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On Sum and Stability of Continuous G -Frames

Azam Yousefzadeheyini¹ and Mohammad Reza Abdollahpour^{2*}

ABSTRACT. In this paper, we give some conditions under which the finite sum of continuous g -frames is again a continuous g -frame. We give necessary and sufficient conditions for the continuous g -frames $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ and $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ and operators U and V on H such that $\Lambda U + \Gamma V = \{\Lambda_w U + \Gamma_w V \in B(H, K_w) : w \in \Omega\}$ is again a continuous g -frame. Moreover, we obtain some sufficient conditions under which the finite sum of continuous g -frames are stable under small perturbations.

1. INTRODUCTION

Frames were first introduced by Duffin and Schaeffer in the study of nonharmonic Fourier series [7] and reintroduced in 1986 by Daubechies, Grossmann and Meyer [6]. In [13], Sun introduced the concept of g -frames in a Hilbert space. The notion of continuous frames was introduced by Kaiser in [8] and independently by Ali, Antoine and Cazeau [2]. In 2008, continuous g -frames were introduced by Abdollahpour and Faroughi [1].

This paper is organized as follows. First, we summarize some facts about continuous g -frames from [1]. By generalizing some results of [5] and [9], in Section 2, we give some conditions that the finite sum of continuous g -frames to be a continuous g -frame and in Section 3, we study some new results in stability of finite sum of continuous g -frames.

Throughout this paper, H is a complex Hilbert space and (Ω, μ) is a measure space with positive measure μ and $\{K_w\}_{w \in \Omega}$ is a family of

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closed subspaces of Hilbert space K . We denote the space of all bounded linear operators from H into K by $B(H, K)$.

Definition 1.1. Let $F \in \prod_{w \in \Omega} K_w$. We say that F is strongly measurable if F as a mapping of Ω to K is measurable, where

$$\prod_{w \in \Omega} K_w = \left\{ f : \Omega \longrightarrow \bigcup_{w \in \Omega} K_w : f(w) \in K_w \right\}.$$

Definition 1.2. We say that $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ is a continuous g -frame for H with respect to $\{K_w\}_{w \in \Omega}$ (or simply continuous g -frame) if

- (i) for each $f \in H$, $\{\Lambda_w f\}_{w \in \Omega}$ is strongly measurable,
- (ii) there are two constants $0 < A_\Lambda \leq B_\Lambda < \infty$ such that

$$(1.1) \quad A_\Lambda \|f\|^2 \leq \int_{\Omega} \|\Lambda_w f\|^2 d\mu(w) \leq B_\Lambda \|f\|^2, \quad f \in H.$$

We call A_Λ and B_Λ the lower and upper continuous g -frame bounds, respectively.

Λ is called a tight continuous g -frame if $A_\Lambda = B_\Lambda$ and a Parseval continuous g -frame if $A_\Lambda = B_\Lambda = 1$. If the right hand inequality of (1.1) holds for all $f \in H$, then we say that Λ is a continuous g -Bessel family for H with respect to $\{K_w\}_{w \in \Omega}$ (or simply continuous g -Bessel family). In this case, B_Λ is called the continuous g -Bessel constant. We denote by A_Λ and B_Λ the lower and upper bounds of continuous g -frame Λ , respectively.

Proposition 1.3 ([1]). *Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g -frame for H with continuous g -frame bounds A_Λ, B_Λ . Then, there exists a unique positive and invertible operator $S_\Lambda : H \longrightarrow H$ such that for each $f, g \in H$,*

$$\langle S_\Lambda f, g \rangle = \int_{\Omega} \langle \Lambda_w^* \Lambda_w f, g \rangle d\mu(w),$$

and $A_\Lambda I_H \leq S_\Lambda \leq B_\Lambda I_H$.

The operator S_Λ in Proposition 1.3 is called the continuous g -frame operator of Λ . Also, we have

$$(1.2) \quad \begin{aligned} \langle f, g \rangle &= \int_{\Omega} \langle S_\Lambda^{-1} f \Lambda_w^* \Lambda_w g \rangle d\mu(w) \\ &= \int_{\Omega} \langle f, \Lambda_w^* \Lambda_w S_\Lambda^{-1} g \rangle d\mu(w), \quad f, g \in H. \end{aligned}$$

Let

$$\hat{K} = \left\{ F \in \prod_{w \in \Omega} K_w : F \text{ is strongly measurable, } \int_{\Omega} \|F(w)\|^2 d\mu(w) < \infty \right\}.$$

It is clear that, \widehat{K} is a Hilbert space with pointwise operations and the inner product given by

$$\langle F, G \rangle = \int_{\Omega} \langle F(w), G(w) \rangle d\mu(w), \quad F, G \in \widehat{K}.$$

Proposition 1.4 ([1]). *Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g -Bessel family. Then the mapping $T_{\Lambda} : \widehat{K} \rightarrow H$ defined by*

$$\langle T_{\Lambda} F, g \rangle = \int_{\Omega} \langle \Lambda_w^* F(w), g \rangle d\mu(w), \quad F \in \widehat{K}, \quad g \in H,$$

is linear and bounded with $\|T_{\Lambda}\| \leq \sqrt{B_{\Lambda}}$. Also, for each $g \in H$ we have

$$T_{\Lambda}^*(g)(w) = \Lambda_w g, \quad w \in \Omega.$$

The operators T_{Λ} and T_{Λ}^* in Proposition 1.4 are called the synthesis and analysis operators of Λ , respectively.

2. THE SUM OF CONTINUOUS G -FRAMES

The authors in [12] have given some conditions under which the finite sum of frames can be also frames. In [10], Madadian and Rahmani have discussed that the finite sum of continuous g -frames can be a continuous g -frame under some conditions.

In this section, we study the sum of continuous g -frames and generalize some results of [5] and [9] to continuous g -frames.

The following example shows that the sum of two continuous g -frames is not necessarily a continuous g -frame.

Example 2.1. Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g -frame. Let $\Gamma_w = -\Lambda_w$ for all $w \in \Omega$, then $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a continuous g -frame and $\Lambda + \Gamma = \{\Lambda_w + \Gamma_w \in B(H, K_w) : w \in \Omega\}$ is not a continuous g -frame.

Here we give some conditions under which $\Lambda + \Gamma$ is a continuous g -frame for H .

Theorem 2.2. *Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g -frame and $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g -Bessel family. For non-zero constants a, b , if*

$$A_{\Lambda}|a|^2 - 2B_{\Gamma}|b|^2 > 0,$$

then $a\Lambda + b\Gamma = \{a\Lambda_w + b\Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a continuous g -frame.

Proof. By the Cauchy-Schwarz inequality, for any $f \in H$, we have

$$\int_{\Omega} \|(a\Lambda_w + b\Gamma_w) f\|^2 d\mu(w)$$

$$\begin{aligned}
&= \int_{\Omega} \langle a\Lambda_w f + b\Gamma_w f, a\Lambda_w f + b\Gamma_w f \rangle d\mu(w) \\
&= \int_{\Omega} \|a\Lambda_w f\|^2 d\mu(w) + \int_{\Omega} 2\operatorname{Re}\langle a\Lambda_w f, b\Gamma_w f \rangle d\mu(w) \\
&\quad + \int_{\Omega} \|b\Gamma_w f\|^2 d\mu(w) \\
&\leq \int_{\Omega} \|a\Lambda_w f\|^2 d\mu(w) + 2 \int_{\Omega} |\langle a\Lambda_w f, b\Gamma_w f \rangle| d\mu(w) \\
&\quad + \int_{\Omega} \|b\Gamma_w f\|^2 d\mu(w) \\
&\leq \int_{\Omega} \|a\Lambda_w f\|^2 d\mu(w) + 2 \int_{\Omega} \|a\Lambda_w f\| \|b\Gamma_w f\| d\mu(w) \\
&\quad + \int_{\Omega} \|b\Gamma_w f\|^2 d\mu(w) \\
&\leq \int_{\Omega} \|a\Lambda_w f\|^2 d\mu(w) + 2 \left(\int_{\Omega} \|a\Lambda_w f\|^2 d\mu(w) \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{\Omega} \|b\Gamma_w f\|^2 d\mu(w) \right)^{\frac{1}{2}} + \int_{\Omega} \|b\Gamma_w f\|^2 d\mu(w) \\
&= \left[\left(\int_{\Omega} \|a\Lambda_w f\|^2 d\mu(w) \right)^{\frac{1}{2}} + \left(\int_{\Omega} \|b\Gamma_w f\|^2 d\mu(w) \right)^{\frac{1}{2}} \right]^2 \\
&\leq 2 \int_{\Omega} |a|^2 \|\Lambda_w f\|^2 d\mu(w) + 2 \int_{\Omega} |b|^2 \|\Gamma_w f\|^2 d\mu(w) \\
&= 2|a|^2 \int_{\Omega} \|\Lambda_w f\|^2 d\mu(w) + 2|b|^2 \int_{\Omega} \|\Gamma_w f\|^2 d\mu(w) \\
&\leq 2(|a|^2 B_{\Lambda} + |b|^2 B_{\Gamma}) \|f\|^2.
\end{aligned}$$

On the other hand, for each $f \in H$,

$$\begin{aligned}
&\int_{\Omega} \|a\Lambda_w f\|^2 d\mu(w) \\
&= \int_{\Omega} \|(a\Lambda_w + b\Gamma_w) f - b\Gamma_w f\|^2 d\mu(w) \\
&\leq 2 \int_{\Omega} \|a\Lambda_w f + b\Gamma_w f\|^2 d\mu(w) + 2 \int_{\Omega} \|b\Gamma_w f\|^2 d\mu(w),
\end{aligned}$$

so

$$\begin{aligned}
&2 \int_{\Omega} \|a\Lambda_w f + b\Gamma_w f\|^2 d\mu(w) \\
&\geq \int_{\Omega} \|a\Lambda_w f\|^2 d\mu(w) - 2 \int_{\Omega} \|b\Gamma_w f\|^2 d\mu(w)
\end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega} |a|^2 \|\Lambda_w f\|^2 d\mu(w) - 2 \int_{\Omega} |b|^2 \|\Gamma_w f\|^2 d\mu(w) \\
&= |a|^2 \int_{\Omega} \|\Lambda_w f\|^2 d\mu(w) - 2|b|^2 \int_{\Omega} \|\Gamma_w f\|^2 d\mu(w) \\
&\geq (|a|^2 A_{\Lambda} - 2|b|^2 B_{\Gamma}) \|f\|^2, \quad f \in H.
\end{aligned}$$

□

Corollary 2.3. *Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g -frame and $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g -Bessel family. If $B_{\Gamma} < \frac{A_{\Lambda}}{2}$, then $\Lambda + \Gamma = \{\Lambda_w + \Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a continuous g -frame.*

Proof. It is sufficient to put $a = b = 1$, in Theorem 2.2. □

Theorem 2.4. *Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ and $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ be two continuous g -frames. Let $U, V \in B(H)$. If $T_{\Lambda} T_{\Gamma}^* = 0$ and U or V is a self-adjoint surjective operator, then $\Lambda U + \Gamma V = \{\Lambda_w U + \Gamma_w V \in B(H, K_w) : w \in \Omega\}$ is a continuous g -frame.*

Proof. Since $T_{\Lambda} T_{\Gamma}^* = 0$, for any $f \in H$, we have

$$\begin{aligned}
&\int_{\Omega} \|(\Lambda_w U + \Gamma_w V) f\|^2 d\mu(w) \\
&= \int_{\Omega} \|\Lambda_w U f\|^2 d\mu(w) + \int_{\Omega} \langle \Gamma_w^* \Lambda_w U f, V f \rangle d\mu(w) \\
&\quad + \int_{\Omega} \langle \Lambda_w^* \Gamma_w V f, U f \rangle d\mu(w) + \int_{\Omega} \|\Gamma_w V f\|^2 d\mu(w) \\
&= \int_{\Omega} \|\Lambda_w U f\|^2 d\mu(w) + \langle T_{\Gamma} T_{\Lambda}^* U f, V f \rangle \\
&\quad + \langle T_{\Lambda} T_{\Gamma}^* V f, U f \rangle + \int_{\Omega} \|\Gamma_w V f\|^2 d\mu(w) \\
&\leq B_{\Lambda} \|U f\|^2 + B_{\Gamma} \|V f\|^2 \\
&\leq (B_{\Lambda} \|U\|^2 + B_{\Gamma} \|V\|^2) \|f\|^2, \quad f \in H.
\end{aligned}$$

Now, suppose that U is a self-adjoint surjective operator. By Lemma 2.4.1 of [3], there exists a constant $C > 0$ such that

$$\|U f\|^2 \geq C \|f\|^2, \quad f \in H.$$

Then

$$\begin{aligned}
&\int_{\Omega} \|(\Lambda_w U + \Gamma_w V) f\|^2 d\mu(w) \\
&= \int_{\Omega} \|\Lambda_w U f\|^2 d\mu(w) + \int_{\Omega} \|\Gamma_w V f\|^2 d\mu(w)
\end{aligned}$$

$$\begin{aligned}
&\geq \int_{\Omega} \|\Lambda_w U f\|^2 d\mu(w) \\
&\geq A_{\Lambda} \|U f\|^2 \\
&\geq A_{\Lambda} C \|f\|^2, \quad f \in H.
\end{aligned}$$

Therefore, $\Lambda U + \Gamma V = \{\Lambda_w U + \Gamma_w V \in B(H, K_w) : w \in \Omega\}$ is a continuous g -frame. \square

Corollary 2.5. *Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ and $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ be two continuous g -frames. If $T_{\Lambda} T_{\Gamma}^* = 0$, then $\Lambda + \Gamma = \{\Lambda_w + \Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a continuous g -frame.*

Proof. It is sufficient to put $U = V = I_H$, in Theorem 2.4. \square

Corollary 2.6. *Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ and $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ be two continuous g -frames. If $T_{\Lambda} T_{\Gamma}^* = 0$ and $U \in B(H)$, then $\Lambda U + \Gamma = \{\Lambda_w U + \Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a continuous g -frame.*

Proof. It is sufficient to put $V = I_H$, in Theorem 2.4. \square

Theorem 2.7. *Let $\Lambda^i = \{\Lambda_w^i \in B(H, K_w) : w \in \Omega\}$ be a continuous g -frame for $i \in I = \{1, 2, \dots, M\}$. Let $\{\alpha_i\}_{i \in I}$ be a sequence of scalars. Then*

$$\sum_{i \in I} \alpha_i \Lambda^i = \left\{ \sum_{i \in I} \alpha_i \Lambda_w^i \in B(H, K_w) : w \in \Omega \right\},$$

is a continuous g -frame if and only if there exist $\beta > 0$ and some $j \in I$ such that

$$\beta \int_{\Omega} \|\Lambda_w^j f\|^2 d\mu(w) \leq \int_{\Omega} \left\| \sum_{i \in I} \alpha_i \Lambda_w^i f \right\|^2 d\mu(w), \quad f \in H.$$

Proof. First, suppose that $\sum_{i \in I} \alpha_i \Lambda^i$ is a continuous g -frame for H . Then

(2.1)

$$A_{\sum_{i \in I} \alpha_i \Lambda^i} \|f\|^2 \leq \int_{\Omega} \left\| \sum_{i \in I} \alpha_i \Lambda_w^i f \right\|^2 d\mu(w) \leq B_{\sum_{i \in I} \alpha_i \Lambda^i} \|f\|^2, \quad f \in H.$$

Since for $j \in I$, Λ^j is a continuous g -frame for H , we have

$$(2.2) \quad A_{\Lambda^j} \|f\|^2 \leq \int_{\Omega} \|\Lambda_w^j f\|^2 d\mu(w) \leq B_{\Lambda^j} \|f\|^2, \quad f \in H, \quad j \in I.$$

By the inequalities (2.1) and (2.2), for any $f \in H$, we have

$$\int_{\Omega} \left\| \sum_{i \in I} \alpha_i \Lambda_w^i f \right\|^2 d\mu(w) \geq A_{\sum_{i \in I} \alpha_i \Lambda^i} \|f\|^2$$

$$\geq \frac{A_{\sum_{i \in I} \alpha_i \Lambda^i}}{B_{\Lambda^j}} \int_{\Omega} \|\Lambda_w^j f\|^2 d\mu(w),$$

thus, it is sufficient to put $\beta = \frac{A_{\sum_{i \in I} \alpha_i \Lambda^i}}{B_{\Lambda^j}}$.

Conversely, we suppose that there exists $\beta > 0$ such that

$$\beta \int_{\Omega} \|\Lambda_w^j f\|^2 d\mu(w) \leq \int_{\Omega} \left\| \sum_{i \in I} \alpha_i \Lambda_w^i f \right\|^2 d\mu(w), \quad f \in H,$$

for some $j \in I$. Thus

$$\begin{aligned} \int_{\Omega} \left\| \sum_{i \in I} \alpha_i \Lambda_w^i f \right\|^2 d\mu(w) &\geq \beta \int_{\Omega} \|\Lambda_w^j f\|^2 d\mu(w) \\ &\geq \beta A_{\Lambda^j} \|f\|^2, \quad f \in H. \end{aligned}$$

On the other hand, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \int_{\Omega} \left\| \sum_{i \in I} \alpha_i \Lambda_w^i f \right\|^2 d\mu(w) &\leq \int_{\Omega} \left(\sum_{i \in I} \|\alpha_i \Lambda_w^i f\| \right)^2 d\mu(w) \\ &\leq \int_{\Omega} M \sum_{i \in I} |\alpha_i|^2 \|\Lambda_w^i f\|^2 d\mu(w) \\ &\leq \int_{\Omega} M^2 \left(\max_{i \in I} |\alpha_i|^2 \right) \sum_{i \in I} \|\Lambda_w^i f\|^2 d\mu(w) \\ &= M^2 \left(\max_{i \in I} |\alpha_i|^2 \right) \sum_{i \in I} \int_{\Omega} \|\Lambda_w^i f\|^2 d\mu(w) \\ &\leq M^2 \left(\max_{i \in I} |\alpha_i|^2 \right) \sum_{i \in I} B_{\Lambda^i} \|f\|^2 \\ &\leq M^3 \left(\max_{i \in I} |\alpha_i|^2 \right) \left(\max_{i \in I} B_{\Lambda^i} \right) \|f\|^2, \quad f \in H. \end{aligned}$$

So, $\sum_{i \in I} \alpha_i \Lambda^i$ is a continuous g -frame for H . \square

3. THE STABILITY OF CONTINUOUS G -FRAMES

In [4], Christensen has discussed the stability of frames in the Hilbert spaces under perturbations. Also, Sun has proved that g -frames are stable under small perturbations [14]. The perturbation result was generalized to continuous g -frames in [1]. In this section, we study the stability of continuous g -frames.

Theorem 3.1. *Suppose that $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a family of operators such that for each $f \in H$, $\{\Gamma_w f\}_{w \in \Omega}$ is strongly measurable.*

Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g -frame and $\Lambda + \Gamma = \{\Lambda_w + \Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a continuous g -Bessel family. If $A_\Lambda - 2B_{\Lambda+\Gamma} > 0$, then Γ is a continuous g -frame for H .

Proof. We have

$$\begin{aligned} \int_{\Omega} \|\Gamma_w f\|^2 d\mu(w) &= \int_{\Omega} \|(\Gamma_w + \Lambda_w) f - \Lambda_w f\|^2 d\mu(w) \\ &\leq 2 \left(\int_{\Omega} \|(\Gamma_w + \Lambda_w) f\|^2 d\mu(w) + \int_{\Omega} \|\Lambda_w f\|^2 d\mu(w) \right) \\ &\leq 2(B_{\Lambda+\Gamma} + B_\Lambda) \|f\|^2, \quad f \in H. \end{aligned}$$

Also,

$$\begin{aligned} 2 \int_{\Omega} \|\Gamma_w f\|^2 d\mu(w) &\geq \int_{\Omega} \|\Lambda_w f\|^2 d\mu(w) - 2 \int_{\Omega} \|(\Lambda_w + \Gamma_w) f\|^2 d\mu(w) \\ &\geq A_\Lambda \|f\|^2 - 2B_{\Lambda+\Gamma} \|f\|^2 \\ &= (A_\Lambda - 2B_{\Lambda+\Gamma}) \|f\|^2, \quad f \in H. \end{aligned}$$

So, Γ is a continuous g -frame for H . \square

Theorem 3.2. Suppose that $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a family of operators such that for each $f \in H$, $\{\Gamma_w f\}_{w \in \Omega}$ is strongly measurable. Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g -frame. Then Γ is a continuous g -Bessel family for H if and only if there exists a constant $\lambda > 0$ such that

$$\int_{\Omega} \|(\Lambda_w - \Gamma_w) f\|^2 d\mu(w) \leq \lambda \int_{\Omega} \|\Lambda_w f\|^2 d\mu(w), \quad f \in H.$$

Proof. First, suppose that Γ is a continuous g -Bessel family for H . Since Λ is a continuous g -frame for H , we have

$$\|f\|^2 \leq \frac{1}{A_\Lambda} \int_{\Omega} \|\Lambda_w f\|^2 d\mu(w), \quad f \in H,$$

thus

$$\begin{aligned} \int_{\Omega} \|\Gamma_w f\|^2 d\mu(w) &\leq B_\Gamma \|f\|^2 \\ &\leq \frac{B_\Gamma}{A_\Lambda} \int_{\Omega} \|\Lambda_w f\|^2 d\mu(w), \quad f \in H. \end{aligned}$$

So

$$\begin{aligned} \int_{\Omega} \|(\Lambda_w - \Gamma_w) f\|^2 d\mu(w) &\leq 2 \int_{\Omega} \|\Lambda_w f\|^2 d\mu(w) + 2 \int_{\Omega} \|\Gamma_w f\|^2 d\mu(w) \\ &\leq 2 \left(1 + \frac{B_\Gamma}{A_\Lambda} \right) \int_{\Omega} \|\Lambda_w f\|^2 d\mu(w), \quad f \in H. \end{aligned}$$

Conversely, for any $f \in H$, we have

$$\begin{aligned} \int_{\Omega} \|\Gamma_w f\|^2 d\mu(w) &= \int_{\Omega} \|(\Gamma_w - \Lambda_w) f + \Lambda_w f\|^2 d\mu(w) \\ &\leq 2 \int_{\Omega} \|(\Gamma_w - \Lambda_w) f\|^2 d\mu(w) + 2 \int_{\Omega} \|\Lambda_w f\|^2 d\mu(w) \\ &\leq 2 \left(\lambda \int_{\Omega} \|\Lambda_w f\|^2 d\mu(w) + \int_{\Omega} \|\Lambda_w f\|^2 d\mu(w) \right) \\ &\leq 2(\lambda + 1) B_{\Lambda} \|f\|^2, \end{aligned}$$

therefore Γ is a continuous g -Bessel family for H . \square

Theorem 3.3. *Suppose that $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a family of operators such that for each $f \in H$, $\{\Gamma_w f\}_{w \in \Omega}$ is strongly measurable. Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g -frame and a and b be non-zero constants. Suppose that there exist constants $0 \leq \lambda, \mu < \frac{1}{2}$ such that for any $f \in H$,*

$$\begin{aligned} \int_{\Omega} \|(a\Lambda_w - b\Gamma_w) f\|^2 d\mu(w) \\ \leq \lambda \int_{\Omega} \|a\Lambda_w f\|^2 d\mu(w) + \mu \int_{\Omega} \|b\Gamma_w f\|^2 d\mu(w). \end{aligned}$$

Then Γ is a continuous g -frame for H .

Proof. For any $f \in H$,

$$\begin{aligned} \int_{\Omega} \|b\Gamma_w f\|^2 d\mu(w) &= \int_{\Omega} \|(b\Gamma_w - a\Lambda_w) f + a\Lambda_w f\|^2 d\mu(w) \\ &\leq 2 \int_{\Omega} \|(b\Gamma_w - a\Lambda_w) f\|^2 d\mu(w) + 2 \int_{\Omega} \|a\Lambda_w f\|^2 d\mu(w) \\ &\leq 2\lambda \int_{\Omega} \|a\Lambda_w f\|^2 d\mu(w) + 2\mu \int_{\Omega} \|b\Gamma_w f\|^2 d\mu(w) \\ &\quad + 2 \int_{\Omega} \|a\Lambda_w f\|^2 d\mu(w) \\ &= 2(\lambda + 1) \int_{\Omega} \|a\Lambda_w f\|^2 d\mu(w) + 2\mu \int_{\Omega} \|b\Gamma_w f\|^2 d\mu(w). \end{aligned}$$

So

$$(1 - 2\mu) \int_{\Omega} \|b\Gamma_w f\|^2 d\mu(w) \leq 2(\lambda + 1) \int_{\Omega} \|a\Lambda_w f\|^2 d\mu(w), \quad f \in H,$$

therefore, for any $f \in H$,

$$(1 - 2\mu) |b|^2 \int_{\Omega} \|\Gamma_w f\|^2 d\mu(w) \leq 2(\lambda + 1) |a|^2 \int_{\Omega} \|\Lambda_w f\|^2 d\mu(w).$$

Thus

$$\begin{aligned} \int_{\Omega} \|\Gamma_w f\|^2 d\mu(w) &\leq \frac{2(\lambda + 1) |a|^2}{(1 - 2\mu) |b|^2} \int_{\Omega} \|\Lambda_w f\|^2 d\mu(w) \\ &\leq \frac{2(\lambda + 1) |a|^2}{(1 - 2\mu) |b|^2} B_{\Lambda} \|f\|^2, \quad f \in H. \end{aligned}$$

On the other hand, for any $f \in H$, we have

$$\begin{aligned} \int_{\Omega} \|a\Lambda_w f\|^2 d\mu(w) &= \int_{\Omega} \|(a\Lambda_w - b\Gamma_w) f + b\Gamma_w f\|^2 d\mu(w) \\ &\leq 2\lambda \int_{\Omega} \|a\Lambda_w f\|^2 d\mu(w) + 2\mu \int_{\Omega} \|b\Gamma_w f\|^2 d\mu(w) \\ &\quad + 2 \int_{\Omega} \|b\Gamma_w f\|^2 d\mu(w). \end{aligned}$$

Also, for each $f \in H$, we have

$$(1 - 2\lambda) |a|^2 \int_{\Omega} \|\Lambda_w f\|^2 d\mu(w) \leq 2(1 + \mu) |b|^2 \int_{\Omega} \|\Gamma_w f\|^2 d\mu(w).$$

Thus

$$\int_{\Omega} \|\Gamma_w f\|^2 d\mu(w) \geq \frac{(1 - 2\lambda) |a|^2}{2(1 + \mu) |b|^2} A_{\Lambda} \|f\|^2, \quad f \in H,$$

therefore, Γ is a continuous g -frame for H . \square

Theorem 3.4. For $i \in I = \{1, 2, \dots, M\}$, let $\Lambda^i = \{\Lambda_w^i \in B(H, K_w) : w \in \Omega\}$ be a continuous g -Bessel family. Let $\Gamma^i = \{\Gamma_w^i \in B(H, K_w) : w \in \Omega\}$ be a continuous g -Bessel family such that

$$\int_{\Omega} \|(\Lambda_w^i - \Gamma_w^i) f\|^2 d\mu(w) \leq \lambda \int_{\Omega} \|\Lambda_w^i f\|^2 d\mu(w), \quad f \in H,$$

for $i \in I$ and $\lambda \geq 0$. If for some $j \in I$, there exists $A_{\Lambda^j} > 0$ such that

$$\int_{\Omega} \|\Lambda_w^j f\|^2 d\mu(w) \geq A_{\Lambda^j} \|f\|^2, \quad f \in H,$$

and

$$2(M - 1) \sum_{i \neq j} \|T_{\Lambda^i}\|^2 + 4M\lambda \sum_{i \in I} \|T_{\Lambda^i}\|^2 < A_{\Lambda^j},$$

then,

$$\sum_{i \in I} \Gamma^i = \left\{ \sum_{i \in I} \Gamma_w^i \in B(H, K_w) : w \in \Omega \right\}$$

is a continuous g -frame.

Proof. For each $f \in H$,

$$\begin{aligned} \int_{\Omega} \|\Lambda_w^j f\|^2 d\mu(w) &= \int_{\Omega} \left\| \sum_{i \in I} \Lambda_w^i f - \sum_{i \neq j} \Lambda_w^i f \right\|^2 d\mu(w) \\ &\leq 2 \int_{\Omega} \left\| \sum_{i \in I} \Lambda_w^i f \right\|^2 d\mu(w) + 2 \int_{\Omega} \left\| \sum_{i \neq j} \Lambda_w^i f \right\|^2 d\mu(w), \end{aligned}$$

thus for any $f \in H$,

$$(3.1) \quad \int_{\Omega} \left\| \sum_{i \in I} \Lambda_w^i f \right\|^2 d\mu(w) \geq \frac{1}{2} \int_{\Omega} \|\Lambda_w^j f\|^2 d\mu(w) - \int_{\Omega} \left\| \sum_{i \neq j} \Lambda_w^i f \right\|^2 d\mu(w).$$

Also,

$$(3.2) \quad \begin{aligned} \int_{\Omega} \|\Lambda_w^i f\|^2 d\mu(w) &= \|(T_{\Lambda^i})^* f\|^2 \\ &\leq \|(T_{\Lambda^i})^*\|^2 \|f\|^2 \\ &= \|T_{\Lambda^i}\|^2 \|f\|^2, \quad f \in H. \end{aligned}$$

Then, by the inequalities (3.1) and (3.2), for any $f \in H$, we have

$$\begin{aligned} &\int_{\Omega} \left\| \sum_{i \in I} \Gamma_w^i f \right\|^2 d\mu(w) \\ &\geq \frac{1}{2} \left(\int_{\Omega} \left\| \sum_{i \in I} \Lambda_w^i f \right\|^2 d\mu(w) - 2 \int_{\Omega} \left\| \sum_{i \in I} (\Gamma_w^i - \Lambda_w^i) f \right\|^2 d\mu(w) \right) \\ &\geq \frac{1}{2} \left(\frac{1}{2} \int_{\Omega} \|\Lambda_w^j f\|^2 d\mu(w) - \int_{\Omega} \left\| \sum_{i \neq j} \Lambda_w^i f \right\|^2 d\mu(w) \right. \\ &\quad \left. - 2 \int_{\Omega} \left\| \sum_{i \in I} (\Gamma_w^i - \Lambda_w^i) f \right\|^2 d\mu(w) \right) \\ &\geq \frac{1}{2} \left(\frac{1}{2} \int_{\Omega} \|\Lambda_w^j f\|^2 d\mu(w) - (M-1) \sum_{i \neq j} \int_{\Omega} \|\Lambda_w^i f\|^2 d\mu(w) \right. \\ &\quad \left. - 2M \sum_{i \in I} \int_{\Omega} \|(\Gamma_w^i - \Lambda_w^i) f\|^2 d\mu(w) \right) \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2} \left(\frac{1}{2} A_{\Lambda^j} \|f\|^2 - (M-1) \sum_{i \neq j} \|T_{\Lambda^i}\|^2 \|f\|^2 - 2M\lambda \sum_{i \in I} \|T_{\Lambda^i}\|^2 \|f\|^2 \right) \\
&= \frac{1}{4} \left(A_{\Lambda^j} - 2(M-1) \sum_{i \neq j} \|T_{\Lambda^i}\|^2 - 4M\lambda \sum_{i \in I} \|T_{\Lambda^i}\|^2 \right) \|f\|^2.
\end{aligned}$$

For any $i \in I$ and $f \in H$, we have

$$\begin{aligned}
&\int_{\Omega} \left\| \sum_{i \in I} \Gamma_w^i f \right\|^2 d\mu(w) \\
&= \int_{\Omega} \left\| \sum_{i \in I} (\Gamma_w^i - \Lambda_w^i) f + \sum_{i \in I} \Lambda_w^i f \right\|^2 d\mu(w) \\
&\leq 2 \left(\int_{\Omega} \left\| \sum_{i \in I} (\Gamma_w^i - \Lambda_w^i) f \right\|^2 d\mu(w) + \int_{\Omega} \left\| \sum_{i \in I} \Lambda_w^i f \right\|^2 d\mu(w) \right) \\
&\leq 2M \left(\int_{\Omega} \sum_{i \in I} \|(\Gamma_w^i - \Lambda_w^i) f\|^2 d\mu(w) + \int_{\Omega} \sum_{i \in I} \|\Lambda_w^i f\|^2 d\mu(w) \right) \\
&\leq 2M \left(\lambda \int_{\Omega} \sum_{i \in I} \|\Lambda_w^i f\|^2 d\mu(w) + \int_{\Omega} \sum_{i \in I} \|\Lambda_w^i f\|^2 d\mu(w) \right) \\
&= 2M(1+\lambda) \sum_{i \in I} \int_{\Omega} \|\Lambda_w^i f\|^2 d\mu(w) \\
&\leq 2M(1+\lambda) \sum_{i \in I} B_{\Lambda^i} \|f\|^2 \\
&\leq 2M^2(1+\lambda) \max_{i \in I} B_{\Lambda^i} \|f\|^2.
\end{aligned}$$

Therefore,

$$\sum_{i \in I} \Gamma^i = \left\{ \sum_{i \in I} \Gamma_w^i \in B(H, K_w) : w \in \Omega \right\},$$

is a continuous g -frame. \square

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