On Preserving Properties of Linear Maps on $C^*$-algebras

Fatemeh Golfarshchi and Ali Asghar Khalilzadeh
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Abstract. Let $A$ and $B$ be two unital $C^*$-algebras and $\varphi : A \rightarrow B$ be a linear map. In this paper, we investigate the structure of linear maps between two $C^*$-algebras that preserve a certain property or relation. In particular, we show that if $\varphi$ is unital, $B$ is commutative and $V(\varphi(a)^*\varphi(b)) \subseteq V(ab^*)$ for all $a, b \in A$, then $\varphi$ is a $*$-homomorphism. It is also shown that if $\varphi([ab]) = [\varphi(a)\varphi(b)]$ for all $a, b \in A$, then $\varphi$ is a unital $*$-homomorphism.

1. Introduction

In 1970, Kaplansky asked the following question:
Let $\varphi : A \rightarrow B$ be a unital and invertibility preserving linear map between unital Banach algebras $A$ and $B$. Is $\varphi$ a Jordan homomorphism?[6].

The Kaplansky’s question was originated by Gleason-Kahane-Zelazko Theorem which states that every invertibility preserving unital linear functional on a unital complex Banach algebra is multiplicative [18]. In this paper we explain and prove a Gleason-Kahane-Zelazko type Theorem and show that if $A$ is a unital $C^*$-algebra and $\varphi$ is a unital linear functional on $A$ such that $V(\varphi(a)^*\varphi(b)) \subseteq V(ab^*)$ for all $a, b \in A$, then $\varphi$ is a $*$-homomorphism.

Another kind of linear preserver problems is absolute value preserving linear maps and in this paper, we will characterize this kind of maps.

Let $(X,d)$ and $(Y,d')$ be metric spaces. A map $f : X \rightarrow Y$ is said to be a contraction if there exist $0 \leq k < 1$ such that
$$d'(f(x_1), f(x_2)) \leq kd(x_1, x_2); \quad x_1, x_2 \in X.$$
Let $A$ be a complex unital normed algebra, and
\[ D(A, 1) = \{ f \in A', f(1) = \| f \| = 1 \}, \]
where $A'$ is the dual space of $A$. The elements of $D(A, 1)$ are called the normalized states on $A$. For $a \in A$ let,
\[ V(a) = \{ f(a) : f \in D(A, 1) \}, \quad v(a) = \sup \{ |\lambda| : \lambda \in V(A) \}. \]
The sets $V(a)$ and $v(a)$ are called the numerical range and numerical radius of $a$ respectively and the spectrum of $a$ is denoted by $\sigma(a)$, namely
\[ \sigma(a) = \{ \lambda \in \mathbb{C} : \lambda - a \in \text{sing}(A) \}. \]
Also the convex hull of $\sigma(x)$ is denoted by $\text{cos}(x)$. If $H$ is a Hilbert space, for every $T \in B(H)$, the numerical range of $T$ is the set
\[ W(T) = \{ \langle T(x), x \rangle : x \in H, \| x \| = 1 \}, \]
and the numerical radius of $T$ is defined by $w(T) = \sup \{ |\lambda| : \lambda \in W(T) \}$. When $A = B(H)$ and $T \in A$, the set $V(T)$ becomes the closure of $W(T)$.

Let $A$ and $B$ be complex unital normed algebras. A linear map $\varphi : A \to B$ is said to be numerical range preserving if $V(\varphi(x)) \subseteq V(x)$, numerical range preserving if $V(\varphi(x)) = V(x)$ and Jordan homomorphism if $\varphi(x^2) = \varphi(x)^2$ for all $x \in A$. Also $\varphi$ is said to be unital if $\varphi(1_A) = 1_B$.

Let $A$ be a unital $C^*$-algebra. An element $a$ of $A$ is said to be positive if $V(a) \subseteq \mathbb{R}^+$ or $a^* = a$ and $\sigma(a) \subseteq \mathbb{R}^+$. We denote by $A^+$ the set of all positive elements of $A$. Also $a$ is called normal and unitary, if $a^*a = aa^*$ and $a^*a = aa^* = 1$, respectively. We recall that if $a$ is a unitary element of $A$, then $\sigma(a) = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$. \[ \text{[17]} \]

A linear map $\varphi$ from a unital $C^*$-algebra $A$ into a unital $C^*$-algebra $B$ is said unitary preserving if $\varphi(u)$ is unitary whenever $u$ is unitary in $A$. We say that $\varphi$ preserves absolute values if $\varphi(|x|) = |\varphi(x)|$, where $|x|^2 = xx^*$ and $*$-homomorphism, if $\varphi(xy) = \varphi(x)\varphi(y)$ and $\varphi(x^*) = \varphi(x)^*$ for every $x, y \in A$. For any positive integer $n$ we define $\varphi_n : M_n(A) \to M_n(B)$ by $\varphi_n((a_{i,j}))_{i,j} = \varphi((a_{i,j}))_{i,j}$, where $M_n(A)$ denotes the set of all $n \times n$ matrices with entries in $A$. The map $\varphi$ is called positive if $\varphi(a) \geq 0$ for all $a \in A^+$ and $n-$positive if $\varphi_n$ is positive. Also $\varphi$ is called completely positive if $\varphi$ is $n-$positive for all $n$. Every positive linear map is not necessarily completely positive, for example if $\varphi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is defined by $\varphi(A) = A^t$, then $\varphi$ is positive but is not necessarily completely positive. \[ \text{[10], Example 4.2]. } \]

Every $*$-homomorphism on a $*$-algebras is completely positive \[ \text{[2], Example II.6.9.3]} \] but the converse is false. For example if $\varphi : M_2(\mathbb{C}) \to \mathbb{C}$ is defined by $\varphi((a_{i,j}))_{i,j} = \sum a_{i,i}$, then $\varphi$ is completely positive by \[ \text{[10], } \]
Exercise 3.5. But \( \varphi \) is not a homomorphism because

\[
\varphi \left( \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right)^2 = 9, \quad \varphi \left( \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right) = 5.
\]

A subspace \( S \) of a unital \( C^* \)-algebra \( A \) is called operator system if it is self-adjoint (\( S = S^* \)) and contains the unit of \( A \).

Numerical range of operators is very important and was studied by many authors, see, e.g., [1, 3, 4, 11, 13]. In this paper, we characterize a linear map \( \varphi \) from a unital \( C^* \)-algebra \( A \) into a unital commutative \( C^* \)-algebra \( B \) and show that if \( \varphi \) is unital and \( V(\varphi(a)^* \varphi(b)) \subseteq V(a^* b) \) or \( V(\varphi(a) \varphi(b) \varphi(a)) \subseteq V(aba) \) for all \( a, b \in A \), then \( \varphi \) is a *-homomorphism. Also if \( V(\varphi(a) \varphi(b) \varphi(a)^*) \subseteq V(aba^*) \) for all \( a, b \in A \), then \( \varphi \) is a unital *-homomorphism. We show that every Jordan homomorphism from a complex Banach algebra into \( \mathbb{C} \) is a numerical range compressing, but the converse is false.

Also in this paper, we discuss about absolute value preserving linear maps and show that if \( \varphi \) is a linear map from a unital \( C^* \)-algebra \( A \) into a unital commutative \( C^* \)-algebra \( B \) such that \( |\varphi(a) \varphi(b)| = \varphi(|ab|) \) for all \( a, b \in A \), then \( \varphi \) is a unital *-homomorphism.

2. Preliminaries

Let \( A \) be a unital \( C^* \)-algebra. If \( a \) and \( b \) are positive elements of \( A \) such that \( a^2 = b^2 \), then \( \sigma(a^2 - b^2) = -\sigma(b^2 - a^2) = \sigma(0) = \{0\} \). Since \( a^2 - b^2 \) and \( b^2 - a^2 \) are self-adjoint, \( a^2 \geq b^2 \) and \( a^2 \leq b^2 \), so by [3, Theorem 2.2.6], \( a \geq b \) and \( a \leq b \). This implies that \( a = b \). [2, Proposition II.3.1.2]. Let \( X \) be a compact Hausdorff space. We denote by \( C(X) \) the algebra of all continuous complex functionals on \( X \).

**Theorem 2.1.** Let \( A \) be a unital \( C^* \)-algebra and \( \varphi : A \to C(X) \) be a positive linear map. If \( \varphi \) is a unitary preserving map, then \( \varphi \) is a unital *-homomorphism.

**Proof.** Since \( \varphi \) is positive and preserves unitary elements, \( \varphi(1_A) \) is positive and \( \varphi(1_A)^2 = \varphi(1_A)^* \varphi(1_A) = 1_{C(X)} \), so \( \varphi(1_A) = 1_{C(X)} \).

Since \( \varphi \) is unital and positive by [2, Proposition II.6.9.4], \( \varphi \) is a contraction. Also, since \( \varphi \) is a bounded linear map and \( A \) is an operator system, \( \varphi \) is completely positive [10, Proposition 3.9].

Now, let \( a \) and \( b \) be elements of \( A \) such that \( \|a\| < 1 \) and \( \|b\| < 1 \). Since \( -1_A \) is a unitary element of \( A \) by [2, Proposition II.3.2.13], there exist unitary elements \( u_1, u_2, v_1, v_2 \) of \( A \) such that \( a - 1_A = u_1 + u_2 \) and \( b - 1_A = v_1 + v_2 \). So

\[
\varphi(ab) = \varphi(1_A) + \varphi(v_1) + \varphi(v_2) + \varphi(u_1) + \varphi(u_2)
\]
Remark 2.2. Since by [4, Theorem 2.1.10], every non-zero commutative $C^*$-algebra $B$ is isomorphic to $C(\Omega(B))$, where $\Omega(B)$ is the set of all linear homomorphisms from $B$ into $\mathbb{C}$, every unitary preserving positive linear map from a unital $C^*$-algebra $A$ into a unital commutative $C^*$-algebra $B$ is a unital $*$-homomorphism.

3. Numerical Range Preserving Maps

Let $H$ and $K$ be complex Hilbert spaces, $A, B \in B(H)$ and $\varphi : B(H) \to B(K)$ be a surjective map. Theorem 2.1 in [5] shows that $W(\varphi(A)\varphi(B)) = W(AB)$ if and only if there exists unitary operator $U$ in $B(H, K)$ such that $\varphi$ is of the form $\varphi(A) = \epsilon UAU^*$ for all $A \in B(H)$, where $\epsilon = \pm 1$. Similarly Theorem 2.2 in [5] states that
Let the converse of Lemma 3.2 is false. To see that, let
\[
W(\varphi(A)\varphi(B)\varphi(A)) = W(ABA) \quad \text{if and only if there exist a scalar } \lambda \text{ with } \lambda^3 = 1 \text{ and a unitary operator } U : H \to K \text{ such that either } \varphi(A) = \lambda UAU \text{ or } \varphi(A) = \lambda U^A U, \text{ where } A^t \text{ is the transpose of } A \text{ with respect to an arbitrary fixed orthonormal basis of } H. \text{ Also } W(\varphi(A)^{\ast}\varphi(B)) = W(A^\ast B) \text{ if and only if there exist unitary operators } U \text{ and } V \text{ in } B(H, K) \text{ such that } \varphi \text{ is of the form } \varphi(A) = UA^\ast [3, Corollary 4.3].

Let \(A\) and \(B\) be unital \(C^\ast\)-algebras and \(\varphi\) be a unital linear mapping from \(A\) onto \(B\). Theorem 2.3 in [3] states that if \(W(\varphi(a)) = W(a)\) for all \(a \in A\), then \(\varphi\) is a Jordan \(*\)-isomorphism. Furthermore, if \(B\) is prime, then \(\varphi\) is a \(C^\ast\)-isomorphism or \(C^\ast\)-anti-isomorphism.

In this section, we characterize the numerical range of a map from a unital \(C^\ast\)-algebra into a unital commutative \(C^\ast\)-algebra.

**Lemma 3.1.** Let \(A\) be a unital complex Banach algebra and \(\varphi : A \to \mathbb{C}\) be a linear functional. If \(V(\varphi(a)) = V(a)\) for all \(a \in A\), then \(\varphi\) is a unital monomorphism.

**Proof.** Since \(V(\varphi(1_A)) = V(1_A) = \{1\}\), \(v(\varphi(1) - 1) = 0\) and by [3, Theorem 1.4.1] \(\varphi(1) - 1 = 0\), therefore \(\varphi\) is unital. Let \(a \in A\) be invertible but \(\varphi(a)\) is singular, then \(\varphi(a) = 0\). Thus \(V(a) = V(\varphi(a)) = 0\), so \(a = 0\) by [3, Theorem, 1.4.1]. But this is a contraction, so \(\varphi\) is an invertibility preserving functional, therefore by Gleason-Kahane-Zelazko Theorem, \(\varphi\) is multiplicative.

If \(a \in A\) and \(\varphi(a) = 0\), then \(V(a) = V(\varphi(a)) = 0\) and by [3, Theorem 1.4.1] \(a = 0\), so \(\varphi\) is injective. \(\square\)

**Lemma 3.2.** Let \(A\) be a unital complex Banach algebra and \(\varphi : A \to \mathbb{C}\) be a linear functional. If \(\varphi\) is a Jordan homomorphism, then \(\varphi\) is a unital numerical range compressing.

**Proof.** Since \(\varphi\) is a Jordan homomorphism, then by [1, Proposition, II.16.6] \(\varphi\) is a homomorphism, so \(\varphi\) is continuous and \(\|\varphi\| = \varphi(1_A) = 1\) [1, Proposition II.16.3]. Let \(a \in A\) and \(\lambda \in V(\varphi(a))\), then there exists \(f \in D(\mathbb{C}, 1)\) such that \(\lambda = f(\varphi(a))\). Let \(g = f\varphi\). Then \(g\) is linear, continuous and for all \(x \in A\)

\[
|g(x)| = |f(\varphi(x))| \leq \|f\|\|\varphi\|\|x\| = \|x\|,
\]

so \(\|g\| \leq 1\), also \(g(1_A) = (f\varphi)(1_A) = 1\), thus \(\|g\| = 1\) and it follows that \(g \in D(A, 1_A)\), but \(\lambda = g(1^A)\). Therefore \(\lambda \in V(a)\). \(\square\)

**Remark 3.3.** The converse of Lemma 3.2 is false. To see that, let \(A = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : a, b, c \in \mathbb{C} \right\} \) and define \(\varphi : A \to \mathbb{C}\) by \(\varphi \left( \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \right) = \frac{a + c}{2}\), then

\[
V\left( \varphi \left( \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \right) \right) = V \left( \frac{a + c}{2} \right)
\]
\[
\left\{ \frac{a+c}{2} \right\}.
\]

Also \( a, c \in \sigma \left( \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \right) \), so \( a, c \in V \left( \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \right) \) [11, Theorem I.2.6]. Also by [11, Proposition 1.10.4], \( V \left( \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \right) \) is convex, so \( \frac{a+c}{2} \in V \left( \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \right) \).

Therefore \( V(\varphi(x)) \subseteq V(x) \) for all \( x \in A \). But \( \varphi \) is not a Jordan homomorphism, because \( \varphi \left( \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right) = 9 \) and \( \varphi \left( \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \right) = 5 \).

\section*{Theorem 3.4.}
Let \( A \) and \( B \) be unital \( C^* \)-algebras and \( \varphi : A \to B \) be a unital linear map. If \( B \) is commutative and \( V(\varphi(a)^*\varphi(b)) \subseteq V(ab) \) for all \( a, b \in A \), then \( \varphi \) is a \(*\)-homomorphism.

\begin{proof}
Let \( a \in A \) be positive. Then \( V(\varphi(a)) = V(\varphi(1A)^*\varphi(a)) \subseteq V(1Aa) \subseteq \mathbb{R}^+ \), so \( \varphi \) is a positive map. Let \( u \in A \) be unitary, then
\[
V(\varphi(u)^*\varphi(u)) \subseteq V(u^*u)
\]
\[
= V(1)
\]
\[
= \{1\},
\]
thus \( \varphi(u)^*\varphi(u) = 1_B \), so \( \varphi \) is a unitary preserving positive linear map. Therefore \( \varphi \) is a \(*\)-homomorphism by Remark 2.2. \qed
\end{proof}

\section*{Corollary 3.5.}
Let \( A \) and \( B \) be unital \( C^* \)-algebras and \( \varphi : A \to B \) be a surjective unital linear map. If \( B \) is commutative and \( V(\varphi(a)^*\varphi(b)) = V(ab) \) for all \( a, b \in A \), then \( \varphi \) is a \(*\)-isomorphism.

\begin{proof}
If \( a \in A \) and \( \varphi(a) = 0 \), then
\[
v(a) = v(\varphi(1A)^*\varphi(a))
\]
\[
= v(\varphi(a))
\]
\[
= v(0)
\]
\[
= 0,
\]
and by [11, Theorem 1.10.14] \( a = 0 \), so \( \varphi \) is injective. Also \( \varphi \) is a \(*\)-homomorphism, by Theorem 3.4, so \( \varphi \) is a \(*\)-isomorphism. \qed
\end{proof}

\section*{Theorem 3.6.}
Let \( A \) be a unital \( C^* \)-algebra. If \( \varphi \) is a linear functional on \( A \) such that \( V(\varphi(a)\varphi(b)) \subseteq V(ab) \) for all \( a, b \in A \), then \( \varphi \) is a scaler of a \(*\)-homomorphism.

\begin{proof}
\( V(\varphi(1A)(1A)) \subseteq V(1) = \{1\} \), so \( \varphi(1A)^2 = 1 \), thus \( \varphi(1A) = 1 \) or \( \varphi(1A) = -1 \). Also if \( a \in A \) is self-adjoint, then
\[
V(\varphi(a)) = V(\varphi(a)\varphi(1))
\]

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\begin{align*}
\subseteq V(a) \\
\subseteq \mathbb{R},
\end{align*}

or

\begin{align*}
V(\varphi(a)) &= V(\varphi(a)\varphi(-1)) \\
\subseteq V(-a) \\
\subseteq \mathbb{R},
\end{align*}

so $\varphi(a)$ is self-adjoint. Now let $x \in A$, then there exist self-adjoint elements $a, b \in A$ such that $x = a + ib$, then

\begin{align*}
\varphi(x^*) &= \varphi(a - ib) \\
&= \varphi(a) - i\varphi(b) \\
&= (\varphi(a) + i\varphi(b))^* \\
&= \varphi(x)^*,
\end{align*}

so $V(\varphi(a)^*\varphi(b)) = V(\varphi(a)^*) \subseteq V(a^*)b$ for all $a, b \in A$. Then, by Theorem 3.9, $\varphi$ is $*$-preserving and $\varphi(ab) = \pm \varphi(a)\varphi(b)$ for all $a, b \in A$.

**Corollary 3.7.** Let $A$ and $B$ be unital $C^*$-algebras and $\varphi : A \to B$ be a linear map. If $B$ is commutative and $V(\varphi(a)\varphi(b)) \subseteq V(ab)$ for all $a, b \in A$, then $\varphi$ is a scaler of a $*$-homomorphism.

**Proof.** Let $\tau$ be a multiplicative functional on $B$ and $\psi = \tau \varphi$. If $a, b \in A$ and $\lambda \in V(\psi(a)\psi(b))$, then there exists $f \in D(\mathbb{C}, 1)$ such that $\lambda = f(\psi(a)\psi(b)) = f\tau(\varphi(a)\varphi(b))$. But $f\tau(1) = \|f\tau\| = 1$, so $\lambda \in V(\varphi(a)\varphi(b))$, thus $\lambda \in V(ab)$. Therefore $V(\psi(a)\psi(b)) \subseteq V(ab)$ and by Theorem 3.9, for all $x, y \in A$, we have

\begin{align*}
\tau(\varphi(xy)) &= \psi(xy) \\
&= \pm \psi(x)\psi(y) \\
&= \pm \tau(\varphi(x))\tau(\varphi(y)) \\
&= \tau(\pm \varphi(x)\varphi(y)),
\end{align*}

and

\begin{align*}
\tau(\varphi(x^*)) &= \psi(x^*) \\
&= \psi(x)^* \\
&= \tau(\varphi(x)^*).
\end{align*}

Since $B$ is semi-simple, $\Omega(B)$ separates the points of $B$ [II.17.7], so $\varphi(xy) = \pm \varphi(x)\varphi(y)$ and $\varphi(x^*) = \varphi(x)^*$.

□
Corollary 3.8. Let $A$ and $B$ be unital $C^*$-algebras and $\varphi : A \to B$ be a surjective linear map. If $B$ is commutative and $V(\varphi(a)\varphi(b)) = \varphi(ab)$ for all $a, b \in A$, then $\varphi$ is a scaler of a $*$-isomorphism.

Proof. The map $\varphi$ is injective. Also $\pm \varphi$ is a $*$-homomorphism, by Corollary 3.7, so $\varphi$ is a scaler of a $*$-isomorphism. □

Theorem 3.9. Let $A$ be a unital $C^*$-algebra and $\varphi$ be a unital linear functional on $A$. If $V(\varphi(a)\varphi(b)\varphi(a)) \subseteq \varphi(aba)$ for all $a, b \in A$, then $\varphi$ is a $*$-homomorphism.

Proof. Let $a \in A$ be positive, then $V(\varphi(a)) \subseteq \varphi(a) \subseteq \mathbb{R}^+$, so $\varphi$ is positive. Therefore $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. Since $\varphi$ is unital and positive, then by [2 Proposition II.6.9.4] $\varphi$ is a contraction. Also since $\varphi$ is a bounded linear functional and $A$ is an operator system, then $\varphi$ is completely positive [11, Proposition 3.8], thus $\varphi$ is 2-positive.

Let $a, b \in A$ and $ab = 0$, then $V(\varphi(a)\varphi(b)\varphi(a)) \subseteq \varphi(aba) = \{0\}$, so $\varphi(a) = 0$ or $\varphi(b) = 0$. Thus $\varphi$ preserves zero product elements, therefore $\varphi$ is a homomorphism [11, Theorem 2]. □

Theorem 3.10. Let $A$ and $B$ be unital $C^*$-algebras and $\varphi : A \to B$ be a unital linear map. If $B$ is commutative and $V(\varphi(a)\varphi(b)\varphi(a)) \subseteq \varphi(aba)$ for all $a, b \in A$, then $\varphi$ is a $*$-homomorphism.

Proof. Let $\tau$ be a multiplicative functional on $B$. Then $\tau(1) = \|\tau\| = 1$ by [11 Proposition I.16.3]. Let $\psi = \tau \varphi$, then $\psi(1) = 1$. Also if $a, b \in A$ and $\lambda \in V(\psi(a)\psi(b)\psi(a))$, then there exists $f \in D(\mathbb{C}, 1)$ such that

$$\lambda = f(\psi(a)\psi(b)\psi(a)) = f\tau(\varphi(a)\varphi(b)\varphi(a)).$$

But $f\tau(1) = \|f\tau\| = 1$, so $\lambda \in V(\varphi(a)\varphi(b)\varphi(a))$, thus $\lambda \in V(aba)$. Therefore $V(\psi(a)\psi(b)\psi(a)) \subseteq \varphi(aba)$, so by Theorem 3.3, for all $x, y \in A$, we have

$$\tau(\varphi(xy)) = \psi(xy) = \psi(x)\psi(y) = \tau(\varphi(x))\tau(\varphi(y)) = \tau(\varphi(x)\varphi(y)),$$

and

$$\tau(\varphi(x^*)) = \psi(x^*) = \psi(x)^* = \tau(\varphi(x)^*).$$

Since $B$ is semi-simple, $\varphi(xy) = \varphi(x)\varphi(y)$ and $\varphi(x^*) = \varphi(x)^*$. □
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Theorem 3.11. Let $A$ be a unital $C^*$-algebra and $\varphi$ be a linear functional on $A$. If $V(\varphi(a)\varphi(b)\varphi(a)^*) \subseteq V(aba^*)$ for all $a, b \in A$, then $\varphi$ is a unital $*$-homomorphism.

Proof. Since $V(\varphi(1_A)\varphi(1_A)\varphi(1_A)^*) \subseteq V(1) = \{1\}$, $\varphi(1_A)\varphi(1_A)\varphi(1_A)^* = 1$, thus $\varphi(1_A) = 1$. Let $a \in A$ be positive. Then

$$V(\varphi(a)) = V(\varphi(1_A)\varphi(a)\varphi(1_A)^*) \subseteq V(a) \subseteq \mathbb{R}^+,$$

so $\varphi(a)$ is positive. Let $u \in A$ be unitary. Then

$$V(\varphi(u)\varphi(u)^*) = V(\varphi(u)\varphi(1)\varphi(u)^*) \subseteq V uu^* = \{1\},$$

so $\varphi(u)\varphi(u)^* = 1$. Therefore, $\varphi$ is a unital $*$-homomorphism by Remark 2.2. □

Lemma 3.12. Let $A$ and $B$ be unital $C^*$-algebras and $\varphi : A \to B$ be a linear map. If $B$ is commutative and $V(\varphi(a)\varphi(b)\varphi(a)^*) \subseteq V(aba^*)$ for all $a, b \in A$, then $\varphi$ is a unital $*$-homomorphism.

Proof. Let $\tau$ be a multiplicative functional on $B$. Then $\tau(1) = \|\tau\| = 1$ by [3, Proposition I.16.3].

Let $\psi = \tau \varphi$. If $a, b \in A$ and $\lambda \in V(\psi(a)\psi(b)\psi(a)^*)$, then there exists $f \in D(\mathbb{C}, 1)$ such that $\lambda = f(\psi(a)\psi(b)\psi(a)^*)$. Since $B$ is a $C^*$-algebra, then by [3, Theorem 2.1.9], $\tau(x^*) = \tau(x)^*$ for all $x \in B$, so $\lambda = f \tau(\varphi(a)\varphi(b)\varphi(a)^*)$. But $f \tau(1) = \|f\tau\| = 1$, so $\lambda \in V(\varphi(a)\varphi(b)\varphi(a)^*)$, thus $\lambda \in V(aba^*)$. Therefore $V(\psi(a))\psi(b)\psi(a)^* \subseteq V(aba^*)$, so $\psi$ is a unital $*$-homomorphism by Theorem 3.11. Therefore, for all $x, y \in A$, we have

$$\tau(\varphi(xy)) = \psi(xy) = \psi(x)\psi(y) = \tau(\varphi(x))\tau(\varphi(y)) = \tau(\varphi(x)\varphi(y)).$$

Similarly

$$\tau(\varphi(x^*)) = \psi(x^*) = \psi(x)^* = \tau(\varphi(x)^*),$$
and
\[
\tau(\varphi(1)) = \psi(1) \\
= 1 \\
= \tau(1).
\]
Since \(B\) is semi-simple, \(\varphi(xy) = \varphi(x)\varphi(y)\) and \(\varphi(x^*) = \varphi(x)^*\) and \(\varphi(1) = 1.\)

4. Absolute Value Preserving Maps

Let \(H\) and \(K\) be Hilbert spaces and \(\varphi : B(H) \to B(K)\) be an additive map. Theorem 2 in [12] states that if \(\varphi(|A|) = |\varphi(A)|\) for every \(A \in B(H)\), \(\varphi(iI)K \subset \varphi(I)K\) and \(\varphi(I)\) is a projection, then \(\varphi\) is the sum of two \(*\)-homomorphisms which one is \(\mathbb{C}\)-linear and the other is \(\mathbb{C}\)-antilinear.

Let \(A\) and \(B\) be unital \(C^*\)-algebras and \(\varphi : A \to B\) be a map satisfying \(\varphi(|a|) = |\varphi(a)|\) for every \(a \in A\). Theorem 2 in [10] states that, if \(\varphi\) is linear, then \(\varphi\) is positive and \(\varphi(a_1a_2) = \varphi(1)\varphi(a_1)\varphi(a_2)\) for all \(a_1, a_2 \in A\). Also Theorem 2.2 in [13] says that, if \(\varphi\) is additive and surjective and \(\varphi(1)\) is a projection, then \(\varphi\) is unital and the restriction of \(\varphi\) to both \(A_s\) and \(A_{sk}\) is a Jordan \(*\)-homomorphism onto the corresponding set in \(B\) where \(A_s\) is the set of all self-adjoint elements of \(A\) and \(A_{sk}\) is the set of all skew-self-adjoint elements of \(A\). Furthermore, if \(B\) is a \(C^*\)-algebra of real-rank zero, then \(\varphi\) is a \(\mathbb{C}\)-linear or \(\mathbb{C}\)-antilinear \(*\)-homomorphism on \(A\) [13, Theorem 2.5].

Theorem 2.9 in [13] states that, if \(\varphi : A \to B\) is a additive map which satisfies \(\varphi(|ab|) = |\varphi(a)\varphi(b)|\) for every \(a, b \in A\) and \(\varphi(c) = 1\) for some \(c \in A\), then \(\varphi\) is unital and the restriction of \(\varphi\) to \(A_s\) is a Jordan homomorphism. Moreover, if \(\varphi\) is surjective and \(B\) is a real rank zero, then \(\varphi\) is a \(\mathbb{C}\)-linear or \(\mathbb{C}\)-antilinear \(*\)-homomorphism.

Molnar in [8, Theorem 3] was proved if \(A\) and \(B\) are von Neumann algebras, \(A \neq \mathbb{C}I\) is a factor and \(\varphi : A \to B\) is a bijective map which satisfies \(\varphi(|ab|) = |\varphi(a)\varphi(b)|\) for every \(a, b \in A\). Then \(\varphi\) is of the form \(\varphi(a) = \tau(a)\psi(a)\) for all \(a \in A\), where \(\psi : A \to B\) is either a linear or a conjugate-linear \(*\)-algebra isomorphism and \(\tau : A \to \mathbb{C}\) is a scalar function of modulus 1.

In this section, we show that if \(\varphi\) is a linear map from a unital \(C^*\)-algebra into a unital commutative \(C^*\)-algebra \(B\) such that \(|\varphi(a)\varphi(b)| = \varphi(|ab|)\) for all \(a, b \in A\), then \(\varphi\) is a unital \(*\)-homomorphism.

\textbf{Remark 4.1.} Let \(A\) and \(B\) be unital \(C^*\)-algebras. If \(B\) is commutative and \(\varphi : A \to B\) is a unital linear map such that \(\varphi(|a|) = |\varphi(a)|\) for all \(a \in A\), then by using Remark 4.1 we can show that \(\varphi\) is a unital \(*\)-homomorphism which compares with Theorem 2 in [10].
Theorem 4.2. Let $A$ be a unital $C^*$-algebra and $\varphi$ be a linear functional on $A$. If $\varphi(|ab|) = |\varphi(a)\varphi(b)|$ for all $a, b \in A$, then $\varphi$ is a unital $\ast$-homomorphism.

Proof. If $a \in A^+$, then $|a|^2 = a^2$ and since $a$ and $|a|$ are positive, $|a| = a$, thus $\varphi(a) = \varphi(|a|) = |\varphi(a)| \geq 0$, so $\varphi$ is a positive. Since $\varphi(1_A)$ is positive, $\varphi(1_A) = \varphi(|1_A|) = |\varphi(1_A)| = \varphi(1_A)^2$. So $\varphi(1_A) = 0$ or $\varphi(1_A) = 1$. If $\varphi(1_A) = 0$, then $\varphi(a) = \varphi(|a|) = |\varphi(a)\varphi(1_A)| = 0$ for all $a \in A^+$ and by [10, Remark 2.2.2] $\varphi(a) = 0$ for all $a \in A$, so $\varphi = 0$ and it is a contraction, therefore $\varphi(1) = 1$. If $u \in A$ is unitary, then $|u| = 1$, so $|\varphi(u)| = |\varphi(u)\varphi(1)| = \varphi(|u|) = \varphi(1) = 1$. Thus $\varphi(u)^\ast\varphi(u) = |\varphi(u)|^2 = 1$, so $\varphi$ is unitary preserving. Therefore $\varphi$ is a unital $\ast$-homomorphism by Remark 2.2. □

Corollary 4.3. Let $A$ and $B$ be unital $C^*$-algebras and $\varphi : A \to B$ be a linear map. If $B$ is commutative and $\varphi(|ab|) = |\varphi(a)\varphi(b)|$ for all $a, b \in A$, then $\varphi$ is a unital $\ast$-homomorphism.

Proof. Let $\tau$ be a multiplicative functional on $B$ and $\psi = \tau\varphi$. Since $\tau$ is $\ast$-homomorphism, then for all $a \in A$, we have:

$$|\tau(a)|^2 = \tau(a^\ast a) = \tau(|a|^2) = (\tau(|a|))^2.$$ Since $|\tau(a)|$ and $\tau(|a|)$ are positive, $|\tau(a)| = \tau(|a|)$. Now let $a, b \in A$. Then

$$\psi(|ab|) = \tau\varphi(|ab|) = \tau(\varphi(|ab|)) = \tau(|\varphi(a)\varphi(b)|) = |\tau(\varphi(a)\varphi(b))| = |\tau(\varphi(a))\tau(\varphi(b))| = |\psi(a)\psi(b)|.$$ So $\psi$ is a unital $\ast$-homomorphism by Theorem 4.2. Therefore, for all $x, y \in A$, we have

$$\tau(\varphi(xy)) = \psi(xy) = \psi(x)\psi(y) = \tau(\varphi(x))\tau(\varphi(y)) = \tau(\varphi(x)\varphi(y)).$$
Similarly
\[
\tau(\varphi(x^*)) = \psi(x^*)
\]
\[
= \psi(x)^*
\]
\[
= \tau(\varphi(x)^*)
\]
and
\[
\tau(\varphi(1)) = \psi(1)
\]
\[
= 1
\]
\[
= \tau(1).
\]
Since \( B \) is semi-simple, \( \varphi(xy) = \varphi(x)\varphi(y) \) and \( \varphi(x^*) = \varphi(x)^* \) and \( \varphi(1) = 1 \). \( \square \)

References

7. L. Molnar, *Some characterizations of the automorphisms of \( B(H) \) and \( C(X) \)*, Amer. Math. Soc., 130 (2001), pp. 111-120.


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