted by $\sigma(a)$, namely

$$\sigma(a)=\{\lambda\in\mathbb{C}:\lambda-a\in \operatorname{sing}(A)\}.$$

Also the convex hull of $\sigma(x)$ is denoted by $co\sigma(x)$. If H is a Hilbert space, for every $T\in B(H)$, the numerical range of T is the set

$$W(T)=\{\langle T(x),x\rangle:x\in H,\|x\|=1\},$$

and the numerical radius of T is defined by $w(T)=sup\{|\lambda|:\lambda\in W(T)\}$. When $A=B(H)$ and $T\in A$, the set $V(T)$ becomes the closure of $W(T)$. Let A and B be complex unital normed algebras. A linear map $\varphi:A\to B$ is said to be numerical range compressing if $V(\varphi(x))\subseteq V(x)$, numerical range preserving if $V(\varphi(x))=V(x)$ and Jordan homomorphism if $\varphi(x^2)=\varphi(x)^2$ for all $x\in A$. Also φ is said to be unital if $\varphi(1_A)=1_B$.

Let A be a unital C^*-algebra. An element a of A is said to be positive if $V(a)\subseteq\mathbb{R}^+$ or $a^*=a$ and $\sigma(a)\subseteq\mathbb{R}^+$. We denote by A^+ the set of all positive elements of A. Also a is called normal and unitary, if $a^*a=aa^*$ and $a^*a=aa^*=1$, respectively. We recall that if a is a unitary element of A, then $\sigma(a)=\{\lambda\in\mathbb{C}:|\lambda|=1\}$ [17].

A linear map φ from a unital C^*-algebra A into a unital C^*-algebra B is said unitary preserving if $\varphi(u)$ is unitary whenever u is unitary in A. We say that φ preserves absolute values if $\varphi(|x|)=|\varphi(x)|$, where $|x|^2=x^*x$ and $*$-homomorphism, if $\varphi(xy)=\varphi(x)\varphi(y)$ and $\varphi(x^*)=\varphi(x)^*$ for every $x,y\in A$. For any positive integer n we define $\varphi_n: M_n(A)\to M_n(B)$ by $\varphi_n((a_{i,j})_{i,j})=\varphi((a_{i,j}))_{i,j}$, where $M_n(A)$ denotes the set of all $n\times n$ matrices with entries in A. The map φ is called positive if $\varphi(a)\geq 0$ for all $a\in A^+$ and $n-$positive if φ_n is positive. Also φ is called completely positive if φ is $n-$positive for all n. Every positive linear map is not necessary completely positive, for example if $\varphi:M_2(\mathbb{C})\to M_2(\mathbb{C})$ is defined by $\varphi(A)=A^t$, then φ is positive but is not necessary completely positive, [10, Example 4.2]. Every $*$-homomorphism on a $*$-algebras is completely positive [2, Example II.6.9.3] but the converse is false. For example if $\varphi:M_2(\mathbb{C})\to\mathbb{C}$ is defined by $\varphi\left((a_{i,j})_{i,j}\right)=\sum a_{i,i}$, then φ is completely positive by [10, Exercise 3.5]. But φ is not a homomorphism because

$$\varphi\left(\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\right)^2 = 9 \qquad , \varphi\left(\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}^2\right) = 5.$$

A subspace S of a unital C^*-algebra A is called operator system if it is self-adjoint ($S = S^*$) and contains the unit of A.

Numerical range of operators is very important and was studied by many authors, see, e.g., [1, 5, 7, 11, 13]. In this paper, we characterize a linear map φ from a unital C^*-algebra A into a unital commutative C^*-algebra B and show that if φ is unital and $V(\varphi(a)^*\varphi(b)) \subseteq V(a^*b)$ or $V(\varphi(a)\varphi(b)\varphi(a)) \subseteq V(aba)$ for all $a, b \in A$, then φ is a $*$-homomorphism. Also if $V(\varphi(a)\varphi(b)\varphi(a)^*) \subseteq V(aba^*)$ for all $a, b \in A$, then φ is a unital $*$-homomorphism. We show that every Jordan homomorphism from a complex Banach algebra into $\mathbb{C}$ is a numerical range compressing, but the converse is false.

Also in this paper, we discuss about absolute value preserving linear maps and show that if φ is a linear map from a unital C^*-algebra A into a unital commutative C^*-algebra B such that $|\varphi(a)\varphi(b)| = \varphi(|ab|)$ for all $a, b \in A$, then φ is a unital $*$-homomorphism.

2. Preliminaries

Let A be a unital C^*-algebra. If a and b are positive elements of A such that $a^2 = b^2$, then $\sigma\left(a^2 - b^2\right) = -\sigma\left(b^2 - a^2\right) = \sigma(0) = \{0\}$. Since $a^2 - b^2$ and $b^2 - a^2$ are self-adjoint, $a^2 \geq b^2$ and $a^2 \leq b^2$, so by [9, Theorem 2.2.6], $a \geq b$ and $a \leq b$. This implies that $a = b$, [2, Proposition II.3.1.2]. Let X be a compact Hausdorff space. We denote by $C(X)$ the algebra of all continuous complex functionals on X.

Theorem 2.1. *Let A be a unital C^*-algebra and $\varphi : A \to C(X)$ be a positive linear map. If φ is a unitary preserving map, then φ is a unital $*$-homomorphism.*

Proof. Since φ is positive and preserves unitary elements, $\varphi(1_A)$ is positive and $\varphi(1_A)^2 = \varphi(1_A)^*\varphi(1_A) = 1_{C(X)}$, so $\varphi(1_A) = 1_{C(x)}$.

Since φ is unital and positive by [2, Proposition II.6.9.4], φ is a contraction. Also, since φ is a bounded linear map and A is an operator system, φ is completely positive [10, Proposition 3.9].

Now, let a and b be elements of A such that $\|a\| < 1$ and $\|b\| < 1$. Since -1_A is a unitary element of A by [2, Poroposition II.3.2.13], there exist unitary elements u_1,u_2,v_1,v_2 of A such that $a - 1_A = u_1 + u_2$ and $b - 1_A = v_1 + v_2$. So

$$\varphi(ab) = \varphi(1_A) + \varphi(v_1) + \varphi(v_2) + \varphi(u_1) + \varphi(u_2)$$

$$+\varphi(u_1v_1)+\varphi(u_1v_2)+\varphi(u_2v_1)+\varphi(u_2v_2).$$

Since v_1 is a unitary element of A and φ preserves unitary elements, we have

$$\begin{aligned}\varphi(v_1)^*\varphi(v_1) &= 1_{C(X)}\\ &= \varphi(1_A)\\ &= \varphi(v_1^*v_1),\end{aligned}$$

and therefore $\varphi(xv_1)=\varphi(x)\varphi(v_1)$ for all $x\in A$ by [2, Poroposition II.6.9.18]. Similarly, we can show that $\varphi(xv_2)=\varphi(x)\varphi(v_2)$ for all $x\in A$. Therefore

$$\begin{aligned}\varphi(ab) &= 1_{C(X)}+\varphi(v_1)+\varphi(v_2)+\varphi(u_1)+\varphi(u_2)+\varphi(u_1)\varphi(v_1)\\ &\quad+\varphi(u_1)\varphi(v_2)+\varphi(u_2)\varphi(v_1)+\varphi(u_2)\varphi(v_2)\\ &= (1_{C(X)}+\varphi(u_1)+\varphi(u_2))(1+\varphi(v_1)+\varphi(v_2))\\ &= \varphi(a)\varphi(b).\end{aligned}$$

Let $\|a\|\geq 1$ and $\|b\|\geq 1$ and let $a^{'}=\frac{a}{1+\|a\|}$ and $b^{'}=\frac{b}{1+\|b\|}$. Then $\|a^{'}\|<1$ and $\|b^{'}\|<1$. Therefore $\varphi(a^{'}b^{'})=\varphi(a^{'})\varphi(b^{'})$ and hence $\varphi(ab)=\varphi(a)\varphi(b)$. Also if $a\in A$ is self-adjoint, then there exist positive elements $b,c\in A$ such that $a=b-c$. Thus $\varphi(a^*)=\varphi(b)-\varphi(c)=(\varphi(b)-\varphi(c))^*=\varphi(a)^*$. Let $a\in A$ be arbitrary. Then there exist self-adjoint elements $x,y\in A$ such that $a=x+iy$, so

$$\begin{aligned}\varphi(a^*) &= \varphi(x-iy)\\ &= \varphi(x)-i\varphi(y)\\ &= (\varphi(x)+i\varphi(y))^*\\ &= \varphi(a)^*.\end{aligned}$$

□

Remark 2.2. Since by [9, Theorem 2.1.10], every non-zero commutative C^*-algebra B is isomorphic to $C(\Omega(B))$, where $\Omega(B)$ is the set of all linear homomorphisms from B into $\mathbb{C}$, every unitary preserving positive linear map from a unital C^*-algebra A into a unital commutative C^*-algebra B is a unital $*$-homomorphism.

3. Numerical Range Preserving Maps

Let H and K be complex Hilbert spaces, $A,B\in B(H)$ and $\varphi: B(H)\to B(K)$ be a surjective map. Theorem 2.1 in [5] shows that $W(\varphi(A)\varphi(B))=W(AB)$ if and only if there exists unitary operator U in $B(H,K)$ such that φ is of the form $\varphi(A)=\epsilon UAU^*$ for all $A\in B(H)$, where $\epsilon=\pm 1$. Similarly Theorem 2.2 in [5] states that

$W(\varphi(A)\varphi(B)\varphi(A)) = W(ABA)$ if and only if there exist a scalar λ with $\lambda^3 = 1$ and a unitary operator $U : H \to K$ such that either $\varphi(A) = \lambda UAU$ or $\varphi(A) = \lambda UA^tU$, where A^t is the transpose of A with respect to an arbitrary fixed orthonormal basis of H. Also $W(\varphi(A)^*\varphi(B)) = W(A^*B)$ if and only if there exist unitary operators U and V in $B(H,K)$ such that φ is of the form $\varphi(A) = UAV^*$ [5, Corollary 4.3].

Let A and B be unital C^*-algebras and φ be a unital linear mapping from A onto B. Theorem 2.3 in [13] states that if $W(\varphi(a)) = W(a)$ for all $a \in A$, then φ is a Jordan $*-$isomorphism. Furthermore, if B is prime, then φ is a C^*-isomorphism or C^*-anti-isomorphism.

In this section, we characterize the numerical range of a map from a unital C^*-algebra into a unital commutative C^*-algebra.

Lemma 3.1. *Let A be a unital complex Banach algebra and $\varphi : A \to \mathbb{C}$ be a linear functional. If $V(\varphi(a)) = V(a)$ for all $a \in A$, then φ is a unital monomorphism.*

Proof. Since $V(\varphi(1_A)) = V(1_A) = \{1\}$, $v(\varphi(1) - 1) = 0$ and by [3, Theorem 1.4.1] $\varphi(1) - 1 = 0$, therefore φ is unital. Let $a \in A$ be invertible but $\varphi(a)$ is singular, then $\varphi(a) = 0$. Thus $V(a) = V(\varphi(a)) = 0$, so $a = 0$ by [3, Theorem, 1.4.1]. But this is a contraction, so φ is an invertibility preserving functional, therefore by Gleason-Kahane-Zelazko Theorem, φ is multiplicative.

If $a \in$A and $\varphi(a) = 0$, then $V(a) = V(\varphi(a)) = 0$ and by [3, Theorem 1.4.1] $a = 0$, so φ is injective. □

Lemma 3.2. *Let A be a unital complex Banach algebra and $\varphi : A \to \mathbb{C}$ be a linear functional. If φ is a Jordan homomorphism, then φ is a unital numerical range compressing.*

Proof. Since φ is a Jordan homomorphism, then by [4, Proposition, II.16.6] φ is a homomorphism, so φ is continuous and $\|\varphi\| = \varphi(1_A) = 1$ [4, Proposition II.16.3]. Let $a \in A$ and $\lambda \in V(\varphi(a))$, then there exists $f \in D(\mathbb{C}, 1)$ such that $\lambda = f(\varphi(a))$. Let $g = f\varphi$. Then g is linear, continuous and for all $x \in A$

$$|g(x)| = |f(\varphi(x))| \leq \|f\|\|\varphi\|\|x\| = \|x\|,$$

so $\|g\| \leq 1$, also $g(1_A) = (f\varphi)(1_A) = 1$, thus $\|g\| = 1$ and it follows that $g \in D(A, 1_A)$, but $\lambda = g(a)$. Therefore $\lambda \in V(a)$. □

Remark 3.3. The converse of Lemma 3.2 is false. To see that, let $A = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : a, b, c \in \mathbb{C} \right\}$ and define $\varphi : A \to \mathbb{C}$ by $\varphi\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\right) = \frac{a+c}{2}$, then

$$V\left(\varphi\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\right)\right) = V\left(\frac{a+c}{2}\right)$$

$$= \left\{ \frac{a+c}{2} \right\}.$$

Also $a, c \in \sigma\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\right)$, so $a, c \in V\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\right)$ [3, Theorem I.2.6]. Also by [4, Proposition 1.10.4], $V\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\right)$ is convex, so $\frac{a+c}{2} \in V\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\right)$. Therefore $V(\varphi(x)) \subseteq V(x)$ for all $x \in A$. But φ is not a Jordan homomorphism, because $\varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)^2 = \frac{9}{4}$ and $\varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^2\right) = \varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\right) = \frac{5}{2}$.

Theorem 3.4. *Let A and B be unital C^*-algebras and $\varphi : A \to B$ be a unital linear map. If B is commutative and $V(\varphi(a)^*\varphi(b)) \subseteq V(a^*b)$ for all $a, b \in A$, then φ is a $*$-homomorphism.*

Proof. Let $a \in A$ be positive. Then $V(\varphi(a)) = V(\varphi(1_A)^*\varphi(a)) \subseteq V(1_A a) \subseteq \mathbb{R}^+$, so φ is a positive map. Let $u \in A$ be unitary, then

$$\begin{aligned} V(\varphi(u)^*\varphi(u)) &\subseteq V(u^*u) \\ &= V(1) \\ &= \{1\}, \end{aligned}$$

thus $\varphi(u)^*\varphi(u) = 1_B$, so φ is a unitary preserving positive linear map. Therefore φ is a $*$-homomorphism by Remark 2.2. $\square$

Corollary 3.5. *Let A and B be unital C^*-algebras and $\varphi : A \to B$ be a surjective unital linear map. If B is commutative and $V(\varphi(a)^*\varphi(b)) = V(a^*b)$ for all $a, b \in A$, then φ is a $*$-isomorphism.*

Proof. If $a \in$A and $\varphi(a) = 0$, then

$$\begin{aligned} v(a) &= v(\varphi(1)^*\varphi(a)) \\ &= v(\varphi(a)) \\ &= v(0) \\ &= 0, \end{aligned}$$

and by [4, Theorem 1.10.14] $a = 0$, so φ is injective. Also φ is a $*$-homomorphism, by Theorem 3.4, so φ is a $*$-isomorphism. $\square$

Theorem 3.6. *Let A be a unital C^*-algebra. If φ is a linear functional on A such that $V(\varphi(a)\varphi(b)) \subseteq V(ab)$ for all $a, b \in A$, then φ is a scaler of a $*$-homomorphism .*

Proof. $V(\varphi(1_A)\varphi(1_A)) \subseteq V(1) = \{1\}$, so $\varphi(1_A)^2 = 1$, thus $\varphi(1_A) = 1$ or $\varphi(1_A) = -1$. Also if $a \in A$ is self-adjoint, then

$$V(\varphi(a)) = V(\varphi(a)\varphi(1))$$

$$\begin{aligned} &\subseteq V(a) \\ &\subseteq \mathbb{R}, \end{aligned}$$

or

$$\begin{aligned} V(\varphi(a)) &= V(\varphi(a)\varphi(-1)) \\ &\subseteq V(-a) \\ &\subseteq \mathbb{R}, \end{aligned}$$

so $\varphi(a)$ is self-adjoint. Now let $x \in A$, then there exist self-adjoint elements $a, b \in A$ such that $x = a + ib$, then

$$\begin{aligned} \varphi(x^*) &= \varphi(a - ib) \\ &= \varphi(a) - i\varphi(b) \\ &= (\varphi(a) + i\varphi(b))^* \\ &= \varphi(x)^*, \end{aligned}$$

so $V(\varphi(a)^*\varphi(b)) = V(\varphi(a^*)\varphi(b)) \subseteq V(a^*b)$ for all $a, b \in A$. Then, by Theorem 3.4, φ is $*$-preserving and $\varphi(ab) = \pm\varphi(a)\varphi(b)$ for all $a, b \in A$. □

Corollary 3.7. *Let A and B be unital C^*-algebras and $\varphi : A \to B$ be a linear map. If B is commutative and $V(\varphi(a)\varphi(b)) \subseteq V(ab)$ for all $a, b \in A$, then φ is a scaler of a $*$-homomorphism.*

Proof. Let τ be a multiplicative functional on B and $\psi = \tau\varphi$. If $a, b \in A$ and $\lambda \in V(\psi(a)\psi(b))$, then there exists $f \in D(\mathbb{C}, 1)$ such that $\lambda = f(\psi(a)\psi(b)) = f\tau(\varphi(a)\varphi(b))$. But $f\tau(1) = \|f\tau\| = 1$, so $\lambda \in V(\varphi(a)\varphi(b))$, thus $\lambda \in V(ab)$. Therefore $V(\psi(a)\psi(b)) \subseteq V(ab)$ and by Theorem 3.9, for all $x, y \in A$, we have

$$\begin{aligned} \tau(\varphi(xy)) &= \psi(xy) \\ &= \pm\psi(x)\psi(y) \\ &= \pm\tau(\varphi(x))\tau(\varphi(y)) \\ &= \tau(\pm\varphi(x)\varphi(y), \end{aligned}$$

and

$$\begin{aligned} \tau(\varphi(x^*)) &= \psi(x^*) \\ &= \psi(x)^* \\ &= \tau(\varphi(x)^*). \end{aligned}$$

Since B is semi-simple, $\Omega(B)$ separates the points of B [4, Corollary II.17.7], so $\varphi(xy) = \pm\varphi(x)\varphi(y)$ and $\varphi(x^*) = \varphi(x)^*$. □

Corollary 3.8. *Let A and B be unital C^*-algebras and $\varphi : A \to B$ be a surjective linear map. If B is commutative and $V(\varphi(a)\varphi(b)) = V(ab)$ for all $a, b \in A$, then φ is a scaler of a $*$-isomorphism.*

Proof. The map φ is injective. Also $\pm\varphi$ is a $*$-homomorphism, by Corollary 3.7, so φ is a scaler of a $*$-isomorphism. □

Theorem 3.9. *Let A be a unital C^*-algebra and φ be a unital linear functional on A. If $V(\varphi(a)\varphi(b)\varphi(a)) \subseteq V(aba)$ for all $a, b \in A$, then φ is a $*$-homomorphism.*

Proof. Let $a \in A$ be positive, then $V(\varphi(a)) \subseteq V(a) \subseteq \mathbb{R}^+$, so φ is positive. Therefore $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. Since φ is unital and positive, then by [2, Proposition II.6.9.4] φ is a contraction. Also since φ is a bounded linear functional and A is an operator system, then φ is completely positive [10, Proposition 3.8], thus φ is 2-positive.

Let $a, b \in A$ and $ab = 0$, then $V(\varphi(a)\varphi(b)\varphi(a) \subseteq V(aba) = \{0\}$, so $\varphi(a) = 0$ or $\varphi(b) = 0$. Thus φ preserves zero product elements, therefore φ is a homomorphism [16, Theorem 2]. □

Theorem 3.10. *Let A and B be unital C^*-algebras and $\varphi : A \to B$ be a unital linear map. If B is commutative and $V(\varphi(a)\varphi(b)\varphi(a)) \subseteq V(aba)$ for all $a, b \in A$, then φ is a $*$-homomorphism.*

Proof. Let τ be a multiplicative functional on B. Then $\tau(1) = \|\tau\| = 1$ by [4, Proposition I.16.3]. Let $\psi = \tau\varphi$, then $\psi(1) = 1$. Also if $a, b \in A$ and $\lambda \in V(\psi(a)\psi(b)\psi(a))$, then there exists $f \in D(\mathbb{C}, 1)$ such that

$$\begin{aligned} \lambda &= f(\psi(a)\psi(b)\psi(a)) \\ &= f\tau(\varphi(a)\varphi(b)\varphi(a). \end{aligned}$$

But $f\tau(1) = \|f\tau\| = 1$, so $\lambda \in V(\varphi(a)\varphi(b)\varphi(a))$, thus $\lambda \in V(aba)$. Therefore $V(\psi(a)\psi(b)\psi(a)) \subseteq V(aba)$, so by Theorem 3.9, for all $x, y \in A$, we have

$$\begin{aligned} \tau(\varphi(xy)) &= \psi(xy) \\ &= \psi(x)\psi(y) \\ &= \tau(\varphi(x))\tau(\varphi(y)) \\ &= \tau(\varphi(x)\varphi(y)), \end{aligned}$$

and

$$\begin{aligned} \tau(\varphi(x^*)) &= \psi(x^*) \\ &= \psi(x)^* \\ &= \tau(\varphi(x)^*). \end{aligned}$$

Since B is semi-simple, $\varphi(xy) = \varphi(x)\varphi(y)$ and $\varphi(x^*) = \varphi(x)^*$. □

Theorem 3.11. *Let A be a unital C^*-algebra and φ be a linear functional on A. If $V(\varphi(a)\varphi(b)\varphi(a)^*) \subseteq V(aba^*)$ for all $a, b \in A$, then φ is a unital $*$-homomorphism.*

Proof. Since $V(\varphi(1_A)\varphi(1_A)\varphi(1_A)^*) \subseteq V(1) = \{1\}$, $\varphi(1_A)\varphi(1_A)\varphi(1_A)^* = 1$, thus $\varphi(1_A) = 1$. Let $a \in A$ be positive. Then

$$\begin{aligned} V(\varphi(a)) &= V\left(\varphi(1_A)\varphi(a)\varphi(1_A)^*\right) \\ &\subseteq V(a) \\ &\subseteq \mathbb{R}^+, \end{aligned}$$

so $\varphi(a)$ is positive. Let $u \in A$ be unitary. Then

$$\begin{aligned} V(\varphi(u)\varphi(u)^*) &= V\left(\varphi(u)\varphi(1)\varphi(u)^*\right) \\ &\subseteq V(uu^*) \\ &= \{1\}, \end{aligned}$$

so $\varphi(u)\varphi(u)^* = 1$. Therefore, φ is a unital $*$-homomorphism by Remark 2.2. □

Lemma 3.12. *Let A and B be unital C^*-algebras and $\varphi : A \to B$ be a linear map. If B is commutative and $V(\varphi(a)\varphi(b)\varphi(a)^*) \subseteq V(aba^*)$ for all $a, b \in A$, then φ is a unital $*$-homomorphism.*

Proof. Let τ be a multiplicative functional on B. Then $\tau(1) = \|\tau\| = 1$ by [4, Proposition I.16.3].

Let $\psi = \tau\varphi$. If $a, b \in A$ and $\lambda \in V(\psi(a)\psi(b)\psi(a)^*)$, then there exists $f \in D(\mathbb{C}, 1)$ such that $\lambda = f(\psi(a)\psi(b)\psi(a)^*)$. Since B is a C^*-algebra, then by [9, Theorem 2.1.9.], $\tau(x^*) = \tau(x)^*$ for all $x \in B$, so $\lambda = f\tau(\varphi(a)\varphi(b)\varphi(a)^*)$. But $f\tau(1) = \|f\tau\| = 1$, so $\lambda \in V(\varphi(a)\varphi(b)\varphi(a)^*)$, thus $\lambda \in V(aba^*)$. Therefore $V(\psi(a))\psi(b)\psi(a)^*) \subseteq V(aba^*)$, so ψ is a unital $*$-homomorphism by Theorem 3.11. Therefore, for all $x, y \in A$, we have

$$\begin{aligned} \tau(\varphi(xy)) &= \psi(xy) \\ &= \psi(x)\psi(y) \\ &= \tau(\varphi(x))\tau(\varphi(y)) \\ &= \tau(\varphi(x)\varphi(y)). \end{aligned}$$

Similarly

$$\begin{aligned} \tau(\varphi(x^*)) &= \psi(x^*) \\ &= \psi(x)^* \\ &= \tau(\varphi(x)^*), \end{aligned}$$

and

$$\begin{aligned}\tau(\varphi(1)) &= \psi(1)\\ &= 1\\ &= \tau(1).\end{aligned}$$

Since B is semi-simple, $\varphi(xy) = \varphi(x)\varphi(y)$ and $\varphi(x^*) = \varphi(x)^*$ and $\varphi(1) = 1$. □

4. Absolute Value Preserving Maps

Let H and K be Hilbert spaces and $\varphi : B(H) \to B(K)$ be an additive map. Theorem 2 in [12] states that if $\varphi(|A|) = |\varphi(A)|$ for every $A \in B(H)$, $\varphi(iI)K \subset \varphi(I)K$ and $\varphi(I)$ is a projection, then φ is the sum of two $*$-homomorphisms which one is $\mathbb{C}$-linear and the other is $\mathbb{C}$-antilinear.

Let A and B be unital C^*-algebras and $\varphi : A \to B$ be a map satisfying $\varphi(|a|) = |\varphi(a)|$ for every $a \in A$. Theorem 2 in [16] states that, if φ is linear, then φ is positive and $\varphi(a_1a_2) = \varphi(1_A)\varphi(a_1)\varphi(a_2)$ for all $a_1, a_2 \in A$. Also Theorem 2.2 in [14] says that, if φ is additive and surjective and $\varphi(1)$ is a projection, then φ is unital and the restriction of φ to both A_s and A_{sk} is a Jordan $*$-homomorphism onto the corresponding set in B where A_s is the set of all self-adjoint elements of A and A_{sk} is the set of all skew-self-adjoint elements of A. Furthermore, if B is a C^*-algebra of real-rank zero, then φ is a $\mathbb{C}$-linear or $\mathbb{C}$-antilinear $*$-homomorphism on A [14, Theorem 2.5].

Theorem 2.9 in [15] states that, if $\varphi : A \to B$ is a additive map which satisfies $\varphi(|ab|) = |\varphi(a)\varphi(b)|$ for every $a, b \in A$ and $\varphi(c) = 1$ for some $c \in A$, then φ is unital and the restriction of φ to A_s is a Jordan homomorphism. Moreover, if φ is surjective and B is a real rank zero, then φ is a $\mathbb{C}-$linear or $\mathbb{C}-$antilinear $*$-homomorphism.

Molnar in [8, Theorem 3] was proved if A and B are von Neumann algebras, $A \neq \mathbb{C}I$ is a factor and $\varphi : A \to B$ is a bijective map which satisfies $\varphi(|ab|) = |\varphi(a)\varphi(b)|$ for every $a, b \in A$. Then φ is of the form $\varphi(a) = \tau(a)\psi(a)$ for all $a \in A$, where $\psi : A \to B$ is either a linear or a conjugate-linear $*$-algebra isomorphism and $\tau : A \to \mathbb{C}$ is a scalar function of modulus 1.

In this section, we show that if φ is a linear map from a unital C^*-algebra into a unital commutative C^*-algebra B such that $|\varphi(a)\varphi(b)| = \varphi(|ab|)$ for all $a, b \in A$, then φ is a unital $*$-homomorphism.

Remark 4.1. Let A and B be unital C^*-algebras. If B is commutative and $\varphi : A \to B$ is a unital linear map such that $\varphi(|a|) = |\varphi(a)|$ for all $a \in A$, then by using Remark 2.2 we can show that φ is a unital $*$-homomorphism which compares with Theorem 2 in [16].

Theorem 4.2. *Let A be a unital C^*-algebra and φ be a linear functional on A. If $\varphi(|ab|) = |\varphi(a)\varphi(b)|$ for all $a, b \in A$, then φ is a unital $*$-homomorphism.*

Proof. If $a \in A^+$, then $|a|^2 = a^2$ and since a and $|a|$ are positive, $|a| = a$, thus $\varphi(a) = \varphi(|a|) = |\varphi(a)| \geq 0$, so φ is a positive. Since $\varphi(1_A)$ is positive, $\varphi(1_A) = \varphi(|1_A|) = |\varphi(1_A)^2| = \varphi(1_A)^2$. So $\varphi(1_A) = 0$ or $\varphi(1_A) = 1$. If $\varphi(1_A) = 0$, then $\varphi(a) = \varphi(|a|) = |\varphi(a)\varphi(1_A)| = 0$ for all $a \in A^+$ and by [10, Remark 2.2.2] $\varphi(a) = 0$ for all $a \in A$, so $\varphi = 0$ and it is a contraction, therefore $\varphi(1) = 1$. If $u \in A$ is unitary, then $|u| = 1$, so $|\varphi(u)| = |\varphi(u)\varphi(1)| = \varphi(|u|) = \varphi(1) = 1$. Thus $\varphi(u)^*\varphi(u) = |\varphi(u)|^2 = 1$, so φ is unitary preserving. Therefore φ is a unital $*$-homomorphism by Remark 2.2. $\square$

Corollary 4.3. *Let A and B be unital C^*-algebras and $\varphi : A \to B$ be a linear map. If B is commutative and $\varphi(|ab|) = |\varphi(a)\varphi(b)|$ for all $a, b \in A$, then φ is a unital $*$-homomorphism.*

Proof. Let τ be a multiplicative functional on B and $\psi = \tau\varphi$. Since τ is $*$-homomorphism, then for all $a \in A$, we have:

$$\begin{aligned}
|\tau(a)|^2 &= \tau(a)^*\tau(a) \\
&= \tau(a^*a) \\
&= \tau(|a|^2) \\
&= (\tau(|a|))^2.
\end{aligned}$$

Since $|\tau(a)|$ and $\tau(|a|)$ are positive, $|\tau(a)| = \tau(|a|)$. Now let $a, b \in A$. Then

$$\begin{aligned}
\psi(|ab|) &= \tau\varphi(|ab|) \\
&= \tau(\varphi(|ab|)) \\
&= \tau(|\varphi(a)\varphi(b)|) \\
&= |\tau(\varphi(a)\varphi(b))| \\
&= |\tau(\varphi(a))\tau(\varphi(b))| \\
&= |\psi(a)\psi(b)|.
\end{aligned}$$

So ψ is a unital $*$-homomorphism by Theorem 4.2. Therefore, for all $x, y \in A$, we have

$$\begin{aligned}
\tau(\varphi(xy)) &= \psi(xy) \\
&= \psi(x)\psi(y) \\
&= \tau(\varphi(x))\tau(\varphi(y)) \\
&= \tau(\varphi(x)\varphi(y)).
\end{aligned}$$

Similarly

$$\begin{aligned}\tau(\varphi(x^*)) &= \psi(x^*)\\ &= \psi(x)^*\\ &= \tau(\varphi(x)^*),\end{aligned}$$

and

$$\begin{aligned}\tau(\varphi(1)) &= \psi(1)\\ &= 1\\ &= \tau(1).\end{aligned}$$

Since B is semi-simple, $\varphi(xy) = \varphi(x)\varphi(y)$ and $\varphi(x^*) = \varphi(x)^*$ and $\varphi(1) = 1$. □

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