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A Version of Favard's Inequality for the Sugeno Integral

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Abstract. In this paper, we present a version of Favard's inequality for special case and then generalize it for the Sugeno integral in fuzzy measure space (X, Σ, μ), where μ is the Lebesgue measure. We consider two cases, when our function is concave and when is convex. In addition for illustration of theorems, several examples are given.

1. Introduction

The theory of fuzzy measures and fuzzy integrals was introduced by Sugeno as a tool for modeling nondeterministic problems [26]. From a mathematical point of view, it has very interesting properties which have been studied by many authors, including Ralescu and Adams [22], Román-Flores et al. [23] and Wang and Klir [27], among others. The Sugeno integral is a useful tool in several theoretical and applied statistics. For instance, in decision theory, the Sugeno integral is a median, which is indeed a qualitative counterpart to the averaging operation underlying expected utility.

In 2007, Román-Flores et al. started the study of inequalities for fuzzy integrals [16, 24]. Other authers followed it, for example, Y. Ouang et al. proved an inequality related to Minkowski-type [21] and generalized Chebyshev-type inequality [20]. H. Agahi et al. proved Minkowski-type inequality [3], generalized it [1] and presented Berwald's inequality [2]. J. Caballero and et al. proved Hermite-Hadamard's inequality [4], Carlson's inequality [5] and Sandor's inequality [6] and B. Daraby et al. presented an inequality related to Carlson-type for these integrals [7]

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(see more [8–15]). Recently D. Hong et al. and M. Kaluszka et al. have proved Steffensen's inequality for fuzzy integrals [17, 19].

The following theorem is the classical Favard's inequality.

Theorem 1.1 ([18])**.** *Let f be a concave non-negative function on $[a,b] \subset \mathbb{R}$. If $q > 1$, then*

$$\frac{2^q}{q+1}\left(\frac{1}{b-a}\int_a^b f(x)dx\right)^q \geq \frac{1}{b-a}\int_a^b f^q(x)dx. \tag{1.1}$$

If $0 < q < 1$, the reverse inequality holds in (1.1)

The aim of this paper is to present a fuzzy version of the above theorem. Moreover, we present some examples to illustrate our results.

This paper is organized as follows: In Section 2, some notations and concepts have been introduced. In Section 3, we prove main results and give some examples. In Section 4 we express our conclusions.

2. Preliminaries

As usual, we denote by $\mathbb{R}$, the set of real numbers. Let X be a non-empty set and let Σ be a σ-algebra of subsets of X. Throught this paper, all considered subsets are supposed to belong to Σ.

Definition 2.1. ([22]). A set function $\mu : \Sigma \to [0, +\infty]$ is called a fuzzy measure if the following properties are satisfied:

(F1) $\mu(\emptyset) = 0$;
(F2) $A, B \in \Sigma$ and $A \subset B$ imply $\mu(A) \leq \mu(B)$;
(F3) $\{A_i\} \subset \Sigma$, $A_1 \subset A_2 \subset \ldots$ and $\bigcup_{i=1}^{\infty} A_i \in \Sigma$ imply $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim \mu(A_i)$;
(F4) $\{A_i\} \subset \Sigma$, $A_1 \supset A_2 \supset \ldots$, $\mu(A_1) < \infty$ and $\bigcap_{i=1}^{\infty} A_i \in \Sigma$ imply $\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim \mu(A_i)$.

The triplet (X, Σ, μ) is called a fuzzy measure space.

We denote the set of all non-negative measurable functions with respect to Σ by $\mathcal{F}_+(X)$. Let f be a non-negative real-valued function defined on X. We denote the set $\{x \in X \mid f(x) \geq \alpha\}$ by F_α for $\alpha \geq 0$.

Definition 2.2 ([22, 27])**.** Let μ be a fuzzy measure on (X, Σ). If $f \in \mathcal{F}_+(X)$ and $A \in \Sigma$, then the Sugeno integral (or fuzzy integral) of f on A, with respect to the fuzzy measure μ, is defined as

$$⨍_A f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(A \cap F_\alpha)),$$

where $\vee$, $\wedge$ denotes the operation sup and inf on $[0,\infty)$ respectively. In particular if $A = X$, then

$$\fint_X f d\mu = \fint f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(F_\alpha)).$$

The following properties of the Sugeno integral can be found in [22, 27].

Theorem 2.3. *Let (X, Σ, μ) be a fuzzy measure space, then*

(a) $\mu(A \cap F_\alpha) \geq \alpha \Rightarrow \fint_A f d\mu \geq \alpha$;
(b) $\mu(A \cap F_\alpha) \leq \alpha \Rightarrow \fint_A f d\mu \leq \alpha$;
(c) $\fint_A f d\mu < \alpha \Leftrightarrow$ *there exists* $\gamma < \alpha$ *such that* $\mu(A \cap F_\gamma) < \alpha$;
(d) $\fint_A f d\mu > \alpha \Leftrightarrow$ *there exists* $\gamma > \alpha$ *such that* $\mu(A \cap F_\gamma) > \alpha$;
(e) $f \leq g$ *on* A, *then* $\fint_A f d\mu \leq \fint_A g d\mu$;
(f) $\fint_A k d\mu = k \wedge \mu(A)$, *for* k *non-negative constant;*
(g) $\fint_A f d\mu \leq \mu(A)$.

Remark 2.4. Let F be distribution function associated to f on A, that is,

$$F(\alpha) = \mu(A \cap \{f \geq \alpha\}).$$

Then, due to (a) and (b) of Theorem 2.3, it is very important to note that

$$F(\alpha) = \alpha \quad \Rightarrow \quad \fint_A f d\mu = \alpha.$$

Thus, from a numerical point of view, the Sugeno integral can be calculated by solving the equation $F(\alpha) = \alpha$.

Jensen's inequality for the Sugeno integral is an important inequality which has many application in the proof of some theorems. In the following Lemma, we remark it.

Lemma 2.5 ([25])**.** *Let $f : [0,1] \to [0,\infty)$ be a measurable function with μ the Lebesgue measure and $s \geq 1$. Then*

$$\left(\fint_0^1 f d\mu\right)^s \leq \fint_0^1 f^s d\mu.$$

3. Main results

The following examples show that Inequality (1.1) is not valid for the Sugeno integral.

Example 3.1. Let $f(x) = \sqrt{x}$, $a = 0$, $b = 1$ and $q = 1.2$. If μ is the Lebesgue measure, then

$\frac{1}{1-0} \fint_0^1 \sqrt{x}^{1.2} d\mu = \fint_0^1 x^{0.6} \approx 0.5877$,

$\frac{2^{1.2}}{2.2}(⨍_0^1 \sqrt{x}d\mu)^{1.2} \approx 1.0443 \times 0.5613 \approx 0.5861$,
but $0.5877 \not\leq 0.5861$.

Example 3.2. Let $f(x) = x$, $a = 0$, $b = 1$ and $q = 0.5$. If μ is the Lebesgue measure, then
$\frac{1}{1-0} ⨍_0^1 x^{0.5}d\mu \approx 0.6180$,
$\frac{2^{0.5}}{1.5}(⨍_0^1 xd\mu)^{0.5} \approx 0.9428 \times 0.7071 \approx 0.6667$,
but $0.6180 \not\geq 0.6667$.

In the sequel, we prove Favard's inequality for the Sugeno integral. First, we present Favard's inequality for special case and then generalize it.

Theorem 3.3. *Let $f : [0,1] \to [0,\infty)$ be a concave function and μ be the Lebesgue measure on $\mathbb{R}$. Then for any $q > 0$,*

(a) *if $f(1) > f(0)$, then*

$$⨍_0^1 f^q d\mu \geq \min\left\{\frac{2^q}{q+1}\left(⨍_0^1 fd\mu\right)^q, 1 - \frac{\frac{2}{(q+1)^{\frac{1}{q}}} ⨍_0^1 fd\mu - f(0)}{f(1)-f(0)}\right\}, \tag{3.1}$$

(b) *if $f(1) = f(0)$, then*

$$⨍_0^1 f^q d\mu \geq \min\{f^q(0), 1\}, \tag{3.2}$$

(c) *if $f(0) > f(1)$, then*

$$⨍_0^1 f^q d\mu \geq \min\left\{\frac{2^q}{q+1}\left(⨍_0^1 fd\mu\right)^q, \frac{f(0) - \frac{2}{(q+1)^{\frac{1}{q}}} ⨍_0^1 fd\mu}{f(0)-f(1)}\right\}. \tag{3.3}$$

Proof. We know that $x = (1-x).0 + x.1$, for any $x \in [0,1]$. So by concavity of f, we have

$$f(x) \geq (1-x)f(0) + xf(1) = h(x).$$

Using Theorem 2.3 (e), we get $⨍_0^1 f^q d\mu \geq ⨍_0^1 h^q d\mu$.

(a) If $f(1) > f(0)$,

$$\begin{aligned} ⨍_0^1 f^q d\mu &\geq ⨍_0^1 h^q d\mu \\ &= \bigvee_{\alpha\geq 0}\left(\alpha \wedge \mu\left([0,1] \cap \left\{x|h(x) \geq \alpha^{\frac{1}{q}}\right\}\right)\right) \end{aligned}$$

$$= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\mu\left([0,1]\cap\left\{x|x\geq\frac{\alpha^{\frac{1}{q}}-f(0)}{f(1)-f(0)}\right\}\right)\right)$$
$$= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\left(1-\frac{\alpha^{\frac{1}{q}}-f(0)}{f(1)-f(0)}\right)\right).$$

If we assume that $\alpha = \frac{2^q}{q+1}(⨍_0^1 fd\mu)^q$, then

$$\bigvee_{\alpha\geq 0}\left(\alpha\wedge\left(1-\frac{\alpha^{\frac{1}{q}}-f(0)}{f(1)-f(0)}\right)\right)\geq\frac{2^q}{q+1}\left(⨍_0^1 fd\mu\right)^q\wedge\left(1-\frac{\frac{2}{(q+1)^{\frac{1}{q}}}⨍_0^1 fd\mu-f(0)}{f(1)-f(0)}\right).$$

It follows that

$$⨍_0^1 f^q d\mu\geq\min\left\{\frac{2^q}{q+1}\left(⨍_0^1 fd\mu\right)^q, 1-\frac{\frac{2}{(q+1)^{\frac{1}{q}}}⨍_0^1 fd\mu-f(0)}{f(1)-f(0)}\right\}$$

(b) If $f(0)=f(1)$, then $h(x)=f(0)=f(1)$ and using Theorem 2.3 (e,f), we have

$$⨍_0^1 f^q d\mu\geq⨍_0^1 h^q d\mu$$
$$=⨍_0^1 f^q(0)d\mu$$
$$=f^q(0)\wedge 1.$$

(c) If $f(0)>f(1)$,

$$⨍_0^1 f^q d\mu\geq⨍_0^1 h^q d\mu$$
$$= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\mu\left([0,1]\cap\left\{x|h(x)\geq\alpha^{\frac{1}{q}}\right\}\right)\right)$$
$$= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\mu\left([0,1]\cap\left\{x|x\leq\frac{f(0)-\alpha^{\frac{1}{q}}}{f(0)-f(1)}\right\}\right)\right)$$
$$= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\frac{f(0)-\alpha^{\frac{1}{q}}}{f(0)-f(1)}\right).$$

Again, if we assume that $\alpha = \frac{2^q}{q+1}(⨍_0^1 fd\mu)^q$, then

$$\bigvee_{\alpha\geq 0}\left(\alpha\wedge\frac{f(0)-\alpha^{\frac{1}{q}}}{f(0)-f(1)}\right)\geq\frac{2^q}{q+1}\left(⨍_0^1 fd\mu\right)^q\wedge\left(\frac{f(0)-\frac{2}{(q+1)^{\frac{1}{q}}}⨍_0^1 fd\mu}{f(0)-f(1)}\right).$$

It follows that

$$\fint_0^1 f^q d\mu \geq \left\{ \frac{2^q}{q+1} \left(\fint_0^1 f d\mu \right)^q , \frac{f(0) - \frac{2}{(q+1)^{\frac{1}{q}}} \fint_0^1 f d\mu}{f(0) - f(1)} \right\}$$

Which completes the proof. □

For illustration of Theorem 3.3 we present some examples.

Example 3.4. Let $f(x) = \sqrt{x}$ and $q = 2$. In this case, $f(1) > f(0)$ and simple calculations show that $\fint_0^1 \sqrt{x} d\mu \approx 0.6180$, $\fint_0^1 \sqrt{x}^2 d\mu = \fint_0^1 x d\mu = 0.5$ and

$$\begin{aligned} 0.5 \geq \min & \left\{ \frac{4}{3}(0.6180)^2, 1 - \frac{\frac{2}{\sqrt{3}} \times 0.6180 - 0}{1-0} \right\} \\ & \approx \min\{0.5092, 0.2864\} \\ & = 0.2864. \end{aligned}$$

Example 3.5. Let $f(x) = (1+x)(2-x)$ and $q = 1$. Then $f(0) = f(1)$ and $\fint_0^1 (1+x)(2-x) d\mu = 1$ and we know that $1 \geq 2 \wedge 1 = 1$.

Example 3.6. Let $f(x) = 1 - x^2$ and $q = 3$. We have $f(0) > f(1)$ and $\fint_0^1 1 - x^2 d\mu \approx 0.6180$, $\fint_0^1 (1-x^2)^3 d\mu \approx 0.4711$, so

$$\begin{aligned} 0.4711 \geq \min & \left\{ \frac{8}{4}(0.6180)^3, \frac{1 - \frac{2}{\sqrt[3]{4}} \times 0.6180}{1-0} \right\} \\ & \approx \min\{0.7638, 0.2214\} \\ & = 0.2214. \end{aligned}$$

The following example shows that concavity of f in Theorem 3.3 is necessary.

Example 3.7. Suppose that $f(x) = x^4$ and $q = 0.5$. In this case, $f(1) > f(0)$ and simple calculations show that
$\fint_0^1 x^4 d\mu \approx 0.2755$ and $\fint_0^1 (x^4)^{0.5} d\mu = \fint_0^1 x^2 d\mu \approx 0.3820$, but

$$\begin{aligned} 0.3820 \ngeq \min & \left\{ \frac{\sqrt{2}}{1.5}(0.2755)^{0.5}, 1 - \frac{\frac{2}{\sqrt{1.5}} \times 0.2755 - 0}{1-0} \right\} \\ & \approx \min\{0.4948, 0.5501\} \\ & = 0.4948. \end{aligned}$$

Now, we replace the domain of f in Theorem 3.3 by $[a, b] \subseteq \mathbb{R}$.

Theorem 3.8. *Let $f : [a, b] \to [0, \infty)$ be a concave function and μ be the Lebesgue measure on $\mathbb{R}$. Then for any $q > 0$,*

(a) *if* $f(b) > f(a)$, *then*

$$\fint_a^b f^q d\mu \geq \min\left\{ \frac{2^q(b-a)}{q+1}\left(\frac{1}{b-a}\fint_a^b f d\mu\right)^q, \right. \tag{3.4}$$
$$\left. b - \frac{2(\frac{b-a}{q+1})^{\frac{1}{q}} \fint_a^b f d\mu + af(b) - bf(a)}{f(b)-f(a)} \right\},$$

(b) *if* $f(b) = f(a)$, *then*

$$\fint_a^b f^q d\mu \geq \min\{f^q(a), b-a\}, \tag{3.5}$$

(c) *if* $f(a) > f(b)$, *then*

$$\fint_a^b f^q d\mu \geq \min\left\{ \frac{2^q(b-a)}{q+1}\left(\frac{1}{b-a}\fint_a^b f d\mu\right)^q, \right. \tag{3.6}$$
$$\left. \frac{bf(a) - af(b) - 2(\frac{b-a}{q+1})^{\frac{1}{q}} \fint_a^b f d\mu}{f(a)-f(b)} - a \right\}.$$

Proof. Assume that $x \in [a, b]$. If we set $x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b$, then by concavity of f we have

$$f(x) \geq \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b) = h(x).$$

By Theorem 2.3 (e), we have $\fint_a^b f^q d\mu \geq \fint_a^b h^q d\mu$.

(a) If $f(b) > f(a)$,

$$\begin{aligned}
\fint_a^b f^q d\mu &\geq \fint_a^b h^q d\mu \\
&= \bigvee_{\alpha \geq 0} \left(\alpha \wedge \mu\left([a,b] \cap \left\{x | h(x) \geq \alpha^{\frac{1}{q}}\right\}\right)\right) \\
&= \bigvee_{\alpha \geq 0} \left(\alpha \wedge \mu\left([a,b] \cap \left\{x | x \geq \frac{\alpha^{\frac{1}{q}}(b-a) + af(b) - bf(a)}{f(b)-f(a)}\right\}\right)\right) \\
&= \bigvee_{\alpha \geq 0} \left(\alpha \wedge \left(b - \frac{\alpha^{\frac{1}{q}}(b-a) + af(b) - bf(a)}{f(b)-f(a)}\right)\right).
\end{aligned}$$

If we assume that $\alpha = \frac{2^q(b-a)}{(q+1)}\left(\frac{1}{b-a}\fint_a^b f d\mu\right)^q$, then

$$\bigvee_{\alpha \geq 0} \left(\alpha \wedge \left(b - \frac{\alpha^{\frac{1}{q}}(b-a) + af(b) - bf(a)}{f(b)-f(a)}\right)\right)$$

$$\geq \frac{2^q(b-a)}{q+1}\left(\frac{1}{b-a}\fint_a^b f d\mu\right)^q$$
$$\wedge\left(b-\frac{2(\frac{b-a}{q+1})^{\frac{1}{q}}\fint_a^b fd\mu+af(b)-bf(a)}{f(b)-f(a)}\right).$$

It follows that

$$\fint_a^b f^q d\mu \geq \min\left\{\frac{2^q(b-a)}{q+1}\left(\frac{1}{b-a}\fint_a^b f d\mu\right)^q, b-\frac{2(\frac{b-a}{q+1})^{\frac{1}{q}}\fint_a^b fd\mu+af(b)-bf(a)}{f(b)-f(a)}\right\}.$$

(b) If $f(a)=f(b)$, then $h(x)=f(a)=f(b)$ and using Theorem 2.3 (e,f), we have

$$\begin{aligned}\fint_a^b f^q d\mu &\geq \fint_a^b h^q d\mu\\ &= \fint_a^b f^q(a) d\mu\\ &= f^q(a)\wedge(b-a).\end{aligned}$$

(c) If $f(a)>f(b)$,

$$\begin{aligned}\fint_a^b f^q d\mu &\geq \fint_a^b h^q d\mu\\ &= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\mu\left([a,b]\cap\left\{x|h(x)\geq\alpha^{\frac{1}{q}}\right\}\right)\right)\\ &= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\mu\left([a,b]\cap\left\{x|x\leq\frac{bf(a)-af(b)-\alpha^{\frac{1}{q}}(b-a)}{f(a)-f(b)}\right\}\right)\right)\\ &= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\left(\frac{bf(a)-af(b)-\alpha^{\frac{1}{q}}(b-a)}{f(a)-f(b)}-a\right)\right).\end{aligned}$$

Again, if we assume that $\alpha=\frac{2^q(b-a)}{(q+1)}\left(\frac{1}{b-a}\fint_a^b fd\mu\right)^q$, then

$$\bigvee_{\alpha\geq 0}\left(\alpha\wedge\left(\frac{bf(a)-af(b)-\alpha^{\frac{1}{q}}(b-a)}{f(a)-f(b)}-a\right)\right)$$
$$\geq \frac{2^q(b-a)}{q+1}\left(\frac{1}{b-a}\fint_a^b f d\mu\right)^q$$
$$\wedge\left(\frac{bf(a)-af(b)-2(\frac{b-a}{q+1})^{\frac{1}{q}}\fint_a^b fd\mu}{f(a)-f(b)}-a\right).$$

It follows that

$$\fint_a^b f^q d\mu \geq \min\left\{\frac{2^q(b-a)}{q+1}\left(\frac{1}{b-a}\fint_a^b f d\mu\right)^q, \frac{bf(a)-af(b)-2(\frac{b-a}{q+1})^{\frac{1}{q}}\fint_a^b f d\mu}{f(a)-f(b)} - a\right\}.$$

Now, the proof is complete. □

In the sequel, we consider convex functions and prove reverse of the mentioned inequalities in Theorems 3.3 and 3.8.

Theorem 3.9. *Let $f : [0,1] \to [0,\infty)$ be a convex function and μ be the Lebesgue measure on $\mathbb{R}$. Then for any $q > 0$,*

(a) *if $f(1) > f(0)$, then*

(3.7)
$$\fint_0^1 f^q d\mu \leq \max\left\{\frac{2^q}{q+1}\left(\fint_0^1 f d\mu\right)^q, 1 - \frac{\frac{2}{(q+1)^{\frac{1}{q}}}\fint_0^1 f d\mu - f(0)}{f(1)-f(0)}\right\},$$

(b) *if $f(1) = f(0)$, then*

$$\fint_0^1 f^q d\mu \leq \min\{f^q(0), 1\}, \tag{3.8}$$

(c) *if $f(0) > f(1)$, then*

$$\fint_0^1 f^q d\mu \leq \max\left\{\frac{2^q}{q+1}\left(\fint_0^1 f d\mu\right)^q, \frac{f(0) - \frac{2}{(q+1)^{\frac{1}{q}}}\fint_0^1 f d\mu}{f(0)-f(1)}\right\}. \tag{3.9}$$

Proof. We know that $x = (1-x).0 + x.1$, for any $x \in [0,1]$. So by convexity of f, we have

$$f(x) \leq (1-x)f(0) + xf(1) = h(x),$$

Using Theorem 2.3 (e), we get $\fint_0^1 f^q d\mu \leq \fint_0^1 h^q d\mu$.

(a) If $f(1) > f(0)$,

$$\begin{aligned}\fint_0^1 f^q d\mu &\leq \fint_0^1 h^q d\mu \\ &= \bigvee_{\alpha\geq 0}\left(\alpha \wedge \mu\left([0,1] \cap \left\{x | h(x) \geq \alpha^{\frac{1}{q}}\right\}\right)\right) \\ &= \bigvee_{\alpha\geq 0}\left(\alpha \wedge \mu\left([0,1] \cap \left\{x | x \geq \frac{\alpha^{\frac{1}{q}} - f(0)}{f(1)-f(0)}\right\}\right)\right)\end{aligned}$$

$$\leq \bigwedge_{\alpha \geq 0} \left(\alpha \vee \mu \left([0,1] \cap \left\{ x | x \geq \frac{\alpha^{\frac{1}{q}} - f(0)}{f(1) - f(0)} \right\} \right) \right)$$
$$= \bigwedge_{\alpha \geq 0} \left(\alpha \vee \left(1 - \frac{\alpha^{\frac{1}{q}} - f(0)}{f(1) - f(0)} \right) \right).$$

If we assume that $\alpha = \frac{2^q}{q+1} (⨍_0^1 f d\mu)^q$, then

$$\bigwedge_{\alpha \geq 0} \left(\alpha \vee \left(1 - \frac{\alpha^{\frac{1}{q}} - f(0)}{f(1) - f(0)} \right) \right) \leq \frac{2^q}{q+1} \left(⨍_0^1 f d\mu \right)^q$$
$$\vee \left(1 - \frac{\frac{2}{(q+1)^{\frac{1}{q}}} ⨍_0^1 f d\mu - f(0)}{f(1) - f(0)} \right).$$

It follows that

$$⨍_0^1 f^q d\mu \leq \max \left\{ \frac{2^q}{q+1} \left(⨍_0^1 f d\mu \right)^q, 1 - \frac{\frac{2}{(q+1)^{\frac{1}{q}}} ⨍_0^1 f d\mu - f(0)}{f(1) - f(0)} \right\}.$$

(b) If $f(0) = f(1)$, then $h(x) = f(0) = f(1)$ and by using Theorem 2.3 (e, f), we have

$$⨍_0^1 f^q d\mu \leq ⨍_0^1 h^q d\mu$$
$$= ⨍_0^1 f^q(0) d\mu$$
$$= f^q(0) \wedge 1.$$

(c) If $f(0) > f(1)$,

$$⨍_0^1 f^q d\mu \leq ⨍_0^1 h^q d\mu$$
$$= \bigvee_{\alpha \geq 0} \left(\alpha \wedge \mu \left([0,1] \cap \left\{ x | h(x) \geq \alpha^{\frac{1}{q}} \right\} \right) \right)$$
$$= \bigvee_{\alpha \geq 0} \left(\alpha \wedge \mu \left([0,1] \cap \left\{ x | x \leq \frac{f(0) - \alpha^{\frac{1}{q}}}{f(0) - f(1)} \right\} \right) \right)$$
$$\leq \bigwedge_{\alpha \geq 0} \left(\alpha \vee \mu \left([0,1] \cap \left\{ x | x \leq \frac{f(0) - \alpha^{\frac{1}{q}}}{f(0) - f(1)} \right\} \right) \right)$$

$$= \bigwedge_{\alpha \geq 0} \left(\alpha \vee \frac{f(0) - \alpha^{\frac{1}{q}}}{f(0) - f(1)} \right).$$

Again if we assume that $\alpha = \frac{2^q}{q+1} (⨍_0^1 f d\mu)^q$, then

$$\bigwedge_{\alpha \geq 0} \left(\alpha \vee \frac{f(0) - \alpha^{\frac{1}{q}}}{f(0) - f(1)} \right) \leq \frac{2^q}{q+1} \left(⨍_0^1 f d\mu \right)^q \vee \left(\frac{f(0) - \frac{2}{(q+1)^{\frac{1}{q}}} ⨍_0^1 f d\mu}{f(0) - f(1)} \right).$$

Hence

$$⨍_0^1 f^q d\mu \leq \max \left\{ \frac{2^q}{q+1} \left(⨍_0^1 f d\mu \right)^q, \frac{f(0) - \frac{2}{(q+1)^{\frac{1}{q}}} ⨍_0^1 f d\mu}{f(0) - f(1)} \right\}.$$

Which completes the proof. □

Now, for illustration of Theorem 3.9, we present some examples.

Example 3.10. Suppose that $f(x) = x^2$ and $q = 0.5$. Then, $f(1) > f(0)$ and a straightforward calculus shows that
$⨍_0^1 x^2 d\mu \approx 0.3820$,
$⨍_0^1 (x^2)^{0.5} d\mu = ⨍_0^1 x d\mu = 0.5$.
On the other hand, we have

$$\begin{aligned} 0.5 &\leq \max \left\{ \frac{\sqrt{2}}{1.5} (0.3820)^{0.5}, 1 - \frac{\frac{2}{(1.5)^2} 0.3820 - 0}{1 - 0} \right\} \\ &\approx \max\{0.5827, 0.6604\} \\ &= 0.6604. \end{aligned}$$

Example 3.11. If we assume that $f(x) = x^2 - x + 1$ and $q = 1$, then $f(0) = f(1)$ and a simple calculation shows that $⨍_0^1 (x^2 - x + 1) d\mu = 0.75$. Thus

$$0.75 \leq f(0) \wedge 1 = 1 \wedge 1 = 1.$$

Example 3.12. By assumption of $f(x) = e^{-x}$ and $q = 2$, we have $f(0) > f(1)$. Simple calculations give
$⨍_0^1 e^{-x} d\mu \approx 0.5671$,
$⨍_0^1 (e^{-x})^2 d\mu = ⨍_0^1 e^{-2x} d\mu \approx 0.4263$.
It follows that

$$\begin{aligned} 0.4263 &\leq \max \left\{ \frac{4}{3} (0.5671)^2, \frac{1 - \frac{2}{\sqrt{3}} 0.5671}{1 - e^{-1}} \right\} \\ &\approx \max\{0.4288, 0.5461\} \\ &= 0.5461. \end{aligned}$$

The next example shows that the convexity of f in Theorem 3.9 is inevitable.

Example 3.13. Let $f(x) = \ln(1+x)$ and $q = 3$. Then $f(1) > f(0)$ and simple calculations show that
$⨍_0^1 \ln(1+x)d\mu \approx 0.4429$,
$⨍_0^1 (\ln(1+x))^3 d\mu \approx 0.2020$.
And we know that

$$\begin{aligned} 0.2020 &\nleq \max\left\{\frac{8}{4}(0.4429)^3, 1 - \frac{\frac{2}{\sqrt[3]{4}}0.4429 - 0}{\ln 2 - 0}\right\} \\ &= \max\{0.1738, 0.1950\} \\ &= 0.1950. \end{aligned}$$

Theorem 3.14. *Let $f : [a,b] \to [0,\infty)$ be a convex function and μ be the Lebesgue measure on $\mathbb{R}$. Then for any $q > 0$,*

(a) *if $f(b) > f(a)$, then*

$$⨍_a^b f^q d\mu \leq \max\left\{\frac{2^q(b-a)}{q+1}\left(\frac{1}{b-a}⨍_a^b f d\mu\right)^q, b - \frac{2(\frac{b-a}{q+1})^{\frac{1}{q}} ⨍_a^b f d\mu + af(b) - bf(a)}{f(b) - f(a)}\right\}, \tag{3.10}$$

(b) *if $f(b) = f(a)$, then*

$$⨍_a^b f^q d\mu \leq \min\{f^q(a), b - a\}, \tag{3.11}$$

(c) *if $f(a) > f(b)$, then*

$$⨍_a^b f^q d\mu \leq \max\left\{\frac{2^q(b-a)}{q+1}\left(\frac{1}{b-a}⨍_a^b f d\mu\right)^q, \frac{bf(a) - af(b) - 2(\frac{b-a}{q+1})^{\frac{1}{q}} ⨍_a^b f d\mu}{f(a) - f(b)} - a\right\}. \tag{3.12}$$

Proof. The proof proceeding is similar to the proof of Theorem 3.8 and so we leave it. □

Remark 3.15. Let f be a measurable function and μ be the Lebesgue measure on $\mathbb{R}$. By Lemma 2.5 we know that for any $q \geq 1$,

$$⨍_0^1 f^q d\mu \geq \left(⨍_0^1 f d\mu\right)^q,$$

on the other hand, when $q \geq 1$, we have $2^q \geq q + 1$. Since the product operation is increasing on positive numbers, so

$$⨍_0^1 f^q d\mu \geq \frac{q+1}{2^q}\left(⨍_0^1 f d\mu\right)^q.$$

Remark 3.16. If we let $0 < q < 1$ in Theorems 3.3, 3.8, 3.9 and 3.14, then the mentioned inequalities in these theorems will be the special case of Berwald's inequality that H. Agahi et al. have proven in [2].

4. Conclusion

In the present paper, we have proven a version of Favard's inequality for the Sugeno integral and illustrated this version by some examples. In the future we will try to prove this inequality for pseudo-integrals.

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