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ABSTRACT. Let \mathcal{X} be a partially ordered set and d be a generalized metric on \mathcal{X} . We obtain some results in coupled and coupled coincidence of g -monotone functions on \mathcal{X} , where g is a function from \mathcal{X} into itself. Moreover, we show that a nonexpansive mapping on a partially ordered Hilbert space has a fixed point lying in the unit ball of the Hilbert space. Some applications for linear and nonlinear matrix equations are given.

1. INTRODUCTION

Let (\mathcal{V}, \preceq) be an ordered Banach space. The cone $\mathcal{V}_+ = \{v \in \mathcal{V} : \theta \preceq v\}$, where θ is the zero-vector of \mathcal{V} , satisfies the usual properties

- (i) $\mathcal{V}_+ \cap -\mathcal{V}_+ = \{\theta\}$;
- (ii) $\mathcal{V}_+ + \mathcal{V}_+ \subset \mathcal{V}_+$;
- (iii) $\alpha\mathcal{V}_+ \subset \mathcal{V}_+$, for $\alpha \geq 0$.

Let \mathcal{X} be a nonempty set. A mapping $d : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{V}$ is called a vector-valued metric on \mathcal{X} , if the following properties are satisfied:

- (i) $d(x, y) \succeq \theta$ for each $x, y \in \mathcal{X}$, if $d(x, y) = \theta$, then $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;
- (iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in \mathcal{X}$.

The pair (\mathcal{X}, d) is called the vector-valued metric space. Similarly, we can define a generalized normed space.

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A set \mathcal{X} equipped with a vector-valued metric d is called a generalized metric space and denoted by (\mathcal{X}, d) . By $M_{m,m}(\mathbb{R}^+)$, we mean the set of all $m \times m$ matrixes with positive elements. We denote by I the identity $m \times m$ matrix. Let $A \in M_{m,m}(\mathbb{R}^+)$, A is said to be convergent to zero if and only if $A^n \rightarrow 0$ as $n \rightarrow \infty$ (for more details see [10]).

Let $\alpha, \beta \in \mathbb{R}^m$, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$, $\beta = (\beta_1, \beta_2, \dots, \beta_m)$ and $c \in \mathbb{R}$. Note that $\alpha \leq \beta$ (resp. $\alpha < \beta$) means $\alpha_i \leq \beta_i$ (resp. $\alpha_i < \beta_i$) for each $1 \leq i \leq m$, and also $\alpha \leq c$ (resp. $\alpha < c$) means $\alpha_i \leq c$ (resp. $\alpha_i < c$) for $1 \leq i \leq m$, respectively. We can define addition and multiplication on \mathbb{R}^m as follows:

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_m + \beta_m),$$

and

$$\alpha \cdot \beta = (\alpha_1\beta_1, \alpha_2\beta_2, \dots, \alpha_m\beta_m).$$

In this paper, we need the following equivalent statements:

- (i) A is convergent towards zero;
- (ii) $A^n \rightarrow 0$ as $n \rightarrow \infty$;
- (iii) The eigenvalues of A are located in the open unit disc, that is, $|\lambda| < 1$, for each $\lambda \in \mathbb{C}$ with $\det(A - \lambda I) = 0$;
- (iv) The matrix $I - A$ is nonsingular and

$$(I - A)^{-1} = I + A + \dots + A^n + \dots;$$

- (v) $A^n q^T \rightarrow 0$ and $q A^n \rightarrow 0$ as $n \rightarrow \infty$, for each $q \in \mathbb{R}^m$, where q^T is the transpose of q .

The above statements are the classical results in matrix analysis (for more details see [1, 5, 9]). Denote, by \mathcal{ZM} the set of all matrices $A \in M_{m,m}(\mathbb{R}^+)$ such that A^n converges to zero. Let (\mathcal{X}, d) be a generalized metric space and let $T : \mathcal{X} \rightarrow \mathcal{X}$ be a mapping. For a given $A \in \mathcal{ZM}$, we call the function mapping T is an A -nonexpansive if $d(T(x), T(y)) \leq Ad(x, y)$ for all $x, y \in X$ and T to be said to be \mathcal{ZM} -nonexpansive if for any B in \mathcal{ZM} , T is a B -nonexpansive function.

Clearly, if $A \in \mathcal{ZM}$, then there exists a norm $\|\cdot\|$ such that $\|A\| < 1$, so every \mathcal{ZM} -nonexpansive operator is nonexpansive, but the converse is not true, in general.

Fixed point theorems on spaces endowed with vector-valued metrics considered by Filip and Petruşel in [3] and some new results around this notion are obtained in [4].

Definition 1.1 ([2]). Let (\mathcal{X}, \preceq) be a partially ordered set and let $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$. The mapping F is said to be has the *mixed monotone property* if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y , that is, for every $x, y \in \mathcal{X}$,

- (i) for each $x_1, x_2 \in \mathcal{X}$, if $x_1 \preceq x_2$, then $F(x_1, y) \preceq F(x_2, y)$;

(ii) for each $y_1, y_2 \in \mathcal{X}$, if $y_1 \preceq y_2$, then $F(x, y_1) \succeq F(x, y_2)$.

Let (\mathcal{X}, \preceq) be a partially ordered set and d be a metric on \mathcal{X} such that (\mathcal{X}, d) is a complete metric space. The product space $\mathcal{X} \times \mathcal{X}$ is endowed with the following partial order:

for, $(x, y), (u, v) \in \mathcal{X} \times \mathcal{X}$, $(u, v) \leq (x, y) \Leftrightarrow x \geq u, y \leq v$.

Definition 1.2 ([2]). Let (\mathcal{X}, \preceq) be a partially ordered set and let $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ be a mapping. An element $(x, y) \in \mathcal{X} \times \mathcal{X}$ is said to be a coupled fixed point of the mapping F , if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.3. An element $(x, y) \in \mathcal{X} \times \mathcal{X}$ is called

- (i) a coupled coincidence point of mappings $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ and $g : \mathcal{X} \rightarrow \mathcal{X}$ if $g(x) = F(x, y)$ and $g(y) = F(y, x)$, and (gx, gy) is called a coupled point of coincidence.
- (ii) a common coupled fixed point of mappings $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ and $g : \mathcal{X} \rightarrow \mathcal{X}$ if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

Definition 1.4. Let (\mathcal{X}, \preceq) be a partially ordered set and $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ and $g : \mathcal{X} \rightarrow \mathcal{X}$ be two self mappings. We say F has the mixed g -monotone property if F is monotone g -non-decreasing in its first argument and is monotone g -non-increasing in its second argument, that is, for all $x_1, x_2 \in \mathcal{X}$, $gx_1 \preceq gx_2$ implies $F(x_1, y) \preceq F(x_2, y)$ for any $y \in \mathcal{X}$, and for all $y_1, y_2 \in \mathcal{X}$, $gy_1 \succeq gy_2$ implies $F(x, y_1) \preceq F(x, y_2)$ for all $x \in \mathcal{X}$.

Definition 1.5. Let \mathcal{X} be a non-empty set. We say that the mappings $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ and $g : \mathcal{X} \rightarrow \mathcal{X}$ are commutative if $g(F(x, y)) = F(gx, gy)$, for all $x, y \in \mathcal{X}$.

Bhaskar and Lakshmikantham in [2], studied the existence of coupled fixed points for continuous mapping with the mixed monotone property $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$, where (\mathcal{X}, \preceq) is a partially ordered set. The existence of coupled fixed point for a mapping with the mixed monotone property $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$, where (\mathcal{X}, d) is a complete generalized metric space, is considered in [7].

In this paper, we consider the existence and uniqueness of coupled fixed points for mappings $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$, under some contractive conditions, where (\mathcal{X}, d) is a complete generalized metric space.

2. MAIN RESULTS

We say that \mathcal{X} satisfies in condition (NDI) if \mathcal{X} has the following properties:

- (i) if a non-decreasing sequence $x_n \rightarrow x$, then $x_n \preceq x$ for all n .
- (ii) if a non-increasing sequence $x_n \rightarrow x$, then $x \preceq x_n$ for all n .

Theorem 2.1. *Let (\mathcal{X}, \preceq) be a partially ordered set, (\mathcal{X}, d) be a complete generalized metric space which satisfies the condition (NDI), and for all $x, y, u, v \in \mathcal{X}$, and let $g : \mathcal{X} \rightarrow \mathcal{X}$ with $gx \preceq gu$ and $gv \preceq gy$. Suppose that $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ satisfies the following condition*

$$(2.1) \quad d(F(x, y), F(u, v)) \leq Ad(gx, gu) + Bd(gy, gv),$$

where $A = (a_{ij}), B = (b_{ij})$ are in $M_{m \times m}(\mathbb{R}^+)$, $(A + B) \in \mathcal{ZM}$, A and B are nonzero matrices in \mathcal{ZM} . Furthermore, assume that F and g satisfy the following conditions

- (i) $F(\mathcal{X} \times \mathcal{X}) \subset g(\mathcal{X})$,
- (ii) $g(\mathcal{X})$ is a complete subspace of \mathcal{X} ,
- (iii) F satisfies the mixed g -monotone property.

If there exist $x_0, y_0 \in \mathcal{X}$ such that $g(x_0) \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq g(y_0)$, then F and g has a unique coupled coincidence fixed point.

Proof. Let $x_0, y_0 \in \mathcal{X}$ be such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$. Since $F(\mathcal{X} \times \mathcal{X}) \subset g(\mathcal{X})$, we can choose $x_2, y_2 \in \mathcal{X}$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Since F satisfying the mixed g -monotone property, we have $gx_0 \preceq gx_1 \preceq gx_2$ and $gy_2 \preceq gy_1 \preceq gy_0$. By continuing this process, we can construct two sequences (x_n) and (y_n) in \mathcal{X} such that $gx_n = F(x_{n-1}, y_{n-1}) \preceq gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n) \preceq gy_n = F(y_{n-1}, x_{n-1})$. Further, for $n = 1, 2, \dots$, by (2.1), we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq Ad(gx_{n-1}, gx_n) + Bd(gy_{n-1}, gy_n), \end{aligned}$$

and similarly,

$$\begin{aligned} d(gy_n, gy_{n+1}) &= d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\leq Ad(gy_{n-1}, gy_n) + Bd(gx_{n-1}, gx_n). \end{aligned}$$

Therefore, by letting $d_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})$, we have

$$\begin{aligned} d_n &= d \\ &\leq f(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) \\ &\leq Ad(gx_{n-1}, gx_n) + Bd(gy_{n-1}, gy_n) \\ &\quad + Ad(gy_{n-1}, gy_n) + Bd(gx_{n-1}, gx_n) \\ &\leq (A + B)(d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)) \\ &\leq (A + B)d_{n-1}. \end{aligned}$$

If we set $C = A + B$, then for all $n \in N$, we have

$$(2.2) \quad 0 \leq d_n \leq Cd_{n-1} \leq C^2d_{n-2} \leq \dots \leq C^n d_0.$$

If $d_0 = 0$ then (x_0, y_0) is a coupled fixed point of F . Now, let $d_0 > \theta$. For each $n \geq m$, we have

$$\begin{aligned} d(gx_n, gx_m) &\leq d(gx_n, gx_{n-1}) \\ &\quad + d(gx_{n-1}, gx_{n-2}) + \cdots + d(gx_{m-1}, gx_m), \end{aligned}$$

and

$$\begin{aligned} d(gy_n, gy_m) &\leq d(gy_n, gy_{n-1}) \\ &\quad + d(gy_{n-1}, gy_{n-2}) + \cdots + d(gy_{m-1}, gy_m). \end{aligned}$$

We have

$$\begin{aligned} d(gx_n, gx_m) + d(gy_n, gy_m) &\leq d_{n-1} + d_{n-2} + d_{n-3} + \cdots + d_m \\ &\leq (C^{m-1} + C^{m-2} + \cdots + C^m) d_0 \\ &\leq (C^{m-1} + C^{m-2} + \cdots + C^m + \cdots) d_0 \\ &\leq C^m (I - C)^{-1} d_0. \end{aligned}$$

So

$$d(gx_n, gx_{n+1}) \leq (A + B)^n (d(gx_0, gx_1) + d(gy_0, gy_1)),$$

and

$$d(gy_n, gy_{n+1}) \leq (A + B)^n (d(gx_0, gx_1) + d(gy_0, gy_1)).$$

Let $m, n \in N$ with $m > n$. Since

$$d(gx_n, gx_m) \leq \sum_{i=n}^{m-1} d(gx_i, gx_{i+1}),$$

thus,

$$d(gx_n, gx_m) \leq (I - A - B)^{-1} (A + B)^n (d(gx_0, gx_1) + d(gy_0, gy_1)),$$

which implies that $\{gx_n\}$ is a Cauchy sequence in $g(\mathcal{X})$, and similarly $\{gy_n\}$ is a Cauchy sequence in $g(\mathcal{X})$. Since $g(\mathcal{X})$ is a complete metric space, there exist $gx, gy \in g(\mathcal{X})$ such that $\lim_{n \rightarrow \infty} gx_n = gx$ and $\lim_{n \rightarrow \infty} gy_n = gy$. Also

$$\begin{aligned} d(F(x, y), gx) &\leq d(F(x, y), gx_{n+1}) + d(gx_{n+1}, gx) \\ &= d(F(x, y), F(x_n, y_n)) + d(gx_{n+1}, gx) \\ &\leq Ad(gx_n, gx) + Bd(gy_n, gy) + d(gx_{n+1}, gx). \end{aligned}$$

Therefore, $d(F(x, y), gx) = \theta$, and so $F(x, y) = gx$. Similarly, $F(y, x) = gy$, that is (gx, gy) is a coupled coincidence fixed point of F and g . Now, if (gx', gy') is another coupled coincidence fixed point of F and g , then

$$d(gx', gx) = d(F(x', y'), F(x, y)) \leq Ad(gx', gx) + Bd(gy', gy),$$

and

$$d(gy', gy) = d(F(y', x'), F(y, x)) \leq Ad(gy', gy) + Bd(gx', gx).$$

Then

$$d(gx', gx) + d(gy', gy) \leq (A + B)d(gx', gx) + d(gy', gy).$$

It follows that $d(gx', gx) + d(gy', gy)(I - C) \leq \theta$. Since $C \neq I$, (2.8) implies that $d(gx', gx) + d(gy', gy) = \theta$. Hence, we have $(gx', gy') = (gx, gy)$. \square

It is a worth notice that when the matrices A and B in Theorem 2.1 are equal, we have the following result.

Corollary 2.2. *Let (\mathcal{X}, \preceq) be a partially ordered set and (\mathcal{X}, d) be a complete generalized metric space which satisfies condition (NDI), and for all $x, y, u, v \in \mathcal{X}$, $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ and $g : \mathcal{X} \rightarrow \mathcal{X}$ with $gx \preceq gu, gv \preceq gy$ the following condition is satisfied:*

$$(2.3) \quad d(F(x, y), F(u, v)) \leq \frac{A}{2} [d(gx, gu) + d(gy, gv)],$$

such that $A = (a_{ij}) \in M_{m \times m}(\mathbb{R}^+)$, is a nonzero matrix in \mathcal{ZM} converges to zero. Let F and g satisfy the following conditions

- (i) $F(\mathcal{X} \times \mathcal{X}) \subset g(\mathcal{X})$,
- (ii) $g(\mathcal{X})$ is a complete subspace of \mathcal{X} , and
- (iii) F has the mixed g -monotone property.

If there exist $x_0, y_0 \in \mathcal{X}$ such that $g(x_0) \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq g(y_0)$, then F and g have a unique coupled coincidence fixed point.

Proof. In Theorem 2.1, take $A = B = \frac{A}{2}$. \square

Corollary 2.3. *Let (\mathcal{X}, \preceq) be a partially ordered set and (\mathcal{X}, d) be a complete generalized metric space that satisfies the condition (NDI), and for all $x, y, u, v \in \mathcal{X}$, $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ with the following condition:*

$$(2.4) \quad d(F(x, y), F(u, v)) \leq \frac{A}{2} [d(x, u) + d(y, v)],$$

where $A = (a_{ij}) \in M_{m \times m}(\mathbb{R}^+)$, is a nonzero matrix in \mathcal{ZM} . Also, it is satisfied for some comparable pairs $x \preceq u, v \preceq y$ and F has the mixed monotone property, If there exist $x_0, y_0 \in \mathcal{X}$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then there exist $x, y \in \mathcal{X}$ such that $x = F(x, y)$ and $y = F(y, x)$.

Proof. It follows from Corollary 2.2 by taking $g =$ identity map. \square

Example 2.4. Let $\mathcal{X} = [0, 1] \times [0, 1]$. Define $d : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^2$ with

$$d((x_1, y_1), (x_2, y_2)) = (|x_1 - x_2|, |y_1 - y_2|).$$

Then (\mathcal{X}, d) is a complete generalized metric space. Consider the mapping $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ with $F(U, V) = \left(\frac{x+u}{3}, \frac{y+v}{3}\right)$, where $U = (x, y), V = (u, v)$. Then F satisfies the contractive condition (2.4), for $A = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$, that is,

$$(2.5) \quad d(F(x, y), F(u, v)) \leq \frac{A}{2} [d(x, u) + d(y, v)].$$

Therefore, by Corollary 2.3, F has a unique coupled fixed point, which in this case is $(0, 0)$.

Let $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ be a real Hilbert space, and let $T : \mathcal{X} \rightarrow \mathcal{X}$ be a nonexpansive potential operator such that there is a functional $J : \mathcal{X} \rightarrow \mathbb{R}$ with $J(0) = 0$ and $J' = T$. Consider the measure space (Ω, μ) ($\Omega = [0, 1]$) such that $\mu(\Omega) = 1$, and consider $L^2(\Omega, X)$ that is consists of all μ -strongly measurable functions $u : \Omega \rightarrow \mathcal{X}$ such that $\int_{\Omega} \|u(t)\|^2 d\mu < \infty$ with L^2 -norm. For $r > 0$, define $B_r = \{x \in \mathcal{X} : \|x\| \leq r\}$ and $S_r = \{x \in \mathcal{X} : \|x\| = r\}$. An interesting question that arises here is: when a fixed point of T lies in the interior of B_r ? Ricceri answered this question in [8].

Corollary 2.5. *Let $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ be a partially ordered real Hilbert space with (NDI) property and with generalized norm, let $T : \mathcal{X} \rightarrow \mathcal{X}$ be an $A/2$ -nonexpansive potential operator and $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ such that $F(x, y) = T(x)$. If there exist $x_0, y_0 \in \mathcal{X}$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then T has a fixed point x lying in the interior of B_r and (x, x) is a coupled fixed point of F .*

Proof. Since T is $A/2$ -nonexpansive, so F satisfies (2.4) and T has a unique fixed point x lying in B_r (see [5] or [3, Theorem 1.3]). Thus Corollary 2.3 implies that F has a coupled fixed point $x, y \in \mathcal{X}$ such that $x = F(x, y)$ and $y = F(y, x)$. The uniqueness of fixed point for T caused that $x = y$. \square

Theorem 2.6. *Let (\mathcal{X}, \preceq) be a partially ordered set, (\mathcal{X}, d) be a complete generalized metric space, and for all $x, y, u, v \in \mathcal{X}$, $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ and $g : \mathcal{X} \rightarrow \mathcal{X}$ with $gx \preceq gu$ and $gv \preceq gy$, satisfy the following condition*

$$(2.6) \quad d(F(x, y), F(u, v)) \leq Ad(gx, gu) + Bd(gy, gv),$$

where $A = (a_{ij}), B = (b_{ij}) \in M_{m \times m}(\mathbb{R}^+)$, $\|A + B\| < 1$ where A and B are nonzero matrices in \mathcal{ZM} . Suppose that F has the mixed g -monotone property, $F(\mathcal{X} \times \mathcal{X}) \subset g(\mathcal{X})$, g is continuous and g commutes with F . Also assume that F is continuous or \mathcal{X} satisfies in condition (NDI). If there exist $x_0, y_0 \in \mathcal{X}$ such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$, then F and g have a coupled coincidence point.

Proof. As in the proof of Theorem 2.1, we can construct two Cauchy sequences (gx_n) and (gy_n) in \mathcal{X} . Since (\mathcal{X}, d) is complete, there exist $x, y \in \mathcal{X}$ such that (gx_{n+1}) converges to x and (gy_{n+1}) converges to y . Since g is continuous, we have (ggx_{n+1}) converges to gx and (ggy_{n+1}) converges to gy . But

$$ggx_{n+1} = g(F(x_n, y_n)) = F(gx_n, gy_n),$$

and

$$ggy_{n+1} = g(F(y_n, x_n)) = F(gy_n, gx_n).$$

We complete the proof in two cases: (1) Suppose that F is continuous, then we have $(F(gx_n, gy_n))$ converges to $F(x, y)$ and $(F(gy_n, gx_n))$ converges to $F(y, x)$. Thus (ggx_{n+1}) converges to $F(x, y)$ and (ggy_{n+1}) converges to $F(y, x)$. Therefore,

$$d(ggx_{n+1}, gx) \rightarrow \theta, \quad d(ggy_{n+1}, F(y, x)) \rightarrow \theta.$$

It follows that

$$d(gx, F(x, y)) \leq d(gx, ggx_{n+1}) + d(ggx_{n+1}, F(x, y)).$$

Therefore, $d(gx, F(x, y)) = \theta$ and $gx = F(x, y)$. Similarly, $gy = F(y, x)$. Hence, (x, y) is a coincidence coupled point of F and g .

(2) Suppose that \mathcal{X} satisfies the condition (NDI). Then $gx_n \preceq x$ and $y \preceq gy_n$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned} d(ggx_{n+1}, F(x, y)) &= d(F(gx_n, gy_n), F(x, y)) \\ &\leq Ad(ggx_n, gx) + Bd(ggy_n, gy). \end{aligned}$$

Since (ggx_n) converges to gx and (ggy_n) converges to y , we get (ggx_n) converges to $F(x, y)$. Similarly, (ggy_n) converges to $F(y, x)$. By similar arguments as above, one can show that $gx = F(x, y)$ and $gy = F(y, x)$. Thus, the pair (x, y) is a coupled coincidence point of F and g . \square

Corollary 2.7. *Let (\mathcal{X}, \preceq) be a partially ordered set, (\mathcal{X}, d) be a complete generalized metric space, and, for all $x, y, u, v \in \mathcal{X}$, $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ and $g : \mathcal{X} \rightarrow \mathcal{X}$ with $gx \preceq gu$ and $gv \preceq gy$ satisfy the following condition*

$$(2.7) \quad d(F(x, y), F(u, v)) \leq A[d(gx, gu) + d(gy, gv)],$$

such that $A = (a_{ij}) \in M_{m \times m}(\mathbb{R}^+)$, where A is a nonzero matrix in \mathcal{ZM} . Suppose that F has the mixed g -monotone property, $F(\mathcal{X} \times \mathcal{X}) \subset g(\mathcal{X})$, g is continuous and g commutes with F . Also, assume that either F is continuous or \mathcal{X} has the condition (NDI).

If there exist $x_0, y_0 \in \mathcal{X}$ such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$, then F and g have a coupled coincidence point.

Proof. In Theorem 2.6, take $A = B = \frac{A}{2}$. \square

Theorem 2.8. *In addition to the hypothesis of Theorem 2.1, suppose that for every $(x, y), (x^*, y^*) \in \mathcal{X} \times \mathcal{X}$, there exists $(u, v) \in \mathcal{X} \times \mathcal{X}$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$.*

If (x, y) and (x^, y^*) are coupled coincidence points of F and g , then $F(x, y) = gx = gx^* = F(x^*, y^*)$ and $F(y, x) = gy = gy^* = F(y^*, x^*)$. Moreover, if F and g commutes, then F and g have a unique common fixed point, that is, there exists a unique pair $(x, y) \in \mathcal{X} \times \mathcal{X}$ such that $x = gx = F(x, y)$ and $y = gy = F(y, x)$.*

Proof. Following the proof of Theorem 2.1, there exists $(x, y) \in \mathcal{X} \times \mathcal{X}$ such that $F(x, y) = gx = p$ and $F(y, x) = gy = q$. Thus the existence of a coupled coincidence point is confirmed. Now, let (x^*, y^*) be another coincidence point of F and g ; that is, $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$. By the additional assumption, there is $(u, v) \in \mathcal{X} \times \mathcal{X}$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$.

Let $u_0 = u, v_0 = v, x_0 = x, y_0 = y, x_0^* = x^*$ and $y_0^* = y^*$. Since $F(\mathcal{X} \times \mathcal{X}) \subseteq g\mathcal{X}$, we can construct the sequences $(gu_n), (gv_n), (gx_n), (gy_n), (gx_n^*),$ and (gy_n^*) , such that $gu_{n+1} = F(u_n, v_n), gv_{n+1} = F(v_n, u_n), gx_{n+1} = F(x_n, y_n), gy_{n+1} = F(y_n, x_n), gx_{n+1}^* = F(x_n^*, y_n^*)$ and $gy_{n+1}^* = F(y_n^*, x_n^*)$. Since

$$(gx, gy) = (F(x, y), F(y, x)) = (gx_1, gy_1),$$

and

$$(F(u, v), F(v, u)) = (gu_1, gv_1),$$

are comparable, then $gx \preceq gu_1$ and $gv_1 \preceq gy$. One can show that $gx \preceq gu_n$, and $gv_n \preceq gy$ for all $n \in \mathbb{N}$. From

$$d(gx, gu_{n+1}) = d(F(x, y), F(u_n, v_n)) \leq Ad(gx, gu_n) + Bd(gy, gv_n),$$

and

$$d(gy, gv_{n+1}) = d(F(v_n, u_n), F(y, x)) \leq Ad(gv_n, gy) + Bd(gu_n, gx),$$

we have

$$d(gx, gu_{n+1}) + d(gy, gv_{n+1}) \leq (A + B)(d(gx, gu_n) + d(gy, gv_n)).$$

Since

$$d(gx, gu_{n+1}) \leq d(gx, gu_n) + d(gy, gv_{n+1}),$$

we have

$$\begin{aligned} d(gu_{n+1}, gx) &\leq (A + B)(d(gx, gu_n) + d(gy, gv_n)) \\ &\leq (A + B)^2(d(gx, gu_{n-1}) + d(gy, gv_{n-1})) \end{aligned}$$

$$\begin{aligned} & \vdots \\ & \leq (A + B)^{n+1} (d(gx, gu) + d(gy, gv)). \end{aligned}$$

Thus, gu_{n+1} converges to gx in (\mathcal{X}, d) . Similarly, we may show that gv_{n+1} converges to gy in (\mathcal{X}, d) . Analogously, we can show that gu_{n+1} converges to gx^* and gv_{n+1} converges to gy^* in (\mathcal{X}, d) . Since (gu_{n+1}) converges to gx and gx^* , we get $gx = gx^*$. Also, since (gv_{n+1}) converges to gy and gy^* , we get $gy = gy^*$. Thus, if (x, y) and (x^*, y^*) are coupled coincidence points of F and g , then

$$F(x, y) = gx = gx^* = F(x^*, y^*),$$

and

$$F(y, x) = gy = gy^* = F(y^*, x^*).$$

Assume that F and g commute, then

$$gp = g(gx) = g(F(x, y)) = F(gx, gy) = F(p, q),$$

and

$$gq = g(gy) = g(F(y, x)) = F(gy, gx) = F(q, p).$$

Hence, the pair (p, q) is also a coupled coincidence point of F and g . Thus, we have $gp = gx$ and $gq = gy$. Hence $gp = p$ and $gq = q$. Therefore $p = gp = F(p, q)$ and $q = gq = F(q, p)$.

Thus (p, q) is a coupled common fixed point of F and g . To prove the uniqueness, let (s, t) be any coupled common fixed point of F and g . Then $s = gs = F(s, t)$ and $t = gt = F(t, s)$.

Since the pair (s, t) is a coupled coincidence point of F and g , we have $gs = gx$ and $gt = gy$. Thus $s = gs = gp = p$ and $t = gt = gq = q$. This shows that the coupled fixed point is unique. \square

3. APPLICATION IN LINEAR MATRIX EQUATIONS

Consider the linear matrix equations of the type

$$(3.1) \quad X - A_1^* X A_1 - \cdots - A_m^* X A_m = Q,$$

and

$$(3.2) \quad X + A_1^* X A_1 + \cdots + A_m^* X A_m = Q,$$

where Q is a positive definite matrix and A_1, \dots, A_m are arbitrary $n \times n$ matrices. We denote the set of all $n \times n$ matrices, $n \times n$ Hermitian matrices and $n \times n$ positive definite matrices by $M(n)$, $H(n)$ and $P(n)$, respectively. Clearly, we have the chain $P(n) \subseteq H(n) \subseteq M(n)$. Consider

the spectral norm, $\|A\| = \sqrt{\lambda^+(A^*A)}$ where $\lambda^+(A^*A)$ is the largest eigenvalue of A^*A . Define maps G and K on $H(n)$ by

$$(3.3) \quad G(X) = Q + \sum_{i=1}^m A_i^* X A_i, \quad K(X) = Q - \sum_{i=1}^m A_i^* X A_i.$$

The fixed points of G are solutions of (3.1) and the fixed points of K are solutions of (3.2). Fixed point theorems for these functions are studied in [6]. Now, we extend the above equations as follows:

$$(3.4) \quad \begin{cases} X - A_1^* X A_1 - \cdots - A_m^* X A_m = Q, \\ Y - B_1^* Y B_1 - \cdots - B_t^* Y B_t = Q', \end{cases}$$

and

$$(3.5) \quad \begin{cases} X + A_1^* X A_1 + \cdots + A_m^* X A_m = Q, \\ Y + B_1^* Y B_1 + \cdots + B_t^* Y B_t = Q', \end{cases}$$

where Q and Q' are positive definite matrices, A_1, \dots, A_m and B_1, \dots, B_m are arbitrary $n \times n$ matrices. Define maps $F_1 : H(n) \times H(n) \times H(n) \times H(n) = H(n)^4 \rightarrow H(n) \times H(n)$ and $F_2 : H(n) \times H(n) \times H(n) \times H(n) \rightarrow H(n) \times H(n)$ as follows

$$(3.6) \quad F_1(U, V) = \left(Q + \sum_{i=1}^m A_i^* (X + X') A_i, Q' + \sum_{i=1}^t B_i^* (Y + Y') B_i \right),$$

and

$$(3.7) \quad F_2(U, V) = \left(Q - \sum_{i=1}^m A_i^* (X + X') A_i, Q' - \sum_{i=1}^t B_i^* (Y + Y') B_i \right),$$

where $U = (X, Y)$ and $V = (X', Y')$. We consider the trace norm $\|\cdot\|_1$ on $H(n)$ as $\|A\|_1 = \sum_{i=1}^n s_i(A)$, where $s_i(A)$'s are singular values of A . For $Q \in P(n)$ we define $\|A\|_{1,Q} = \|Q^{\frac{1}{2}} A Q^{\frac{1}{2}}\|_1$. Then $H(n)$ by this norm becomes a complete metric space for any positive definite Q . Let \preceq be a partially order on $H(n)$, as defined in [6] (we use \preceq and \triangleleft instead of \leq and $<$, respectively, on $H(n)$). We consider the partial order on $H(n) \times H(n)$ as follows

$$(3.8) \quad (X, Y) \preceq (X', Y') \Leftrightarrow X \preceq X' \text{ and } Y \preceq Y'.$$

Similarly, we can extend this for $H(n)^4$ as follows

$$(3.9) \quad (U, V) \tilde{\preceq} (U', V') \Leftrightarrow U \preceq U' \text{ and } V \preceq V'.$$

We define $\|\cdot\|_{(1,1),(Q,Q')} : H(n) \times H(n) \rightarrow \mathbb{R}^2$ by

$$\begin{aligned} \|(A, B)\|_{(1,1),(Q,Q')} &= (\|A\|_{1,Q}, \|B\|_{1,Q'}) \\ &= \left(\left\| Q^{\frac{1}{2}} A Q^{\frac{1}{2}} \right\|_1, \left\| Q'^{\frac{1}{2}} B Q'^{\frac{1}{2}} \right\|_1 \right). \end{aligned}$$

Clearly, $H(n) \times H(n)$ equipped with the above metric is a complete metric space for any positive definite Q and Q' .

Theorem 3.1. *Let $Q, Q' \in P(n)$ such that*

$$\begin{aligned} \sum_{i=1}^m A_i^* Q A_i \preceq Q, \quad \sum_{i=1}^t B_i^* Q' B_i \preceq Q', \\ \left\| \sum_{i=1}^m Q^{-\frac{1}{2}} A_i^* Q A_i Q^{-\frac{1}{2}} \right\| < \frac{1}{2} \end{aligned}$$

and

$$\left\| \sum_{i=1}^t Q'^{-\frac{1}{2}} B_i^* Q' B_i Q'^{-\frac{1}{2}} \right\| < \frac{1}{2}.$$

Then F_2 has a unique coupled fixed point in $H(n)$.

Proof. Let $(U, V), (U', V') \in H(n)^4$ such that $(U, V) \widetilde{\preceq} (U', V')$, where

$$U = (X_1, Y_1), \quad V = (X'_1, Y'_1), \quad U' = (X_2, Y_2),$$

and $V' = (X'_2, Y'_2)$. Take $U_0 = (0, 0)$ and $V_0 = (Q, Q')$. Then

$$F_2(U_0, V_0) \preceq V_0, \quad U_0 \preceq F_2(U_0, V_0).$$

Furthermore, for given $U = (X_1, Y_1), V = (X'_1, Y'_1), U' = (X_2, Y_2)$ and $V' = (X'_2, Y'_2)$, we have

$$\begin{aligned} &\|F_2(U', V') - F_2(U, V)\|_{(1,1),(Q,Q')} \\ &= \left\| \left(\sum_{i=1}^m A_i^* (X_2 + X'_2 - X_1 - X'_1) A_i, \sum_{i=1}^t B_i^* (Y_2 + Y'_2 - Y_1 - Y'_1) B_i \right) \right\|_{(1,1),(Q,Q')} \\ &= \left(\operatorname{tr} \left(\sum_{i=1}^m Q^{\frac{1}{2}} A_i^* (X_2 + X'_2 - X_1 - X'_1) A_i Q^{\frac{1}{2}} \right), \right. \\ &\quad \left. \operatorname{tr} \left(\sum_{i=1}^t Q'^{\frac{1}{2}} B_i^* (Y_2 + Y'_2 - Y_1 - Y'_1) B_i Q'^{\frac{1}{2}} \right) \right) \\ &= \left(\operatorname{tr} \left(\sum_{i=1}^m Q^{-\frac{1}{2}} A_i^* Q A_i Q^{-\frac{1}{2}} Q^{\frac{1}{2}} (X_2 + X'_2 - X_1 - X'_1) Q^{\frac{1}{2}} \right), \right. \\ &\quad \left. \operatorname{tr} \left(\sum_{i=1}^t Q'^{-\frac{1}{2}} B_i^* Q' B_i Q'^{-\frac{1}{2}} Q'^{\frac{1}{2}} (Y_2 + Y'_2 - Y_1 - Y'_1) Q'^{\frac{1}{2}} \right) \right) \\ &= \left(\operatorname{tr} \left(\left(\sum_{i=1}^m (Q^{-\frac{1}{2}} A_i^* Q A_i Q^{-\frac{1}{2}}) (Q^{\frac{1}{2}} (X_2 + X'_2 - X_1 - X'_1) Q^{\frac{1}{2}}) \right) \right) \right), \end{aligned}$$

$$\begin{aligned}
& tr \left(\left(\sum_{i=1}^t Q'^{-\frac{1}{2}} B_i^* Q' B_i Q'^{-\frac{1}{2}} \right) \left(Q'^{\frac{1}{2}} (Y_2 + Y_2' - Y_1 - Y_1') Q'^{\frac{1}{2}} \right) \right) \\
\leq & \left(\left\| \sum_{i=1}^m Q^{-\frac{1}{2}} A_i^* Q A_i Q^{-\frac{1}{2}} \right\| \|X_2 + X_2' - X_1 - X_1'\|_{1,Q}, \right. \\
& \left. \left\| \sum_{i=1}^t Q'^{-\frac{1}{2}} B_i^* Q' B_i Q'^{-\frac{1}{2}} \right\| \|Y_2 + Y_2' - Y_1 - Y_1'\|_{1,Q'} \right) \\
= & \begin{bmatrix} \left\| \sum_{i=1}^m Q^{-\frac{1}{2}} A_i^* Q A_i Q^{-\frac{1}{2}} \right\| & 0 \\ 0 & \left\| \sum_{i=1}^t Q'^{-\frac{1}{2}} B_i^* Q' B_i Q'^{-\frac{1}{2}} \right\| \end{bmatrix} \\
& \times (\|X_2 + X_2' - X_1 - X_1'\|_{1,Q}, \|Y_2 + Y_2' - Y_1 - Y_1'\|_{1,Q'}).
\end{aligned}$$

In the above statements, we used Lemma 3.1 of [6]. Set

$$\alpha = \left\| \sum_{i=1}^m Q^{-\frac{1}{2}} A_i^* Q A_i Q^{-\frac{1}{2}} \right\|,$$

and

$$\beta = \left\| \sum_{i=1}^t Q'^{-\frac{1}{2}} B_i^* Q' B_i Q'^{-\frac{1}{2}} \right\|.$$

Then

$$\begin{aligned}
& \|F_2(U', V') - F_2(U, V)\|_{(1,1),(Q,Q')}, \\
& \leq \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} (\|X_2 - X_1\|_{1,Q} + \|X_2' - X_1'\|_{1,Q}, \|Y_2 - Y_1\|_{1,Q'} + \|Y_2' - Y_1'\|_{1,Q'}) \\
& = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} ((\|X_2 - X_1\|_{1,Q}, \|Y_2 - Y_1\|_{1,Q'}) + (\|X_2' - X_1'\|_{1,Q}, \|Y_2' - Y_1'\|_{1,Q'})) \\
& = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} (\|U' - U\|_{(1,1),(Q,Q')}, \|V - V'\|_{(1,1),(Q,Q')}).
\end{aligned}$$

Now apply Theorem 2.1. \square

The above theorem says that, under the conditions of this theorem, the equation system (3.5) has a unique solution.

4. APPLICATION IN NONLINEAR MATRIX EQUATIONS

In this section, we study the following class of nonlinear matrix equations system:

$$(4.1) \quad \begin{cases} X + A_1^* \mathcal{F}(X) A_1 + \cdots + A_m^* \mathcal{F}(X) A_m = Q, \\ Y + B_1^* \mathcal{G}(Y) B_1 + \cdots + B_t^* \mathcal{G}(Y) B_t = Q', \end{cases}$$

where Q and Q' are positive definite matrices, A_1, \dots, A_m and B_1, \dots, B_t are arbitrary $n \times n$ matrices and \mathcal{F} and \mathcal{G} are continuous maps, from $P(n) \times P(n)$ into $P(n)$, where $n \geq 3$. We define $\mathfrak{F} : H(n) \times H(n) \times H(n) \times H(n) = H(n)^4 \rightarrow H(n) \times H(n)$ as follows

$$(4.2) \quad \mathfrak{F}(U, V) = \left(Q - \sum_{i=1}^m A_i^* \mathcal{F}((X, Y)) A_i, Q' - \sum_{i=1}^t B_i^* \mathcal{G}((X', Y')) B_i \right),$$

for every $U = (X, Y), V = (X', Y') \in H(n) \times H(n)$. Clearly, if \mathfrak{F} has a unique coupled fixed point then (4.1) has a unique solution. We consider the norm $\|\cdot\| : H(n) \times H(n) \rightarrow \mathbb{R}^2$ such that

$$(4.3) \quad \|(A, B)\|_{1,1} = (\|A\|_1, \|B\|_1), \quad A, B \in H(n).$$

Before considering the system (4.1), we take into account the result obtained in [6] for the following nonlinear matrix equation:

$$(4.4) \quad X + A_1^* \mathcal{F}(X) A_1 + \dots + A_m^* \mathcal{F}(X) A_m = Q,$$

where Q and A_1, \dots, A_m are as above. By reviewing the proof of Theorem 4.1, we reach to a gap in the assumption. At first, note that the space of all $n \times n$, $M(n)$ is a unital Banach algebra with of unit I_n , where I_n is identity the matrix $n \times n$. Authors in [6] considered the Banach algebra $M(n)$ with two norms $\|A\| = \sqrt{\lambda^+(A^*A)}$ and $\|A\|_1 = \sum_{i=1}^n s_i(A)$ for $A \in M(n)$. It is known that all norms on a finite dimensional Banach algebra are equivalent. It is rutin in the Banach algebra theory that we assume the norm of identity element is equale to 1 (this holds when we use the spectral norm). Now, we use the other norm i.e., $\|I_n\|_1 = \sum_{i=1}^n s_i(I_n) = n$. We rewrite Theorem 4.1 and investigate it.

Let $Q \in P(n)$. Assume that there exists a positive number M for which $\sum_{j=1}^m A_j A_j^* < M \cdot I_n$ and such that for $X \leq Y$ we have

$$(4.5) \quad |\text{tr}(\mathcal{F}(Y) - \mathcal{F}(X))| \leq \frac{1}{M} |\text{tr}(Y - X)|.$$

Then (4.4) has a unique solution in $P(n)$.

In the proof of the above stated result, authors proved the following

$$\|\mathcal{G}(Y) - \mathcal{G}(X)\|_1 \leq \left\| \sum_{i=1}^m A_i A_i^* \right\| \|\mathcal{F}(Y) - \mathcal{F}(X)\|_1.$$

We consider the proof with two norms $\|\cdot\|$ (spectral norm) and $\|\cdot\|_1$. For $\|\cdot\|_1$, by using of (4.5) the condition (1) of Theorem 2.1 does not hold. Because

$$\|\mathcal{G}(Y) - \mathcal{G}(X)\|_1 \leq \left\| \sum_{i=1}^m A_i A_i^* \right\| \|\mathcal{F}(Y) - \mathcal{F}(X)\|_1$$

$$\begin{aligned}
&\leq M \|I_n\|_1 |tr(\mathcal{F}(Y) - \mathcal{F}(X))| \\
&\leq \|I_n\|_1 |tr(Y - X)| \\
&= n |tr(Y - X)|.
\end{aligned}$$

Thus, the above statement does not satisfy condition (1) of Theorem 2.1 of [6]. If we use the spectral norm, then we obtain $\|\mathcal{G}(Y) - \mathcal{G}(X)\|_1 \leq |tr(Y - X)|$, again it dose not satisfy in condition (1), note that in condition (1), there is a $0 < c < 1$. Now consider the following conditions on M and (4.5):

- (i) $M > 1$ and $|tr(\mathcal{F}(Y) - \mathcal{F}(X))| \leq \frac{1}{(n+1)M} |tr(Y - X)|$.
- (ii) $M > n$ and $|tr(\mathcal{F}(Y) - \mathcal{F}(X))| \leq \frac{1}{M^2} |tr(Y - X)|$.

We write the correction of Theorem 4.1 of [6] with our paper notations for partial orders as follows:

Theorem 4.1. *Let $Q \in P(n)$. Assume that there exists a positive number M for which $\sum_{j=1}^m A_j A_j^* \triangleleft M \cdot I_n$ and such that for $X \leq Y$ we have one of the conditions (i) or (ii) holds. Then (4.4) has a unique solution in $P(n)$.*

Now, we consider the nonlinear matrix equation system (4.1) as follows:

Theorem 4.2. *Let $Q, Q' \in P(n)$ and the following statements hold.*

- (i) $\sum_{i=1}^m A_i^* \mathcal{F}((X, Y)) A_i \triangleleft Q$ and $\sum_{i=1}^t B_i^* \mathcal{G}((X', Y')) B_i \triangleleft Q'$.
- (ii) *There exist positive numbers M, M' such that*

$$\sum_{i=1}^m A_i^* A_i \triangleleft M \cdot I_n, \quad \sum_{i=1}^t B_i^* B_i \triangleleft M' \cdot I_t,$$

and satisfy in one of the following conditions:

1. $M, M' > 1$ and for every $(U, V), (U', V') \in H(n)^4$,
$$\begin{aligned}
&(|tr(\mathcal{F}(U) - \mathcal{F}(U'))|, |tr(\mathcal{G}(V) - \mathcal{G}(V'))|) \\
&\leq \left(\frac{1}{(n+1)M} |tr(U - U')|, \frac{1}{(t+1)M'} |tr(V - V')| \right).
\end{aligned}$$
2. $M > n, M' > t$ and for every $(U, V), (U', V') \in H(n)^4$,
$$\begin{aligned}
&(|tr(\mathcal{F}(U) - \mathcal{F}(U'))|, |tr(\mathcal{G}(V) - \mathcal{G}(V'))|) \\
&\leq \left(\frac{1}{M^2} |tr(U - U')|, \frac{1}{M'^2} |tr(V - V')| \right).
\end{aligned}$$

Then \mathfrak{F} has a unique coupled fixed point in $H(n)$.

Proof. Let $(U, V), (U', V') \in H(n)^4$ such that $(U, V) \stackrel{\sim}{\preceq} (U', V')$, where $U = (X_1, Y_1), V = (X'_1, Y'_1), U' = (X_2, Y_2)$ and $V' = (X'_2, Y'_2)$. Take $U_0 = (0, 0)$ and $V_0 = (Q, Q')$. Then $\mathfrak{F}(U_0, V_0) \preceq V_0$ and $U_0 \preceq \mathfrak{F}(U_0, V_0)$. Furthermore, for given $U = (X_1, Y_1), V = (X'_1, Y'_1), U' = (X_2, Y_2)$ and $V' = (X'_2, Y'_2)$ we have

$$\begin{aligned}
& \|\mathfrak{F}(U', V') - \mathfrak{F}(U, V)\|_{(1,1)} \\
&= \left\| \left(\sum_{i=1}^m A_i^* (\mathcal{F}(X_2, Y_2) - \mathcal{F}(X_1, Y_1)) A_i, \sum_{i=1}^t B_i^* (\mathcal{G}(X'_2, Y'_2) - \mathcal{G}(X'_1, Y'_1)) B_i \right) \right\|_{(1,1)} \\
&= \left(\sum_{i=1}^m \text{tr}(A_i A_i^* (\mathcal{F}(X_2, Y_2) - \mathcal{F}(X_1, Y_1))) , \sum_{i=1}^t \text{tr}(B_i^* B_i (\mathcal{G}(X'_2, Y'_2) - \mathcal{G}(X'_1, Y'_1))) \right) \\
&= \left(\text{tr} \left(\sum_{i=1}^m A_i^* A_i (\mathcal{F}(X_2, Y_2) - \mathcal{F}(X_1, Y_1)) \right), \right. \\
&\quad \left. \text{tr} \left(\sum_{i=1}^t B_i^* B_i (\mathcal{G}(X'_2, Y'_2) - \mathcal{G}(X'_1, Y'_1)) \right) \right) \\
&= \left(\text{tr} \left(\sum_{i=1}^m (A_i^* A_i) (\mathcal{F}(X_2, Y_2) - \mathcal{F}(X_1, Y_1)) \right), \right. \\
&\quad \left. \text{tr} \left(\left(\sum_{i=1}^t B_i^* B_i \right) (\mathcal{G}(X'_2, Y'_2) - \mathcal{G}(X'_1, Y'_1)) \right) \right) \\
&\leq \left(\left\| \sum_{i=1}^m A_i^* A_i \right\| \|\mathcal{F}(X_2, Y_2) - \mathcal{F}(X_1, Y_1)\|_1, \right. \\
&\quad \left. \left\| \sum_{i=1}^t B_i^* B_i \right\| \|\mathcal{G}(X'_2, Y'_2) - \mathcal{G}(X'_1, Y'_1)\|_1 \right) \\
&= \begin{bmatrix} \left\| \sum_{i=1}^m A_i^* A_i \right\| & 0 \\ 0 & \left\| \sum_{i=1}^t B_i^* B_i \right\| \end{bmatrix} \\
&\quad \times (\|\mathcal{F}(X_2, Y_2) - \mathcal{F}(X_1, Y_1)\|_1, \|\mathcal{G}(X'_2, Y'_2) - \mathcal{G}(X'_1, Y'_1)\|_1).
\end{aligned}$$

In the above statements, we used Lemma 3.1 of [6]. Suppose that (1) holds. Then,

$$\begin{aligned}
& \|\mathfrak{F}(U', V') - \mathfrak{F}(U, V)\|_{(1,1)} \\
&\leq \begin{bmatrix} M\|I_n\|_1 & 0 \\ 0 & M'\|I_t\|_1 \end{bmatrix} \left(\frac{1}{(n+1)M} (\|X_2 - X_1\|_1 + \|Y_2 - Y_1\|_1), \right. \\
&\quad \left. \frac{1}{(t+1)M'} (\|X'_2 - X'_1\|_1 + \|Y'_2 - Y'_1\|_1) \right) \\
&= \begin{bmatrix} \frac{1}{(n+1)} & 0 \\ 0 & \frac{1}{(t+1)} \end{bmatrix} ((\|X_2 - X_1\|_1 + \|Y'_2 - Y'_1\|_1),
\end{aligned}$$

$$\begin{aligned} & (\|Y_2 - Y_1\|_1 + \|X'_2 - X'_1\|_1) \\ = & \begin{bmatrix} \frac{1}{(n+1)} & 0 \\ 0 & \frac{1}{(t+1)} \end{bmatrix} (\|U' - U\|_{(1,1)}, \|V - V'\|_{(1,1)}). \end{aligned}$$

We obtain similar results for case (2), that provide the required conditions in Theorem 2.1. This completes the proof. \square

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