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Strong Convergence of the Iterations of Quasi ϕ-nonexpansive Mappings and its Applications in Banach Spaces

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Abstract. In this paper, we study the iterations of quasi ϕ-nonexpansive mappings and its applications in Banach spaces. At the first, we prove strong convergence of the sequence generated by the hybrid proximal point method to a common fixed point of a family of quasi ϕ-nonexpansive mappings. Then, we give applications of our main results in equilibrium problems.

1. Introduction and Preliminaries

We denote the dual of a real Banach space E with E^*, its norm with $\|.\|$ and the value of $v \in E^*$ at $x \in E$ by $\langle x, v\rangle$. The mapping $J : E \to 2^{E^*}$ defined by

$$Jx = \left\{v \in E^* : \langle x, v\rangle = \|x\|^2 = \|v\|^2\right\}$$

for $x \in E$, is called the duality mapping. A Banach space E for which $\left\|\frac{x+y}{2}\right\| < 1$ for any $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$, is called strictly convex. Also, it is called uniformly convex if for any $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|\frac{x+y}{2}\| < 1 - \delta$, for any $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$. We know that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is called smooth if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists for all $x, y \in U = \{z \in E : \|z\| = 1\}$. If for all $x, y \in U$, the limit (1.1) is attained uniformly, then E is called the uniformly smooth

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Banach space. The strong convergence of a sequence $\{x_k\}$ in E to $x \in E$ is denoted by $x_k \to x$ and its weak convergence by $x_k \rightharpoonup x$.

For a smooth Banach space E, we will use the following function

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{1.2}$$

for any $x, y \in E$ which is used in [1] by Alber, in [4] by Kamimura and Takahashi and in [10] by Reich. It can be shown from the definition of ϕ that

$$0 \leq (\|x\| - \|y\|)^2 \leq \phi(x,y). \tag{1.3}$$

Since the duality mapping is the identity operator in Hilbert spaces, so $\phi(x,y) = \|x-y\|^2$, if E is a Hilbert space.

Proposition 1.1 ([4]). *Suppose that $\{x_k\}$ and $\{y_k\}$ are two sequences in a uniformly convex and smooth Banach space E. If $\phi(x_k, y_k)$ tends to zero, as $k \to \infty$ and either $\{x_k\}$ or $\{y_k\}$ is bounded, then* $\lim_{k\to\infty} \|x_k - y_k\| = 0$.

Proposition 1.2 ([4]). *Let C be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space E and $x \in E$. Then we can find a unique element $x_0 \in C$, such that*

$$\phi(x_0, x) = \inf\{\phi(z,x) : z \in C\}.$$

Regarding Proposition 1.2, we denote the unique element $x_0 \in C$ by $P_C(x)$, where the mapping P_C is called the generalized projection from E onto C. It is well known that the generalized projection mapping P_C is coincident with the metric projection from E onto C in Hilbert spaces. We also need the following proposition to prove the strong convergence in Section 3.

Proposition 1.3 ([4]). *Let C be a convex subset of a smooth Banach space E, $x \in E$ and $x_0 \in C$. Then*

$$\phi(x_0, x) = \inf\{\phi(z,x) : z \in C\}$$

if and only if

$$\langle z - x_0, Jx - Jx_0\rangle \leq 0, \quad \forall z \in C.$$

Let C be a closed and convex subset of a Banach space E. We denote the set of all fixed points of a mapping $T : C \to C$ by $F(T)$, i.e. $F(T) = \{x \in C : Tx = x\}$.
T is called a nonexpansive mapping if and only if for any $x, y \in C$,

$$\|Tx - Ty\| \leq \|x - y\|$$

and T is called a quasi-nonexpansive mapping, whenever $F(T) \neq \varnothing$ and for any $(q,x) \in F(T) \times C$,

$$\|Tx - q\| \leq \|x - q\|.$$

Regarding the definitions of nonexpansive and quasi-nonexpansive mappings, we recall now the definitions of ϕ-nonexpansive and quasi ϕ-nonexpansive mappings in Banach spaces.

Definition 1.4. A mapping $T : C \to C$ is said to be ϕ-nonexpansive if and only if

$$\phi(Tx, Ty) \leq \phi(x, y), \quad \forall x, y \in C$$

and T is said to be quasi ϕ-nonexpansive, whenever $F(T) \neq \varnothing$ and

$$\phi(q, Tx) \leq \phi(q, x), \quad \forall (q, x) \in F(T) \times C.$$

This paper is organized as follows. In Section 2, we study iterations of quasi ϕ-nonexpansive mappings and prove strong convergence of their iterations to a common fixed point of the sequence of quasi ϕ-nonexpansive mappings. Finally, in Section 3, we give applications of our main results in equilibrium problems.

2. Main Results

Suppose that C is a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $T_k : C \to C$ be a sequence of quasi ϕ-nonexpansive mappings with $\bigcap_k F(T_k) \neq \emptyset$. Now, we are going to study strong convergence of the sequence $\{x_k\}$ generated by the following algorithm.

Algorithm 1

1: **Initialize:**
Take $x_0 \in C$, $n := 0$ and $\gamma_k \in [\varepsilon, \frac{1}{2}]$ for some $\varepsilon \in (0, \frac{1}{2}]$ and $k = 0, 1, 2, \ldots$

2: **Step 1:**

$$z_n = T_n x_n \tag{2.1}$$

3: **Step 2:**
Determine the next approximation x_{n+1} as the projection of x_0 onto the intersection $H_n \cap W_n$, $x_{n+1} = P_{H_n \cap W_n}(x_0)$, where

$$H_n = \{z \in C : \langle z - x_n, Jx_n - Jz_n \rangle \leq -\gamma_n \phi(x_n, z_n)\};$$

$$W_n = \{z \in C : \langle z - x_n, Jx_0 - Jx_n \rangle \leq 0\}.$$

4: **Step 3:**
Set $n := n + 1$ and go back Step 1.

In order to prove the optimality of weak limit points, it is need to define the demiclosedness of a sequence of quasi ϕ-nonexpansive mappings.

Nonexpansive mappings are demiclosed. But for quasi ϕ-nonexpansive mappings, we have to assume this property even in Hilbert spaces. Therefore we introduce the definition of demiclosedness for a sequence of mappings.

Definition 2.1. A sequence $\{T_k\}$ of quasi ϕ-nonexpansive mappings is said to be demiclosed if for each subsequences $\{x_{k_j}\}$ of $\{x_k\}$ and $\{T_{k_j}\}$ of $\{T_k\}$, if

$$x_{k_j} \rightharpoonup p \text{ and } \lim_{j\to\infty} \|T_{k_j}x_{k_j} - x_{k_j}\| = 0, \text{ then } p \in \bigcap_k F(T_k). \tag{2.2}$$

If $T_k \equiv T$, then Definition 2.1 reduces to the defenition of the demiclosedness of T.

Lemma 2.2. *If Algorithm 1 reaches to the iteration step n, then* $\bigcap_k F(T_k) \subset H_n \cap W_n$ *and* x_{n+1} *is well defined.*

Proof. Note that $\bigcap_k F(T_k)$ is closed and convex. Also, H_n and W_n are closed and convex by the definition of them. We now show that $\bigcap_k F(T_k) \subset H_n \cap W_n$ for all $n \geq 0$. Putting

$$C_n = \{z \in C : \phi(z, z_n) \leq \phi(z, x_n)\}.$$

It is easy to see that

$$C_n = \left\{z \in C : \langle z - x_n, Jx_n - Jz_n\rangle \leq \frac{-1}{2}\phi(x_n, z_n)\right\}.$$

Since $\gamma_n \in [\varepsilon, \frac{1}{2}]$, we have $C_n \subseteq H_n$ for all $n \geq 0$. Take $x^* \in \bigcap_k F(T_k)$, Since $\{T_k\}$ is a sequence of quasi ϕ-nonexpansive mapping, then (2.1) shows that $\phi(x^*, z_n) \leq \phi(x^*, x_n)$ for all $x^* \in \bigcap_k F(T_k)$. Therefore $\bigcap_k F(T_k) \subset C_n$ for all $n \geq 0$, that implies $\bigcap_k F(T_k) \subset H_n$ for all $n \geq 0$. Next, we show that $\bigcap_k F(T_k) \subset H_n \cap W_n$, for all $n \geq 0$, by the induction. Indeed, we have $\bigcap_k F(T_k) \subset H_0 \cap W_0$, because $W_0 = C$. Assume that $\bigcap_k F(T_k) \subset H_n \cap W_n$ for some $n \geq 0$. Since $x_{n+1} = P_{H_n \cap W_n}(x_0)$, we have

$$\langle z - x_{n+1}, Jx_0 - Jx_{n+1}\rangle \leq 0, \quad \forall z \in H_n \cap W_n.$$

Since $\bigcap_k F(T_k) \subset H_n \cap W_n$, therefore

$$\langle z - x_{n+1}, Jx_0 - Jx_{n+1}\rangle \leq 0, \quad \forall z \in \bigcap_k F(T_k).$$

$P_{H_n \cap W_n}$ is a projection map, so $\langle z - x_{n+1}, Jx_0 - Jx_{n+1}\rangle \leq 0, \quad \forall z \in \bigcap_k F(T_k)$. Now, the definition of W_{n+1} implies that $\bigcap_k F(T_k) \subset W_{n+1}$ and so $\bigcap_k F(T_k) \subset H_{n+1} \cap W_{n+1}$. From induction it follows that $\bigcap_k F(T_k) \subset C_n \cap W_n \subset H_n \cap W_n$ for all $n \geq 0$. Since $\bigcap_k F(T_k)$ is nonempty, therefore x_{n+1} is well defined. ∎

Lemma 2.3. *Let $\{x_n\}$ and $\{z_n\}$ be the sequences generated by Algorithm 2, then we have $\lim_{n\to\infty} \|x_n - z_n\| = 0$.*

Proof. From the definition of W_n, we have $x_n = P_{W_n}(x_0)$. Let $u \in \bigcap_k F(T_k)$. Since P_{W_n} is the projection map onto W_n, we have

$$\langle u - x_n, Jx_0 - Jx_n\rangle \le 0,$$

this implies that

$$\phi(x_n, x_0) \le \phi(u, x_0). \tag{2.3}$$

Thus, the sequence $\{x_n\}$ is bounded. Moreover, the projection $x_{n+1} = P_{H_n\cap W_n}(x_0)$ implies $x_{n+1} \in W_n$. Similar to the previous case we see that $\phi(x_n, x_0) \le \phi(x_{n+1}, x_0)$. Therefore, the sequence $\{\phi(x_n, x_0)\}$ is non-decreasing and hence convergent. By $x_{n+1} \in W_n$ and $x_n = P_{W_n}(x_0)$, a simple calculation shows that

$$\phi(x_{n+1}, x_n) \le \phi(x_{n+1}, x_0) - \phi(x_n, x_0).$$

Passing to the limit in the above inequality as $n \to \infty$, one gets

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0.$$

Now, by Proposition 1.1, we have $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Since $x_{n+1} \in H_n$, from the definition of H_n, we have

$$\gamma_n\phi(x_n, z_n) \le \langle x_n - x_{n+1}, Jx_n - Jz_n\rangle .$$

Thus $\gamma_n\phi(x_n, z_n) \le \|x_n - x_{n+1}\|\|Jx_n - Jz_n\|$. Since $\gamma_n \ge \varepsilon > 0$ and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, then $\lim_{n\to\infty} \phi(x_n, z_n) = 0$. Now, by Proposition 1.1, we have $\lim_{n\to\infty} \|x_n - z_n\| = 0$. □

Theorem 2.4. *Assume that E is a uniformly convex and uniformly smooth Banach space. Let $T_k : C \to C$ be a sequence of quasi ϕ-nonexpansive mappings and $\{T_k\}$ satisfies* (2.2)*. Then the sequence $\{x_k\}$ generated by Algorithm 2 converges strongly to $P_{\bigcap_k F(T_k)}x_0$.*

Proof. It is clear that $\bigcap_k F(T_k)$ is closed and convex. We define $x^* = P_{\bigcap_k F(T_k)}(x_0)$. Note that from Lemma 2.3, we have

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{2.4}$$

Now, by (2.1), we have $\lim \|T_nx_n - x_n\| = 0$.

On the other hand, note that the sequence $\{x_n\}$ is bounded. Assume that p is any weak limit point of the sequence $\{x_n\}$. Therefore, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p$ as $k \to \infty$. Note that we have $\lim \|T_{n_k}x_{n_k} - x_{n_k}\| = 0$. Now, (2.2) shows that $p \in \bigcap_k F(T_k)$.

In the sequel, we first prove that there exists only one weak limit point of $\{x_n\}$. Finally, we show that $x_n \to x^*$. From the definition of W_n, we have $x_n = P_{W_n}(x_0)$. For $x^* \in \bigcap_k F(T_k) \subset W_n$, since P_{W_n} is the projection map onto W_n, we have $\langle x^* - x_n, Jx_0 - Jx_n \rangle \leq 0$. This implies that $\phi(x_n, x_0) \leq \phi(x^*, x_0)$. Therefore

$$\|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 \leq \phi(x^*, x_0). \tag{2.5}$$

Since $x_{n_k} \rightharpoonup p$ and $\|\cdot\|$ is weak lower semi-continuous, replacing n by n_k in (2.5), we have

$$\begin{aligned}
\phi(p, x_0) &= \|p\|^2 - 2\langle p, Jx_0\rangle + \|x_0\|^2 \\
&\leq \liminf_{n\to\infty} \left(\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_0\rangle + \|x_0\|^2\right) \\
&\leq \phi(x^*, x_0).
\end{aligned}$$

By the definition of x^*, we have $x^* = p$. In the sequel, since $\{x_{n_k}\}$ is a arbitrary subsequence of $\{x_n\}$, we have $x_n \rightharpoonup x^*$. By taking limsup and liminf from (2.5) and by definition of $\phi(x^*, x_0)$, we get $\lim_{n\to\infty} \|x_n\| = \|x^*\|$. Now, note that

$$\begin{aligned}
\lim_{n\to\infty} \phi(x_n, x^*) &= \lim_{n\to\infty} \left(\|x_n\|^2 - 2\langle x_n, Jx^*\rangle + \|x^*\|^2\right) \\
&= 0.
\end{aligned}$$

Therefore by proposition 1.1, we have $x_n \to x^* = P_{\bigcap_k F(T_k)}(x_0)$. □

3. Application in Equilibrium Problems

In this section, we apply our main results to approximate a solution of equilibrium problems. In fact, our main motivation in this section is to approximate a solution of equilibrium problems using quasi-nonexpansive mappings. Let K be a nonempty, closed and convex subset of Banach space E. Suppose that $f : K \times K \to \mathbb{R}$ is a bifunction. An equilibrium problem for f and K (shortly $EP(f;K)$) is to find $x^* \in K$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in K. \tag{3.1}$$

x^* is said to be an equilibrium point. The set of all equilibrium points for (3.1) is denoted by $S(f;K)$. We now recall some monotonicity assumptions on the bifunction $f : K \times K \to \mathbb{R}$:

(i) f is called monotone, whenever $f(x,y) + f(y,x) \leq 0, \forall x, y \in K$.
(ii) f is called pseudomonotone, whenever $f(x,y) \geq 0$ with $x, y \in K$, it holds that $f(y,x) \leq 0$.
(iii) f is called μ-under monotone where μ is the under monotonicity constant of f, if there exists $\mu \geq 0$ such that $f(x,y) + f(y,x) \leq \frac{\mu}{2}(\phi(x,y) + \phi(y,x))$, for all $x, y \in K$.

In [3], the authors have shown that if f satisfies in the following conditions;

(P1) $f(x,x)=0$ for all $x \in K$,
(P2) $f(x,\cdot)$ is lower semi-continuous (lsc) and convex for all $x \in K$,
(P3) $f(\cdot,y)$ is upper semi-continuous for all $y \in K$,
(P4) f is a μ-under monotone bifunction,

then there is the unique point $J_\lambda^f x$ such that

$$f\left(J_\lambda^f x, y\right) + \lambda \left\langle y - J_\lambda^f x, J\left(J_\lambda^f x\right) - Jx \right\rangle \geq 0, \quad \forall y \in K, \tag{3.2}$$

where $x \in E$ and $\lambda > \mu$. In this case, $J_\lambda^f x$ is said to be the resolvent of f of order λ at $x \in E$. Consider $\{\lambda_k\} \subset (\mu, \bar{\lambda}]$, for some $\bar{\lambda} > \mu$ and $x_0 \in E$. The proximal point method for approximating a solution of the equilibrium problem for f defined by $x_{k+1} = J_{\lambda_k}^f x_k$. In (3.2), it is clear that $F(J_\lambda^f) \subseteq S(f;K)$ and if f is pseudomonotone, then we have $S(f;K) \subseteq F(J_\lambda^f)$.

In the sequel, we study the strong convergence of the sequence generated by the hybrid proximal point method. We first propose the algorithm, then we show the sequence generated by the algorithm converges strongly to a solution of the problem.

Algorithm 2

1: **Initialize:**
$x^0 \in E$, $n := 0$, $\mu < \lambda_k \leq \bar{\lambda}$ for some $\bar{\lambda}$, $\gamma_k \in [\varepsilon, \frac{1}{2}]$ for some $\varepsilon \in (0, \frac{1}{2}]$ and $k = 0, 1, 2, \ldots$

2: **Step 1:**
Let z_n be the equilibrium point of $f(x,y) + \lambda_n \langle y - x, Jx - Jx_n \rangle$, i.e.

$$f(z_n, y) + \lambda_n \langle y - z_n, Jz_n - Jx_n \rangle \geq 0, \quad \forall y \in K, \tag{3.3}$$

3: **Step 2:**
Determine the next approximation x_{n+1} as the projection of x_0 onto $H_n \cap W_n$, $x_{n+1} = P_{H_n \cap W_n}(x_0)$, where

$$H_n = \{z \in E : \langle z - x_n, Jx_n - Jz_n \rangle \leq -\gamma_n \phi(x_n, z_n)\},$$
$$W_n = \{z \in E : \langle z - x_n, Jx_0 - Jx_n \rangle \leq 0\}.$$

4: **Step 3:**
Set $n := n + 1$ and go back Step 1.

We show that strong convergence of the sequence generated by Aalgorithm 2 to an equilibrium point of f is a consequence of our main result.

Lemma 3.1. *Suppose that $f : K \times K \to \mathbb{R}$ satisfies P1–P4 and it is pseudomonotone. If $S(f;K) \neq \varnothing$, then $J^f_{\lambda_k}$ is a quasi ϕ-nonexpansive sequence.*

Proof. Let $p \in S(f;K)$ and take $y = p$ in (3.2), we get

$$f\left(J^f_{\lambda_k} x_k, p\right) + \lambda_k \left\langle p - J^f_{\lambda_k} x_k, J\left(J^f_{\lambda_k} x_k\right) - Jx_k \right\rangle \geq 0$$

Since $p \in S(f;K)$ and f is pseudomonotone, therefore we have $f\left(J^f_{\lambda_k} x_k, p\right) \leq 0$. Hence

$$\left\langle p - J^f_{\lambda_k} x_k, J\left(J^f_{\lambda_k} x_k\right) - Jx_k \right\rangle \geq 0$$

which implies that

$$\phi\left(J^f_{\lambda_k} x_k, x_k\right) \leq \phi(p, x_k) - \phi\left(p, J^f_{\lambda_k} x_k\right)$$

Therefore $J^f_{\lambda_k}$ is quasi ϕ-nonexpansive. $\square$

Lemma 3.2. *Suppose that $f : K \times K \to \mathbb{R}$ satisfies P1–P4 and it is pseudomonotone. If $S(f;K) \neq \varnothing$ and $f(\cdot, y)$ is weakly upper semi-continuous for all $y \in K$, then $J^f_{\lambda_k}$ satisfies* (2.2).

Proof. Take $y \in K$. Suppose that the sequence $\{x_k\}$ is arbitrary such that $x_k \rightharpoonup p$ and $\left\| J^f_{\lambda_k} x_k - x_k \right\| \to 0$. We are going to prove $p \in \bigcap_k F(J^f_{\lambda_k})$. We have

$$\begin{aligned} 0 &\leq f\left(J^f_{\lambda_k} x_k, y\right) + \lambda_k \left\langle y - J^f_{\lambda_k} x_k, J\left(J^f_{\lambda_k} x_k\right) - Jx_k \right\rangle \\ &\leq f\left(J^f_{\lambda_k} x_k, y\right) + \lambda_k \|y - J^f_{\lambda_k} x_k\| \|J\left(J^f_{\lambda_k} x_k\right) - Jx_k\|. \end{aligned}$$

Note that $\{\lambda_k\}$ and $\{x_k\}$ are bounded and $\lim_{k\to\infty} \left\| J\left(J^f_{\lambda_k} x_k\right) - Jx_k \right\| = 0$, we have:

$$0 \leq \liminf_{k\to\infty} f\left(J^f_{\lambda_k} x_k, y\right), \quad \forall y \in K. \tag{3.4}$$

Note that since $\lim_{k\to\infty} \|J^f_{\lambda_k} x_k - x_k\| = 0$, we have $J^f_{\lambda_k} x_k \rightharpoonup p$. Now since $f(\cdot, y)$ is weakly upper semi-continuous for all $y \in K$, we obtain

$$0 \leq \liminf_{k\to\infty} f\left(J^f_{\lambda_k} x_k, y\right) \leq \limsup_{k\to\infty} f\left(J^f_{\lambda_k} x_k, y\right) \leq f(p, y),$$

for all $y \in K$. Hence $p \in S(f;K)$ and since f is pseudomonotone, then $p \in \bigcap_k F(J^f_{\lambda_k})$. □

The following theorem is a direct consequence of our main results.

Theorem 3.3. *Suppose that $f : K \times K \to \mathbb{R}$ satisfies P1–P4 and it is pseudomonotone. If $S(f;K) \neq \varnothing$ and $f(\cdot, y)$ is weakly upper semicontinuous for all $y \in K$. Then $\{x_k\}$ generated by the algorithm 2 is strongly convergent to $P_{S(f;K)}x_0$.*

Proof. It follows from Lemmas 3.1, 3.2 and Theorem 2.4. □

Example 3.4. Let $E = \ell^p = \left\{ \xi = (\xi_1, \xi_2, \cdots) : \left(\sum_{i=1}^{\infty} |\xi_i|^p \right)^{\frac{1}{p}} < \infty \right\}$ for $1 < p < \infty$ and let $K = \{\xi = (\xi_1, \xi_2, \xi_3, \ldots) \in \ell^p : \xi_i \geq 0, \forall i \in \mathbb{N}\}$. Define $f : K \times K \to \mathbb{R}$ by $f(x, y) = \langle y - x, Jx \rangle$. It is easy to see that the conditions of Theorem 3.3 are satisfied. Now, if the sequence $\{x_k\}$ is generated by Algorithm 2, then by Theorem 3.3, it converges strongly to an element of $S(f;K)$.

Example 3.5. Suppose that $\psi : E \to \mathbb{R}$ is a convex, proper and lsc function. Define $f(x, y) = \psi(y) - \psi(x)$ and $K = E$. Then the conditions of Theorem 3.3 are satisfied. Now, if the sequence $\{x_k\}$ is generated by Algorithm 2, then by Theorem 3.3, it converges strongly to an element of $S(f;K)$. Finally, It is easy to see that each equilibrium point of f is a minimizer of ψ and vice versa. Therefore Algorithm 2 provides a scheme to approximate a minimizer of ψ.

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