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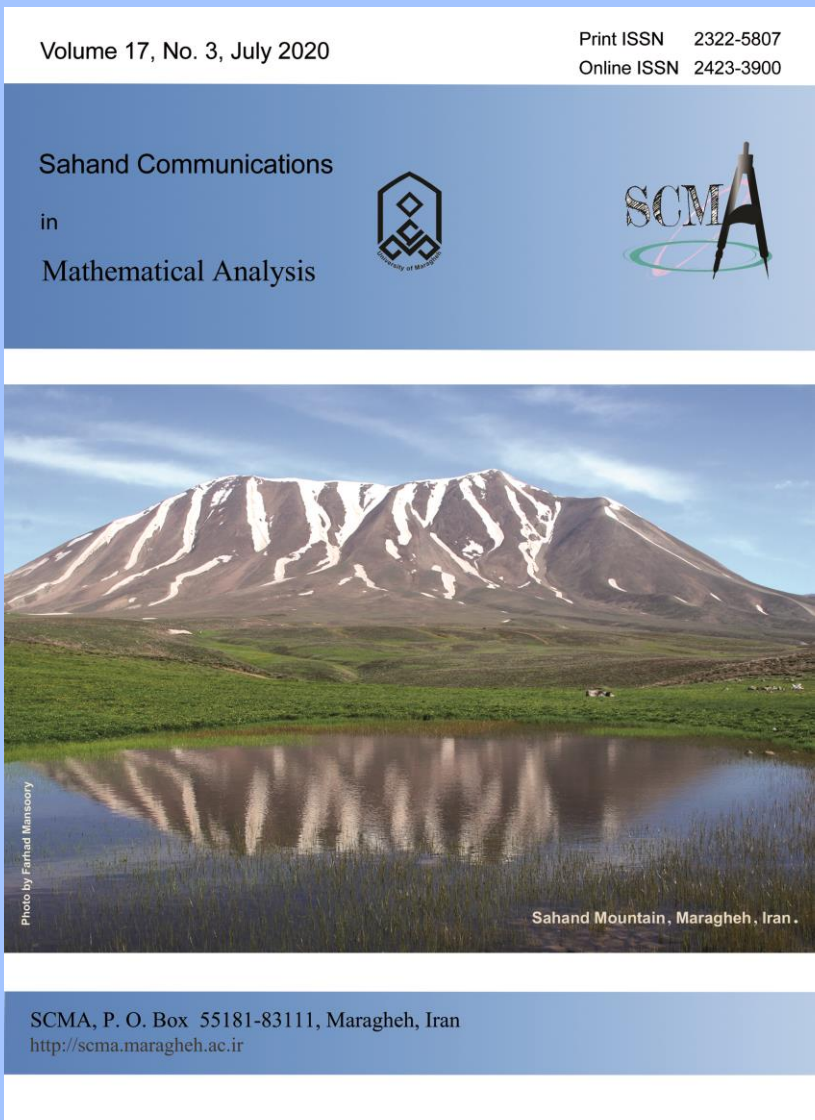

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On Fixed Point Results for Hemicontractive-Type Multi-Valued Mapping, Finite Families of Split Equilibrium and Variational Inequality Problems

Tesfalem Hadush Meche[1] and Habtu Zegeye[2*]

Abstract. In this article, we introduced an iterative scheme for finding a common element of the set of fixed points of a multi-valued hemicontractive-type mapping, the set of common solutions of a finite family of split equilibrium problems and the set of common solutions of a finite family of variational inequality problems in real Hilbert spaces. Moreover, the sequence generated by the proposed algorithm is proved to be strongly convergent to a common solution of these three problems under mild conditions on parameters. Our results improve and generalize many well-known recent results existing in the literature in this field of research.

1. Introduction

Throughout this paper, unless otherwise stated, let H_1 and H_2 be real Hilbert spaces with inner product $\langle ., . \rangle$ and induced norm $\|.\|$ and let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. We denote the strong and weak convergence of any sequence $\{x_n\}$ to x by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Let $S : C \longrightarrow H_1$ be a mapping. We say that the mapping S is $k-$strictly pseudocontractive if there exists $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|x - Sx - (y - Sy)\|^2, \tag{1.1}$$

for all $x, y \in C$. If, in (1.1), $k = 0$ and $k = 1$, the mapping S is said to be nonexpansive and pseudocontractive, respectively. And if there exists

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$L \geq 0$ such that $\|Sx - Sy\| \leq L\|x - y\|$, for all $x, y \in C$, the mapping S is called Lipschitzian.

Observe that the class of nonexpansive mappings is strictly contained in the class of $k-$strictly pseudocontractive mappings which, in turn, strictly contained in the class of pseudocontractive mappings (see [3, 38]).

The mapping S is said to be firmly nonexpansive if

$$\|Sx - Sy\|^2 \leq \langle Sx - Sy, x - y \rangle, \quad \forall x, y \in C.$$

It is well-known that the class of nonexpansive mappings properly includes the class of firmly nonexpansive mappings (see [17]).

A point $x \in C$ is said to be a fixed point of a mapping S if $x = Sx$ and denote by $F(S)$ the set of all fixed points of S.

A mapping $S : C \longrightarrow H_1$ with $F(S) \neq \emptyset$ is said to be demicontractive if there exists a constant $k \in [0, 1)$ such that

$$\|Sx - p\|^2 \leq \|x - p\|^2 + k\|x - Sx\|^2, \quad \forall p \in F(S), x \in C. \tag{1.2}$$

If, in (1.2), $k = 0$ and $k = 1$, the mapping S is said to be quasi-nonexpansive and hemicontractive, respectively.

It is easily observed that the class of hemicontractive mappings properly encloses the classes of pseudocontractive mappings S with $F(S) \neq \emptyset$, quasi-nonexpansive and demicontractive mappings; the class of demicontractive mappings strictly contains the classes of $k-$strictly pseudocontractive mappings S with $F(S) \neq \emptyset$ and quasi-nonexpansive mappings; and the class of quasi-nonexpansive mappings strictly contains the class of nonexpansive mappings S with $F(S) \neq \emptyset$ (see, for example, [15, 35]).

In the sequel, we denote by $CB(C)$ the collection of nonempty, closed and bounded subsets of C.

The Hausdorff metric D on $CB(C)$ is defined by

$$D(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\}, \quad \text{for all } A, B \in CB(C),$$

where $d(x, A) = \inf\{\|x - b\| : b \in A\}$.

A multi-valued mapping $S : C \longrightarrow CB(C)$ is said to be $k-$strictly pseudocontractive if there exists $k \in [0, 1)$ such that

$$D^2(Sx, Sy) \leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2, \tag{1.3}$$

for all $x, y \in C$, $u \in Sx$ and $v \in Sy$.

If, in (1.3), $k = 0$ and $k = 1$, the mapping S is called nonexpansive and pseudocontractive, respectively.

It is not difficult to see from the definitions that every multi-valued nonexpansive mapping is $k-$strictly pseudocontractive mapping and every multi-valued $k-$strictly pseudocontractive mapping is pseudocontractive mapping, however, the inclusions are strict (see [3, 35]).

Recall that the multi-valued mapping S is said to be $L-$Lipschitzian if there exists a constant number $L \geq 0$ such that

$$D(Sx, Sy) \leq L\|x - y\|, \quad \text{for all } x, y \in C.$$

And the set of all fixed points (if exists) of the multi-valued mapping S is denoted by $F(S)$, i.e., $F(S) = \{x \in C : x \in Sx\}$.

Let $S : C \longrightarrow CB(C)$ be a multi-valued mapping with a nonempty fixed point set $F(S)$. The mapping S is said to be quasi-nonexpansive if for all $p \in F(S)$, $x \in C$,

$$D(Sx, Sp) \leq \|x - p\|.$$

If there exists a constant $k \in [0, 1)$ such that

$$D^2(Sx, Sp) \leq \|x - p\|^2 + k\|x - u\|^2,$$

for all $p \in F(S)$, $x \in C$ and $u \in Sx$, the mapping S is called demicontractive-type. Further, the mapping S is said to be hemicontractive-type if

$$D^2(Sx, Sp) \leq \|x - p\|^2 + \|x - u\|^2, \quad \forall p \in F(S), x \in C \text{ and } u \in Sx.$$

The following is an example of hemicontractive-type multi-valued mapping S such that $Sp = \{p\}$ for all fixed point p of S.

Example 1.1. Let $C = [0, \infty)$ and let $S : C \longrightarrow CB(C)$ be defined by

$$Sx = 0 \text{ if } x \leq 2, \qquad Sx = \left[x - \frac{1}{2}, x - \frac{1}{4}\right] \text{ if } x > 2.$$

Then, clearly $F(S) = \{0\}$ and $S0 = \{0\}$. For $0 \leq x \leq 2$, since $Sx = 0$ we have

$$\begin{aligned} D^2(Sx, S0) &= 0 \\ &\leq |x - 0|^2 \\ &\leq |x - 0|^2 + |x - Sx|^2. \end{aligned}$$

And for $x > 2$, we have

$$\begin{aligned} D(Sx, S0) &= \max\left\{\sup_{a \in Sx} d(a, S0), \sup_{b \in S0} d(b, Sx)\right\} \\ &= \max\left\{\sup_{a \in Sx} |a|, d(0, Sx)\right\} \\ &= \max\left\{\left|x - \frac{1}{4}\right|, \left|x - \frac{1}{2}\right|\right\} \end{aligned}$$

$$
\begin{aligned}
&= x - \frac{1}{4} \\
&\le x \\
&= |x - 0|.
\end{aligned}
$$

Thus, $D^2(Sx, S0) \le |x-0|^2 + |x-u|^2$ for all $u \in Sx$ and hence S is a hemicontractive-type multivalued mapping.

We observe that every multi-valued nonexpansive mapping with nonempty set of fixed points is quasi-nonexpansive, every multi-valued $k-$strictly pseudocontractive mapping S with $F(S) \neq \emptyset$ and $S(p) = \{p\}, \quad \forall p \in F(S)$ is demicontractive-type and every multi-valued pseudocontractive mapping S with $F(S) \neq \emptyset$ and $S(p) = \{p\}$, $\forall p \in F(S)$ is hemicontractive-type mapping. It is also easy to see that every multi-valued quasi-nonexpansive mapping is demicontractive-type and every multi-valued demicontractive-type mapping is hemicontractive-type. However, all the inclusions are proper (see, for example, [35, 38]).

The fixed point problem for multi-valued mapping $S : C \longrightarrow CB(C)$ is to find a point $x \in C$ such that $x \in Sx$.

We denote the solution set of the problem by $F(S)$.

Many authors have shown their interest in the existence and approximation of fixed points of nonlinear mappings (including hemicontractive-type mapping) (see, for example, [3, 21, 22, 29, 35] and the references therein).

A mapping $A : C \longrightarrow H_1$ is called monotone if

$$\langle Ax - Ay, x - y\rangle \ge 0, \quad \forall\, x, y \in C.$$

And if there exists a number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \alpha \|Ax - Ay\|^2, \quad \forall\, x, y \in C,$$

then the mapping A is called α-inverse strongly monotone.

It is noticeable that the class of monotone mappings strictly includes the class of α-inverse strongly monotone mappings (see, for example, [38]). Furthermore, every α-inverse strongly monotone mapping is $\frac{1}{\alpha}-$Lipschitzian mapping.

Let $A_m : C \longrightarrow H_1$ be a nonlinear mapping for each $m \in \{1, 2, \ldots, N\}$. The finite family of variational inequality problems is to find a point $u \in C$ such that

$$\langle v - u, A_m u\rangle \ge 0, \quad \text{for all } v \in C,\ m \in \{1, 2, \ldots, N\}. \tag{1.4}$$

The solution set of problem (1.4) is denoted by $VI(C, A_m)$ for each $m \in \{1, 2, \ldots, N\}$. It is easy to see that (1.4) is reduced to the classical variational inequality problem if $N = 1$, which was introduced by Stampacchia [24] as a tool for solving partial differential equations.

Such a problem is related with convex minimization problem, the complementarity problem, the problem of finding a point $x \in C$ satisfying $0 \in Ax$ and etc. Fixed point problems are also closely related to the variational inequality problems. Based on this relationship, iterative methods for finding common solution of variational inequality problem and fixed point problem for some nonlinear mappings have been studied by many authors (see, e.g., [4, 12, 14, 29, 34, 40] and the references therein).

Let $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem, which was initially formulated from variational inequality and optimization by Blum and Oettli [1] in 1994, is to find a point $x \in C$ such that

$$F(x, y) \geq 0, \quad \text{for all } y \in C. \tag{1.5}$$

We denote the set of solutions for problem (1.5) by $EP(F)$. Let $F_m : C \times C \longrightarrow \mathbb{R}$ be a finite family of bifunctions. The finite family of equilibrium problem is to determine common points for the set

$$EP(F_m) = \{p \in C : F_m(p, y) \geq 0, \quad \forall y \in C, m = 1, 2, \ldots, N\},$$

which was studied by Wang and Zhou [33]. Clearly, it is reduced to problem (1.5) when $N = 1$. Various problems arising in physics, optimization, economics, engineering, transportation and etc can be reduced to finding solutions of equilibrium problems. As a result of interaction between different natures of mathematical problems, we now have a variety of methods to analysis several algorithms for finding solutions of equilibrium and related problems. It is also well-known that the equilibrium problems are closely connected with fixed point problems. To find common solutions of these problems, various iterative algorithms have been established and investigated by many researchers in the literature (see, for example, [5, 11, 15, 33, 34, 38, 40] and the references cited therein).

As a generalization of the problem (1.5), Z. He [7] considered the following split equilibrium problem which consists of a pair of equilibrium problems.

Let $F_1 : C \times C \longrightarrow \mathbb{R}$ and $F_2 : Q \times Q \longrightarrow \mathbb{R}$ be two bifunctions and $B : H_1 \longrightarrow H_2$ be a bounded linear operator. The split equilibrium problem (SEP, in short) is the problem of finding a point $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \text{for all } x \in C \ , \tag{1.6}$$

and such that

$$y^* = Bx^* \in Q \text{ solves } F_2(y^*, y) \geq 0, \quad \text{for all } y \in Q \ . \tag{1.7}$$

The solution set of split equilibrium problem (1.6) and (1.7), in this paper, is denoted by Ω. That is,

$$\Omega = \{p \in C : p \in EP(F_1) \text{ and } Bp \in EP(F_2)\}.$$

Split equilibrium problem enable us to solve equilibrium problem (1.6) in H_1 for which the image of its solution under a given bounded linear operator B is a solution of equilibrium problem (1.7) in another Hilbert space H_2. Split variational inequality problem, split zero problem, split fixed point problem, classical equilibrium problem and split feasibility problem are special case of split equilibrium problem, which have already been studied and used in practice, see, e.g, [2, 6, 23, 26, 32, 41].
Let $F_{1,m} : C \times C \longrightarrow \mathbb{R}$ and $F_{2,m} : Q \times Q \longrightarrow \mathbb{R}$ be two finite families of bifunctions and $B_m : H_1 \longrightarrow H_2$ be a finite family of bounded linear operators. The finite family of split equilibrium problems is to find common elements for the following set

$$\Omega_m = \{p \in C : p \in EP(F_{1,m}) \text{ and } B_m p \in EP(F_{2,m}), m = 1, 2, \dots, N\}.$$

If

$$F_{1,m}(x, y) = \langle A_{1,m}x, y - x\rangle, \quad \forall x, y \in C$$

and

$$F_{2,m}(u, v) = \langle A_{2,m}u, v - u\rangle, \quad \forall u, v \in Q,$$

with some nonlinear mappings $A_{1,m} : C \longrightarrow H_1$ and $A_{2,m} : Q \longrightarrow H_2$, then the finite family of split equilibrium problem becomes finite family of split variational inequality problem. For finite families of mappings $S_m : C \longrightarrow C$ and $T_m : Q \longrightarrow Q$, if $F_{1,m}(x, y) = \langle (I - S_m)x, y - x\rangle$ for all $x, y \in C$ and $F_{2,m}(u, v) = \langle (I - T_m)u, v - u\rangle$ for all $u, v \in Q$, then the finite family of split equilibrium problem reduces to finite family of split fixed point problems. Besides, if $H_1 = H_2$, $B_m = I$, $Q = C$ and $F_{2,m} \equiv 0$, for each $m = 1, 2, \dots, N$, then the finite family of split equilibrium problems reduces to the classical finite family of equilibrium problems.

For obtaining a solution of split equilibrium problem, Z. He [7] also proved weak and strong convergence theorems in real Hilbert spaces. Subsequently, to find a common element of the set of fixed points of a nonexpansive single-valued self-mapping S and the sets of solutions of split equilibrium and variational inequality problems, Kazmi and Rizvi [10] proposed the following iterative algorithm:

$$(1.8) \qquad \begin{cases} u_n = T^{F_1}_{\tau_n}(x_n + \gamma B^*(T^{F_2}_{\tau_n} - I)Bx_n, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n v + \beta_n x_n + \gamma_n S y_n, \qquad \forall n \geq 0, \end{cases}$$

where F_2 is upper semi-continuous in the first argument and B is a bounded linear operator. Then, under certain conditions on parameters, the authors proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to a common solution of these three problems. Recently, Meche et al.[16] extended the results of [10] to multi-valued nonexpansive mapping and removed the assumption imposed in [10] that F_2 is upper semi-continuous in the first argument. In particular, the authors considered the following iterative algorithm for obtaining a common solution of a split equilibrium problem, a variational inequality problem for Lipschitz monotone mapping A and a fixed point problem for non-expansive multi-valued mapping S:

$$\begin{cases} x_0 \in C, \\ z_n = T_\sigma^{F_1}(I - \lambda B^*(I - T_\tau^{F_2})B)x_n, \\ u_n = P_C[z_n - \gamma_n A z_n], \\ y_n = P_C[z_n - \gamma_n A u_n], \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n)(\gamma_n x_n + (1-\gamma_n)v_n), \end{cases} \tag{1.9}$$

for all $n \geq 0$, where $B : H_1 \longrightarrow H_2$ is bounded linear operator with its adjoint B^*, $v_n \in Sy_n$, f a contraction mapping and the control sequences satisfy mild conditions. It was proved in [16] that the sequence $\{x_n\}$ generated by (1.9) converges strongly to the same point $p \in \Theta = F(S) \bigcap \Omega \bigcap VI(C, A)$, where $p = P_\Theta f(p)$. Furthermore, Okeke and Mewomo [19] proposed the following iterative algorithm and obtained strong convergence result for approximating a common solution of variational inequality problem, split equilibrium problem and fixed point problem for multi-valued quasi-nonexpansive mapping S in real Hilbert space:

$$\begin{cases} x_1 \in H_1, \\ u_n = T_{\tau_n}^{F_1}(x_n + \gamma_n B^*(T_{\tau_n}^{F_2} - I)Bx_n), \\ y_n = P_C[u_n - \lambda_n A u_n], \\ x_{n+1} = \alpha_n f_n(x_n) + \beta_n x_n + \delta_n(\sigma w_n + (1-\sigma)y_n), \quad \forall n \geq 1, \end{cases}$$

where $w_n \in Sx_n$, B is a bounded linear operator, A is an inverse strongly monotone mapping from C into H_1, F_2 is upper semi-continuous in the first argument and the sequences $\{r_n\}, \{\gamma_n\}, \{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ satisfy some appropriate conditions.

On the other hand, approximating common solution of a family of split equilibrium problem is an important and active research area. Iterative algorithms for finding a common point of a family of split equilibrium problems, variational inequality problems and fixed point of some nonlinear mappings have received vast consideration by several authors [8, 11, 28, 31]. In 2016, Wang et al [31] proposed an iterative algorithm and proved some strong convergence theorems for finding a common element of the set of common solutions of a finite family of split equilibrium problems and the set of common fixed points of a countable family of nonexpansive mappings in Hilbert spaces. In [28], Ugwunnadi and Ali relaxed the results of Wang et al [31] to continuous pseudocontractive mappings and introduced an iterative algorithm for finding common solution of finite family of split equilibrium problem, fixed point problem for finite family of continuous pseudocontractive mappings and variational inequality problem in Hilbert spaces. Under some appropriate conditions on parameters, they also proved that the sequence generated by the algorithm convergence strongly to a common solution of these problems. However, it is worthy to mention that the results in [31] and [28] restricted to single-valued nonlinear mappings.

Motivated and inspired by the above results and recent works [8, 23, 28, 30, 32, 41], we have raised the following research question:

Question: Can we obtain an iterative algorithm which converges strongly to a common solution of fixed point problem for Lipschitz hemicontractive-type multi-valued mapping, finite families of variational inequality and split equilibrium problems?

It is our purpose in this paper to establish an iterative algorithm and prove that the produced sequence converges strongly to a common element of fixed point set of a Lipschitz hemicontractive-type multi-valued mapping, common solution set of a finite family of split equilibrium problems and common solution set of a finite family of variational inequality problems in the framework of real Hilbert spaces. The results presented in this work generalize and improve the recent results of Eslamian [6], Jeong [8], Kazmi and Rizvi [10], Meche et al.[14–16], Okeke and Mewomo [19], Shehu and Iyiola [23], Ugwunnadi and Ali [28], Zegeye and Shahzad [40] and some other results in this area.

2. Preliminaries

In this section, we collect some basic concepts and results from the existing literature which play a vital role in the sequel.

Let $S : C \longrightarrow C$ be a nonexpansive mapping with $F(S) \neq \emptyset$. Then, (see, e.g., [16]), for every $x \in C$ and $y \in F(S)$,

$$\langle x - Sx, y - Sx\rangle \leq \frac{1}{2}\|Sx - x\|^2. \tag{2.1}$$

Since C is a nonempty, closed and convex subset of a real Hilbert space H_1, it is well-known that for every point $x \in H_1$ there exists a unique nearest point $P_C x \in C$ such that

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

It means that the metric projection P_C of H_1 onto C is a single-valued mapping. Moreover, for every $x \in H_1$ and $z \in C$, we have

$$z = P_C x \iff \langle x - z, z - y \rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

Let $S : C \longrightarrow CB(C)$ be a multi-valued mapping and let the sequence $\{x_n\} \subset C$ converges weakly to x. Then, $(I - S)$ is said to be demiclosed at zero if $x \in Sx$ whenever $\lim_{n\to\infty} d(x_n, Sx_n) = 0$, where I is the identity mapping on C. It is well-known that if $S : C \longrightarrow C$ is a single-valued nonexpansive mapping, then $(I - S)$ is demiclosed at zero (see, [15]).

On the other hand, given an $\alpha-$inverse strongly monotone mapping and $\lambda \in (0, 2\alpha]$, then $I - \lambda A$ is a nonexpansive mapping from C into H_1 (see, for example, [27, 38]). But, if $S : C \longrightarrow H_1$ is nonexpansive mapping, then $A := I - S$ is $\frac{1}{2}-$inverse strongly monotone mapping (for more details, see [25]).

The following common assumption will be used in the sequel.

Assumption 2.1. Let $F : C \times C \to \mathbb{R}$ be any given bifunction. We assumed that F satisfies the following conditions:

- $(A1)$ $F(x, x) = 0$, $\forall\, x \in C$;
- $(A2)$ F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$, $\forall\, x, y \in C$;
- $(A3)$ $\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$, $\forall\, x, y, z \in C$;
- $(A4)$ for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

In order to prove our main results, we also need the following familiar lemmas.

Lemma 2.2 ([39])**.** *Let H be a real Hilbert space. Then, for all $y_i \in H$ and $\alpha_i \in [0, 1]$, for $i = 1, 2, \ldots, N$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_N = 1$ the following equality holds:*

$$\|\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_N y_N\|^2 = \sum_{i=1}^{N} \alpha_i \|y_i\|^2 - \sum_{1\leq i,j\leq N} \alpha_i\alpha_j \|y_i - y_j\|^2.$$

Lemma 2.3 ([36])**.** *Let $\{\gamma_n\}$ be a sequence of nonnegative real numbers such that*

$$\gamma_{n+1} \leq (1 - \beta_n)\gamma_n + \beta_n\delta_n, \quad \text{for } n \geq n_0,$$

where $n_0 \in \mathbb{N}$ and the control sequences $\{\beta_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following:

$$\lim_{n\to\infty} \beta_n = 0, \qquad \sum_{n=1}^{\infty} \beta_n = \infty, \qquad \limsup_{n\to\infty} \delta_n \leq 0.$$

Then, $\lim_{n\to\infty} \gamma_n = 0$.

Lemma 2.4. *Let H be a real Hilbert space. Then, it is known that for every $x, y \in H$,*

i) $\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle$.
ii) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$.

Lemma 2.5 ([18])**.** *Let H be a Hilbert space. Let $A, B \in CB(H)$ and $a \in A$. Then, for every $\varepsilon > 0$, there exists a point $b \in B$ such that $\|a-b\| \le D(A,B) + \varepsilon$. In particular, for any $a \in A$ there exists an element $b \in B$ such that $\|a-b\| \le 2D(A,B)$.*

Lemma 2.6 ([37])**.** *Let A be a continuous monotone mapping from C into H_1. Then, for any $\mu > 0$ and $x \in H_1$, there exists $z \in C$ such that*

$$\langle Az, y-z\rangle + \frac{1}{\mu}\langle y-z, z-x\rangle \ge 0, \quad \forall\, y \in C.$$

Moreover, the mapping $J_\mu : H_1 \longrightarrow C$ given by

$$J_\mu x = \left\{ z \in C : \langle Az, y-z\rangle + \frac{1}{\mu}\langle y-z, z-x\rangle \ge 0, \quad \forall\, y \in C \right\}$$

is well-defined and satisfies:

(i) *J_μ is single-valued;*
(ii) *J_μ is firmly nonexpansive, that is,*
$\|J_\mu x - J_\mu y\|^2 \le \langle J_\mu x - J_\mu y, x-y\rangle, \quad \forall\, x, y \in H_1;$
(iii) *$F(J_\mu) = VI(C, A)$;*
(iv) *$VI(C, A)$ is closed and convex.*

Lemma 2.7 ([1, 5])**.** *Let F_1 be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying Assumption 2.1. For any $\sigma > 0$ and for all $x \in H_1$, the mapping $T_\sigma^{F_1} : H_1 \longrightarrow C$ defined by*

$$T_\sigma^{F_1} x = \left\{ z \in C : F_1(z, y) + \frac{1}{\sigma}\langle y-z, z-x\rangle \ge 0,\ \forall\, y \in C \right\},$$

is well-defined and satisfies the following:

(i) *$T_\sigma^{F_1}$ is nonempty and single valued;*
(ii) *$T_\sigma^{F_1}$ is firmly nonexpansive, i.e.,*

$$\|T_\sigma^{F_1} x - T_\sigma^{F_1} y\|^2 \le \left\langle T_\sigma^{F_1} x - T_\sigma^{F_1} y, x-y \right\rangle, \quad \forall\, x, y \in H_1;$$

(iii) *$F(T_\sigma^{F_1}) = EP(F_1)$;*
(iv) *$EP(F_1)$ is closed and convex.*

Let $F_2 : Q \times Q \longrightarrow \mathbb{R}$ satisfies Assumption 2.1. Applying Lemma 2.7, we can define a mapping $T_\tau^{F_2} : H_2 \longrightarrow Q$ by

$$T_\tau^{F_2} u = \left\{ w \in Q : F_2(w, v) + \frac{1}{\tau}\langle v-w, w-u\rangle \ge 0,\ \forall\, v \in Q \right\}$$

for $\tau > 0$ and for all $u \in H_2$. Then $T_\tau^{F_2}$ also satisfies the same properties in the previous Lemma 2.7. It is not difficult to check that Ω is a closed and convex set.

Lemma 2.8 ([13]). *Let $\{\beta_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\beta_{n_j} < \beta_{n_j+1}$, for all $j \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{\delta_k\} \subset \mathbb{N}$ such that $\delta_k \to \infty$ and for all (sufficiently large) numbers $k \in \mathbb{N}$,*

$$\beta_{\delta_k} \leq \beta_{\delta_k+1}, \qquad \beta_k \leq \beta_{\delta_k+1}.$$

In fact, $\delta_k = \max\{i \leq k : \beta_i \leq \beta_{i+1}\}$.

Lemma 2.9 ([20]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{t_n\}$ be a sequence of real numbers. Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n t_n \text{ for all } n \geq 0.$$

If $\limsup_{k\to\infty} t_{n_k} \leq 0$ for every subsequence $\{s_{n_k}\}$ of $\{s_n\}$ satisfying

$$\liminf_{k\to\infty}(s_{n_k+1} - s_{n_k}) \geq 0,$$

then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.10. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $S : C \longrightarrow CB(C)$ be a $L-$Lipschitz multi-valued mapping with $F(S) \neq \emptyset$ and $S(p) = \{p\}$, for all $p \in F(S)$. Then, $F(S)$ is closed subset of C.*

Proof. Let $\{x_n\} \subset F(S)$ be such that $x_n \to x$. We claim that $x \in F(S)$. Now, since C is closed, we have $x \in C$. From the fact that the distance function $d(., Sx)$ is continuous and S is Lipschitz mapping, we have that

$$\begin{aligned} d(x, Sx) &= \lim_{n\to\infty} d(x_n, Sx) \\ &\leq \lim_{n\to\infty} D(Sx_n, Sx) \\ &\leq \lim_{n\to\infty} L\,\|x_n - x\| \\ &= 0. \end{aligned}$$

Thus, since Sx is closed, we get that $x \in Sx$, that is, $x \in F(S)$. Hence, $F(S)$ is closed subset of C. □

3. Main Results

In this section, we give an iterative algorithm and prove its strong convergence theorems for a finite family of split equilibrium and variational inequality problems and a fixed point problem for a Lipschitz hemicontractive-type multi-valued mapping in Hilbert spaces.

Theorem 3.1. *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A_m : C \longrightarrow H_1$ be a continuous monotone mapping and $B_m : H_1 \longrightarrow H_2$ be a bounded linear operator with its adjoint B_m^* for each $m \in \{1, 2, \dots, N\}$. Let $F_{1,m} : C \times C \longrightarrow \mathbb{R}$ and $F_{2,m} : Q \times Q \longrightarrow \mathbb{R}$ be bifunctions satisfying Assumption 2.1 for each $m \in \{1, 2, \dots, N\}$. Let $S : C \longrightarrow CB(C)$ be a $L-$Lipschitz hemicontractive-type multi-valued mapping. Assume that $\Theta = \bigcap_{m=1}^{N} \left(\Omega_m \bigcap VI(C, A_m)\right) \bigcap F(S)$ is nonempty and $Sp = \{p\}$ for all $p \in \Theta$. Given $x_0, u \in C$, for each $m = 1, 2, \dots, N$, let $\{x_n\}$ be a sequence in C defined by*

$$\begin{cases} z_{n,m} = T_\sigma^{F_{1,m}} \left(I - \lambda_m B_m^* \left(I - T_\tau^{F_{2,m}} \right) B_m \right) x_n, \\ e_{n,m} = J_\mu^m z_{n,m}, \\ y_n = \displaystyle\sum_{m=1}^{N} \tau_{n,m} e_{n,m}, \\ u_n = (1 - \alpha_n) y_n + \alpha_n v_n, \\ x_{n+1} = \beta_n u + \gamma_n w_n + \sigma_n y_n, \end{cases} \tag{3.1}$$

for all $n \geq 0$, where $v_n \in Sy_n$ and $w_n \in Su_n$ are such that $\|v_n - w_n\| \leq 2D(Sy_n, Su_n)$ and $\sigma, \tau, \mu > 0$, $\lambda_m \in \left(0, \frac{1}{\eta_m}\right)$, where $\eta_m = \|B_m\|^2$, $\{\beta_n\}, \{\alpha_n\} \subset (0, 1)$, $\{\tau_{n,m}\} \subset (0, 1]$ and $\{\gamma_n\}$,$\{\sigma_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (0, 1)$ satisfying the following conditions:

(i) $\beta_n + \gamma_n + \sigma_n = 1$;

(ii) $\displaystyle\sum_{m=1}^{N} \tau_{n,m} = 1$;

(iii) $\beta_n + \gamma_n \leq \alpha_n \leq \gamma < \frac{1}{\sqrt{1+4L^2}+1}$.

Then, the sequence $\{x_n\}$ is bounded.

Proof. It follows from (ii) of Lemma 2.7 that $T_\tau^{F_{2,m}}$ is firmly nonexpansive for each $m \in \{1, 2, \dots, N\}$ and so it is nonexpansive. Since the nonexpansiveness of $T_\tau^{F_{2,m}}$ implies that $I - T_\tau^{F_{2,m}}$ is $\frac{1}{2}-$inverse strongly monotone mapping, it follows from the hypothesis and Cauchy Schwartz inequality that $B_m^* \left(I - T_\tau^{F_{2,m}} \right) B_m$ is a $\frac{1}{2\eta_m}-$inverse strongly monotone mapping for each $m \in \{1, 2, \dots, N\}$. Since $\lambda_m \in \left(0, \frac{1}{\eta_m}\right)$ for each $m \in \{1, 2, \dots, N\}$, we have $I - \lambda B_m^* \left(I - T_\tau^{F_{2,m}} \right) B_m$ is nonexpansive. Again, by (ii) of Lemma 2.7, $T_\sigma^{F_{1,m}}$ is nonexpansive for each

$m \in \{1, 2, \ldots, N\}$. It then follows that

$$\|T_\sigma^{F_{1,m}} \left(I - \lambda_m B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m\right) x - T_\sigma^{F_{1,m}} \left(I - \lambda_m B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m\right) y\| \leq \|x - y\|. \tag{3.2}$$

Now, let $p \in \Theta$. Then, we have $Sp = p$, $J_\mu^m p = p$, $p \in \Omega_m$ and so $p = T_\sigma^{F_{1,m}} p$ and $B_m p = T_\tau^{F_{2,m}} B_m p$ for each $m \in \{1, 2, \ldots, N\}$. This implies that $T_\sigma^{F_{1,m}} \left(I - \lambda_m B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m\right) p = p$. Thus, using (3.2), we get that

$$\begin{aligned} \|z_{n,m} - p\| &= \left\|T_\sigma^{F_{1,m}} \left(I - \lambda_m B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m\right) x_n - p\right\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{3.3}$$

Using (ii) of Lemma 2.6, J_μ^m is firmly nonexpansive for each $m \in \{1, 2, \ldots, N\}$ and so nonexpansive. Then from (3.3), we find that

$$\begin{aligned} \|e_{n,m} - p\| &= \|J_\mu^m z_{n,m} - J_\mu^m p\| \\ &\leq \|z_{n,m} - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{3.4}$$

Then, by applying triangle inequality, (3.4) and condition (ii), we derive that

$$\begin{aligned} \|y_n - p\| &= \left\|\sum_{m=1}^{N} \tau_{n,m} e_{n,m} - p\right\| \\ &\leq \sum_{m=1}^{N} \tau_{n,m} \|e_{n,m} - p\| \\ &\leq \sum_{m=1}^{N} \tau_{n,m} \|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{3.5}$$

From the hypothesis that S is hemicontractive-type mapping and $w_n \in Su_n$, it is clear that

$$\begin{aligned} \|w_n - p\|^2 &\leq D^2(Su_n, Sp) \\ &\leq \|u_n - p\|^2 + \|u_n - w_n\|^2. \end{aligned} \tag{3.6}$$

Again, since S is hemicontractive-type mapping and $v_n \in Sy_n$, it follows from (3.1), (3.5) and Lemma 2.2 that

$$\begin{aligned} \|u_n - p\|^2 &= \|(1 - \alpha_n) y_n + \alpha_n v_n - p\|^2 \\ &= (1 - \alpha_n)\|y_n - p\|^2 + \alpha_n \|v_n - p\|^2 \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
&- \alpha_n(1-\alpha_n)\|y_n - v_n\|^2 \\
&\leq (1-\alpha_n)\|y_n - p\|^2 + \alpha_n D^2(Sy_n, Sp) \\
&\quad - \alpha_n\|y_n - v_n\|^2 + \alpha_n^2\|y_n - v_n\|^2 \\
&\leq (1-\alpha_n)\|y_n - p\|^2 + \alpha_n\Big(\|y_n - p\|^2 + \|y_n - v_n\|^2\Big) \\
&\quad - \alpha_n\|y_n - v_n\|^2 + \alpha_n^2\|y_n - v_n\|^2 \\
&= \|y_n - p\|^2 + \alpha_n\|y_n - v_n\|^2 - \alpha_n\|y_n - v_n\|^2 \\
&\quad + \alpha_n^2\|y_n - v_n\|^2 \\
&\leq \|x_n - p\|^2 + \alpha_n^2\|y_n - v_n\|^2.
\end{aligned}
$$

Combining (3.6) and (3.7) yields that

$$
\|w_n - p\|^2 \leq \|x_n - p\|^2 + \alpha_n^2\|y_n - v_n\|^2 + \|u_n - w_n\|^2. \tag{3.8}
$$

We easily obtain from (3.1) that

$$
\begin{aligned}
\|y_n - u_n\|^2 &= \|y_n - ((1-\alpha_n)y_n + \alpha_n v_n)\|^2 \\
&= \alpha_n^2\|y_n - v_n\|^2.
\end{aligned} \tag{3.9}
$$

By Lemma 2.2 and the assumption that $\|v_n - w_n\| \leq 2D(Sy_n, Su_n)$, we obtain

$$
\begin{aligned}
\|u_n - w_n\|^2 &= \|(1-\alpha_n)(y_n - w_n) + \alpha_n(v_n - w_n)\|^2 \\
&= (1-\alpha_n)\|y_n - w_n\|^2 + \alpha_n\|v_n - w_n\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\|y_n - v_n\|^2 \\
&\leq (1-\alpha_n)\|y_n - w_n\|^2 + 4\alpha_n D^2(Sy_n, Su_n) \\
&\quad - \alpha_n\|y_n - v_n\|^2 + \alpha_n^2\|y_n - v_n\|^2.
\end{aligned}
$$

Utilizing the hypothesis that S is $L-$Lipschitzian mapping and (3.9), we get

$$
\begin{aligned}
\|u_n - w_n\|^2 &\leq (1-\alpha_n)\|y_n - w_n\|^2 + 4\alpha_n L^2\|y_n - u_n\|^2 \\
&\quad - \alpha_n\|y_n - v_n\|^2 + \alpha_n^2\|y_n - v_n\|^2 \\
&= (1-\alpha_n)\|y_n - w_n\|^2 + 4\alpha_n^3 L^2\|y_n - v_n\|^2 \\
&\quad - \alpha_n\|y_n - v_n\|^2 + \alpha_n^2\|y_n - v_n\|^2 \\
&= (1-\alpha_n)\|y_n - w_n\|^2 + \alpha_n(4L^2\alpha_n^2 \\
&\quad + \alpha_n - 1)\|y_n - v_n\|^2.
\end{aligned} \tag{3.10}
$$

Thus, substituting (3.10) into (3.8) gives that

$$
\begin{aligned}
\|w_n - p\|^2 \leq \|x_n - p\|^2 &+ \alpha_n^2\|y_n - v_n\|^2 + (1-\alpha_n)\|y_n - w_n\|^2 \\
&+ \alpha_n\left(4L^2\alpha_n^2 + \alpha_n - 1\right)\|y_n - v_n\|^2
\end{aligned} \tag{3.11}
$$

$$= \|x_n - p\|^2 + (1-\alpha_n)\|y_n - w_n\|^2 \\ - \alpha_n\left(1 - 4L^2\alpha_n^2 - 2\alpha_n\right)\|y_n - v_n\|^2.$$

Again, using Lemma 2.2 and Condition (i), we find that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\beta_n u + \gamma_n w_n + \sigma_n y_n - p\|^2 \\ &\le \beta_n\|u-p\|^2 + \gamma_n\|w_n - p\|^2 \\ &\quad + \sigma_n\|y_n - p\|^2 - \gamma_n\sigma_n\|y_n - w_n\|^2.\end{aligned}$$

It follows easily from (3.5) and (3.11) that

$$\begin{aligned}\|x_{n+1} - p\|^2 \le \beta_n\|u-p\|^2 + \gamma_n\Big(&\|x_n - p\|^2 + (1-\alpha_n)\|y_n - w_n\|^2 \\ &- \alpha_n\left(1 - 4L^2\alpha_n^2 - 2\alpha_n\right)\|y_n - v_n\|^2\Big) \\ &+ \sigma_n\|x_n - p\|^2 - \gamma_n\sigma_n\|y_n - w_n\|^2.\end{aligned}$$

Combining this fact with Condition (i) yields

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \beta_n\|u-p\|^2 + (1-\beta_n)\|x_n - p\|^2 \\ &\quad - \gamma_n\alpha_n\left(1 - 4L^2\alpha_n^2 - 2\alpha_n\right)\|y_n - v_n\|^2 \\ &\quad + \gamma_n(\beta_n + \gamma_n - \alpha_n)\|y_n - w_n\|^2.\end{aligned} \tag{3.12}$$

Moreover, condition (iii) implies that

$$1 - 4L^2\alpha_n^2 - 2\alpha_n \ge 1 - 4L^2\gamma^2 - 2\gamma > 0, \qquad \beta_n + \gamma_n - \alpha_n \le 0, \tag{3.13}$$

for all $n \ge 0$. Thus, using (3.13) in (3.12), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \beta_n\|u-p\|^2 + (1-\beta_n)\|x_n - p\|^2 \\ &\le \max\{\|u-p\|^2, \|x_n - p\|^2\}.\end{aligned}$$

It then follows from the Mathematical induction principle that

$$\|x_n - p\|^2 \le \max\{\|u-p\|^2, \|x_0 - p\|^2\}.$$

Therefore, the sequence $\{x_n\}$ is bounded. We also obtain that $\{y_n\}$, $\{z_{n,m}\}$ and $\{u_n\}$ are all bounded. The proof is completed. □

Theorem 3.2. *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A_m : C \longrightarrow H_1$ be a continuous monotone mapping and $B_m : H_1 \longrightarrow H_2$ be a bounded linear operator with its adjoint B_m^* for each $m \in \{1, 2, \ldots, N\}$. Let $F_{1,m} : C \times C \longrightarrow \mathbb{R}$ and $F_{2,m} : Q \times Q \longrightarrow \mathbb{R}$ be bifunctions satisfying Assumption 2.1 for each $m \in \{1, 2, \ldots, N\}$. Let $S : C \longrightarrow CB(C)$ be a $L-$Lipschitz hemicontractive-type multi-valued mapping such that $(I - S)$ is demiclosed at zero. Assume that*

$\Theta = \bigcap_{m=1}^{N} \left(\Omega_m \bigcap VI(C, A_m)\right) \bigcap F(S)$ *is nonempty convex and* $Sp = \{p\}$ *for all* $p \in \Theta$ *. Let* $\{\beta_n\}$,$\{\alpha_n\}$,$\{\gamma_n\}$,$\{\sigma_n\}$ *and* $\{\tau_{n,m}\}$ *be real sequences in* $(0,1)$ *such that*

i. $\beta_n + \gamma_n + \sigma_n = 1$ *and* $0 < \alpha \le \gamma_n, \sigma_n \le \beta < 1$;

ii. $\lim_{n\to\infty} \beta_n = 0,\ \sum_{n=0}^{\infty} \beta_n = \infty$;

iii. $\sum_{m=1}^{N} \tau_{n,m} = 1$ *and* $0 < \delta \le \tau_{n,m} \le 1$;

iv. $\beta_n + \gamma_n \le \alpha_n \le \gamma < \frac{1}{\sqrt{1+4L^2}+1}$.

Let $x_0, u \in C$ *be arbitrary. Then, the sequence* $\{x_n\}$ *generated by (3.1) converges strongly to* $q = P_\Theta(u)$.

Proof. Let $p \in \Theta$. Then, using the nonexpansivity of $T_\sigma^{F_{1,m}}$ for each $m \in \{1, 2, \dots, N\}$, we have

$$\begin{aligned}
\|z_{n,m} - p\|^2 &= \Big\|T_\sigma^{F_{1,m}} \left(I - \lambda_m B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m\right) x_n \\
&\quad -T_\sigma^{F_{1,m}} \left(I - \lambda_m B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m\right) p\Big\|^2 \\
&\le \Big\|\left(I - \lambda_m B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m\right) x_n \\
&\quad - \left(I - \lambda_m B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m\right) p\Big\|^2 \\
&= \Big\|(x_n - p) - \lambda_m \Big(B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m x_n \\
&\quad -B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m p\Big)\Big\|^2 \\
&= \|x_n - p\|^2 - 2\lambda_m \Big\langle x_n - p, B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m x_n \\
&\quad -B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m p\Big\rangle \\
&\quad + \lambda_m^2 \Big\|B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m x_n - B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m p\Big\|^2 .
\end{aligned}$$

Because $B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m$ is $\frac{1}{2\eta_m}-$inverse strongly monotone and $B_m p = T_\tau^{F_{2,m}} B_m p$ for each $m \in \{1, 2, \dots, N\}$, we find that

(3.14)

$$\begin{aligned}
\|z_{n,m} - p\|^2 &\le \|x_n - p\|^2 \\
&\quad - \frac{\lambda_m}{\eta_m} \Big\|B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m x_n - B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m p\Big\|^2
\end{aligned}$$

$$+ \lambda_m^2 \left\| B_m^* \left(I - T_\tau^{F_{2,m}} \right) B_m x_n - B_m^* \left(I - T_\tau^{F_{2,m}} \right) B_m p \right\|^2$$
$$= \|x_n - p\|^2 + \lambda_m \left(\lambda_m - \frac{1}{\eta_m} \right) \left\| B_m^* \left(I - T_\tau^{F_{2,m}} \right) B_m x_n \right\|^2 .$$

It follows from (3.5), (3.4), (3.14) and Lemma 2.2 that

(3.15)
$$\begin{aligned}
\|y_n - p\|^2 &\le \sum_{m=1}^{N} \tau_{n,m} \|e_{n,m} - p\|^2 \\
&\le \sum_{m=1}^{N} \tau_{n,m} \|z_{n,m} - p\|^2 \\
&\le \|x_n - p\|^2 \\
&\quad + \sum_{m=1}^{N} \tau_{n,m} \lambda_m \left(\lambda_m - \frac{1}{\eta_m} \right) \left\| B_m^* \left(I - T_\tau^{F_{2,m}} \right) B_m x_n \right\|^2 .
\end{aligned}$$

Using (ii) of Lemma 2.4, we find that

(3.16)
$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\beta_n u + \gamma_n w_n + \sigma_n y_n - p\|^2 \\
&\le \|\gamma_n (w_n - p) + \sigma_n (y_n - p)\|^2 + 2\beta_n \langle u - p, x_{n+1} - p \rangle \\
&\le \gamma_n \|w_n - p\|^2 + \sigma_n \|y_n - p\|^2 - \gamma_n \sigma_n \|y_n - w_n\|^2 \\
&\quad + 2\beta_n \langle u - p, x_{n+1} - p \rangle .
\end{aligned}$$

Therefore, (3.11), (3.15) and (3.16) imply that

(3.17)
$$\begin{aligned}
\|x_{n+1} - p\|^2 \le (1 - \beta_n)\|x_n - p\|^2 &- \gamma_n \alpha_n \left(1 - 4L^2 \alpha_n^2 - 2\alpha_n\right) \\
&\times \|y_n - v_n\|^2 + \gamma_n (\beta_n + \gamma_n - \alpha_n) \|y_n - w_n\|^2 \\
&- \sigma_n \sum_{m=1}^{N} \tau_{n,m} \lambda_m \left(\frac{1}{\eta_m} - \lambda_m\right) \left\| B_m^* \left(I - T_\tau^{F_{2,m}} \right) B_m x_n \right\|^2 \\
&+ 2\beta_n \langle u - p, x_{n+1} - p \rangle .
\end{aligned}$$

We now consider the following two cases.

Case 1: Assume that there exists a natural number n_0 such that $\{\|x_n - p\|\}$ is nonincreasing for all $n \ge n_0$. Then, $\{\|x_n - p\|\}$ is convergent and obviously $\|x_n - p\| - \|x_{n+1} - p\| \to 0$ as $n \to \infty$. It follows from

(3.13) and (3.17) that

(3.18)

$$\sigma_n\tau_{n,m}\lambda_m\left(\frac{1}{\eta_m}-\lambda_m\right)\left\|B_m^*\left(I-T_\tau^{F_{2,m}}\right)B_mx_n\right\|^2$$
$$\leq(1-\beta_n)\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\beta_n\times\langle u-p,x_{n+1}-p\rangle\,,$$

Since $\beta_n\to 0$ as $n\to\infty$, we infer that

$$\left\|B_m^*\left(I-T_\tau^{F_{2,m}}\right)B_mx_n\right\|\to 0 \text{ as } n\to\infty. \tag{3.19}$$

We also obtain

$$\left\|x_n-(x_n-\lambda_mB_m^*\left(I-T_\tau^{F_{2,m}}\right)B_mx_n)\right\|\to 0 \text{ as } n\to\infty. \tag{3.20}$$

Utilizing the firmly nonexpansivity of $T_\sigma^{F_{1,m}}$, nonexpansivity of $\left(I-\lambda_mB_m^*\left(I-T_\tau^{F_{2,m}}\right)B_m\right)$, (3.1) and Lemma 2.4 (i), we get that

$$\begin{aligned}
\|z_{n,m}-p\|^2&=\left\|T_\sigma^{F_{1,m}}\left(I-\lambda_mB_m^*\left(I-T_\tau^{F_{2,m}}\right)B_m\right)x_n-T_\sigma^{F_{1,m}}p\right\|^2\\
&\leq\left\langle z_{n,m}-p,\left(I-\lambda_mB_m^*\left(I-T_\tau^{F_{2,m}}\right)B_m\right)x_n-p\right\rangle\\
&=\frac{1}{2}\Big(\|z_{n,m}-p\|^2+\left\|\left(I-\lambda_mB_m^*\left(I-T_\tau^{F_{2,m}}\right)B_m\right)x_n-p\right\|^2\\
&\quad-\left\|z_{n,m}-\left(I-\lambda_mB_m^*\left(I-T_\tau^{F_{2,m}}\right)B_m\right)x_n\right\|^2\Big)\\
&\leq\frac{1}{2}\Big(\|z_{n,m}-p\|^2+\|x_n-p\|^2-\|z_{n,m}-x_n\|^2\\
&\quad-2\lambda_m\left\langle z_{n,m}-x_n,B_m^*\left(I-T_\tau^{F_{2,m}}\right)B_mx_n\right\rangle\\
&\quad-\lambda_m^2\left\|B_m^*\left(I-T_\tau^{F_{2,m}}\right)B_mx_n\right\|^2\Big).
\end{aligned}$$

So, we get that

$$\begin{aligned}
\|z_{n,m}-p\|^2\leq\|x_n-p\|^2&-\|z_{n,m}-x_n\|^2\\
&+2\lambda_m\left\langle x_n-z_{n,m},B_m^*\left(I-T_\tau^{F_{2,m}}\right)B_mx_n\right\rangle.
\end{aligned}\tag{3.21}$$

Note that

$$\|y_n-p\|^2\leq\sum_{m=1}^N\tau_{n,m}\|z_{n,m}-p\|^2.$$

And so from (3.21), we have

(3.22)

$$\|y_n-p\|^2\leq\|x_n-p\|^2-\sum_{m=1}^N\tau_{n,m}\|z_{n,m}-x_n\|^2$$

$$+ 2\sum_{m=1}^{N}\left(\tau_{n,m}\lambda_m\left\langle x_n - z_{n,m}, B_m^*\left(I - T_\tau^{F_{2,m}}\right)B_m x_n\right\rangle\right).$$

Thus, substituting (3.11) and (3.22) into (3.16), we obtain that

(3.23)

$$\begin{aligned}\|x_{n+1}-p\|^2 \le (1-\beta_n)\|x_n-p\|^2 &- \gamma_n\alpha_n\left(1-4L^2\alpha_n^2-2\alpha_n\right)\|y_n-v_n\|^2\\ &+ \gamma_n(\beta_n+\gamma_n-\alpha_n)\|y_n-w_n\|^2\\ &- \sigma_n\sum_{m=1}^{N}\tau_{n,m}\|z_{n,m}-x_n\|^2 + 2\sigma_n\\ &\times\sum_{m=1}^{N}\left(\tau_{n,m}\lambda_m\left\langle x_n - z_{n,m}, B_m^*\left(I - T_\tau^{F_{2,m}}\right)B_m x_n\right\rangle\right)\\ &+ 2\beta_n\langle u-p, x_{n+1}-p\rangle.\end{aligned}$$

It follows from (3.13) and (3.23) that

$$\begin{aligned}\sigma_n\tau_{n,m}\|z_{n,m}-x_n\|^2 \le (1-\beta_n)\|x_n-p\|^2 &- \|x_{n+1}-p\|^2\\ &+ 2\sigma_n\sum_{m=1}^{N}\left(\tau_{n,m}\lambda_m\|x_n-z_{n,m}\|\right.\\ &\times\left.\left\|B_m^*\left(I-T_\tau^{F_{2,m}}\right)B_m x_n\right\|\right)\\ &+ 2\beta_n\langle u-p, x_{n+1}-p\rangle.\end{aligned}$$

Thus, since $\{x_n\}$ and $\{z_{n,m}\}$ are bounded, $\beta_n \to 0$ as $n \to \infty$, we obtain from (3.19) that

$$\|z_{n,m}-x_n\| \to 0 \text{ as } n \to \infty\ . \tag{3.24}$$

Moreover, from (3.13) and (3.23), it is clear that

$$\begin{aligned}\gamma_n\alpha_n\left(1-4L^2\alpha_n^2-2\alpha_n\right)\|y_n-v_n\|^2 \le (1-\beta_n)\|x_n-p\|^2 &- \|x_{n+1}-p\|^2\\ &+ 2\sigma_n\sum_{m=1}^{N}\left(\tau_{n,m}\lambda_m\|x_n-z_{n,m}\|\right.\\ &\times\left.\left\|B_m^*\left(I-T_\tau^{F_{2,m}}\right)B_m x_n\right\|\right)\\ &+ 2\beta_n\langle u-p, x_{n+1}-p\rangle.\end{aligned}$$

Since $\beta_n \to 0$ as $n \to \infty$, combining this fact with condition (i) and (3.19) yields

$$\|y_n-v_n\| \to 0 \text{ as } n \to \infty, \tag{3.25}$$

and so as $v_n \in Sy_n$, we find that

$$d(y_n, Sy_n) \to 0 \text{ as } n \to \infty. \tag{3.26}$$

On the other hand, since J_μ^m is firmly nonexpansive and $J_\mu^m p = p$ for each $m \in \{1, 2, \ldots, N\}$, from Lemma 2.4 (i), we have

$$\begin{aligned}\|e_{n,m} - p\|^2 &= \|J_\mu^m z_{n,m} - J_\mu^m p\|^2 \\ &\leq \langle e_{n,m} - p, z_{n,m} - p\rangle \\ &= \frac{1}{2}\left(\|e_{n,m} - p\|^2 + \|z_{n,m} - p\|^2 - \|z_{n,m} - e_{n,m}\|^2\right).\end{aligned}$$

This fact with (3.3) gives that

$$\begin{aligned}\|e_{n,m} - p\|^2 &\leq \|z_{n,m} - p\|^2 - \|z_{n,m} - e_{n,m}\|^2 \\ &\leq \|x_n - p\|^2 - \|z_{n,m} - e_{n,m}\|^2,\end{aligned}$$

and then it follows from (3.5) that

$$\begin{aligned}\|y_n - p\|^2 &\leq \sum_{m=1}^{N} \tau_{n,m} \|e_{n,m} - p\|^2 \\ &\leq \|x_n - p\|^2 - \sum_{m=1}^{N} \tau_{n,m} \|z_{n,m} - e_{n,m}\|^2.\end{aligned} \tag{3.27}$$

By substituting (3.11) and (3.27) into (3.16), we get that

(3.28)

$$\begin{aligned}\|x_{n+1} - p\|^2 \leq{}& (1 - \beta_n)\|x_n - p\|^2 - \gamma_n\alpha_n\left(1 - 4L^2\alpha_n^2 - 2\alpha_n\right) \\ &\times \|y_n - v_n\|^2 + \gamma_n(\beta_n + \gamma_n - \alpha_n)\|y_n - w_n\|^2 \\ &- \sigma_n \sum_{m=1}^{N} \tau_{n,m}\|z_{n,m} - e_{n,m}\|^2 + 2\beta_n \langle u - p, x_{n+1} - p\rangle.\end{aligned}$$

Consequently, using (3.13), we find that

$$\begin{aligned}\sigma_n\tau_{n,m}\|z_{n,m} - e_{n,m}\|^2 \leq{}& (1 - \beta_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &+ 2\beta_n \langle u - p, x_{n+1} - p\rangle.\end{aligned}$$

Because $\beta_n \to 0$ as $n \to \infty$, it follows that

$$\|z_{n,m} - e_{n,m}\| \to 0 \text{ as } n \to \infty \tag{3.29}$$

for each $m = 1, 2, \ldots, N$.

It then follows from triangle inequality and (3.24) that

$$\|e_{n,m} - x_n\| \leq \|e_{n,m} - z_{n,m}\| + \|z_{n,m} - x_n\| \to 0$$

as $n \to \infty$ for each $m = 1, 2, \ldots, N$.
As a result, since

$$\|y_n - x_n\| = \left\| \sum_{m=1}^{N} \tau_{n,m} e_{n,m} - x_n \right\| \leq \sum_{m=1}^{N} \tau_{n,m} \|e_{n,m} - x_n\|,$$

we obtain that

$$\|y_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.30}$$

In addition, since S is a $L-$Lipschitzian multi-valued mapping, using the fact that $\|v_n - w_n\| \leq 2D(Sy_n, Su_n)$, (3.9) and (3.25), we get that

$$\begin{aligned} \|y_n - w_n\| &\leq \|y_n - v_n\| + \|v_n - w_n\| \\ &\leq \|y_n - v_n\| + 2L\|y_n - u_n\| \\ &= \|y_n - v_n\| + 2L\alpha_n\|y_n - v_n\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.31}$$

Therefore, since $\{y_n\}$ is bounded and $\beta_n \to 0$ as $n \to \infty$, (3.30) and (3.31) imply that

(3.32)

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|x_{n+1} - y_n\| + \|y_n - x_n\| \\ &= \|\beta_n(u - y_n) + \gamma_n(w_n - y_n)\| + \|y_n - x_n\| \\ &\leq \beta_n\|u - y_n\| + \gamma_n\|w_n - y_n\| + \|y_n - x_n\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Furthermore, it follows from (3.13) and (3.28) that

$$\|x_{n+1} - p\|^2 \leq (1 - \beta_n)\|x_n - p\|^2 + 2\beta_n \langle u - p, x_{n+1} - p \rangle. \tag{3.33}$$

Now, let $q = P_\Theta(u)$. Then, we claim that

$$\limsup_{n\to\infty} \langle u - q, x_{n+1} - q \rangle \leq 0.$$

By Theorem 3.1, the sequence $\{x_{n+1}\}$ is bounded, so we can choose a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ such that $x_{n_i+1} \rightharpoonup w$ as $i \to \infty$ and

$$\limsup_{n\to\infty} \langle u - q, x_{n+1} - q \rangle = \lim_{i\to\infty} \langle u - q, x_{n_i+1} - q \rangle.$$

Clearly, $w \in C$ and (3.32) implies that $x_{n_i} \rightharpoonup w$ as $i \to \infty$.
Consequently, it follows from (3.30) that

$$y_{n_i} \rightharpoonup w \text{ as } i \to \infty.$$

Then the demiclosedness of $(I - S)$ at zero and (3.26) ensure that the weak limit w of $\{y_{n_i}\}$ is a fixed point of the multi-valued mapping S. That is,

$$w \in F(S).$$

Again, since $\left(I-\lambda_m B_m^*\left(I-T_\tau^{F_{2,m}}\right)B_m\right)$ is a nonexpansive for each $m\in\{1,2,\dots,N\}$ and $x_{n_i}\rightharpoonup w$, the demiclosedness principle for nonexpansive and (3.20) implies that

$$w=\left(I-\lambda_m B_m^*\left(I-T_\tau^{F_{2,m}}\right)B_m\right)w$$

for each $m\in\{1,2,\dots,N\}$. This fact with the condition $\lambda_m>0$ implies that

$$B_m^*\left(I-T_\tau^{F_{2,m}}\right)B_m w=0.$$

Therefore, applying (2.1) we see that

$$B_m w=T_\tau^{F_{2,m}}B_m w,\qquad \text{for each } m=1,2,\dots,N.$$

And hence

$$B_m w\in EP(F_{2,m}).$$

In addition, Since $T_\sigma^{F_{1,m}}(I-\lambda_m B_m^*\left(I-T_\tau^{F_{2,m}}\right)B_m$ is nonexpansive for each $m\in\{1,2,\dots,N\}$, from the demiclosedness principle of nonexpansive mapping and (3.24), we obtain that

$$w=T_\sigma^{F_{1,m}}\left(I-\lambda_m B_m^*\left(I-T_\tau^{F_{2,m}}\right)B_m\right)w.$$

Since $B_m w=T_\tau^{F_{2,m}}B_m w$, we get that $w=T_\sigma^{F_{1,m}}w$ and so $w\in EP(F_{1,m})$ for each $m=1,2,\dots,N$. Therefore,

$$w\in\bigcap_{m=1}^{N}\Omega_m.$$

On the other hand, the fact that $x_{n_i}\rightharpoonup w$ as $i\to\infty$ and (3.24) implies that $z_{n_i,m}\rightharpoonup w$ as $i\to\infty$ for each $m\in\{1,2,\dots,N\}$. Furthermore, (3.29) imply that

$$\begin{aligned}\lim_{n\to\infty}\left\|z_{n_i,m}-J_\mu^m z_{n_i,m}\right\| &= \lim_{n\to\infty}\|z_{n_i,m}-e_{n_i,m}\|\\ &=0.\end{aligned}$$

Hence, the demiclosedness principle of nonexpansive guarantees that the weak limit w of the sequence $\{z_{n_i,m}\}$ is a fixed point of the mapping J_μ^m for each $m\in\{1,2,\dots,N\}$, that is, $w=J_\mu^m w$. This fact with Lemma 2.6 gives that

$$w\in\bigcap_{m=1}^{N}VI(C,A_m).$$

Therefore,

$$w\in\Theta=\bigcap_{m=1}^{N}\left(\Omega_m\bigcap VI(C,A_m)\right)\bigcap F(S).$$

From the fact that $q = P_\Theta(u)$ and $x_{n_i+1} \rightharpoonup w$ as $i \to \infty$ and (2.2), we have

$$\limsup_{n\to\infty} \langle u - q, x_{n+1} - q\rangle = \lim_{i\to\infty} \langle u - q, x_{n_i+1} - q\rangle = \langle u - q, w - q\rangle \leq 0. \tag{3.34}$$

Moreover, since $p \in \Theta$ was arbitrary and $q \in \Theta$, it follows from (3.33), (3.34) and Lemma 2.3 that

$$\|x_n - q\| \to 0 \text{ as } n \to \infty.$$

That is, $x_n \to q = P_\Theta(u)$.

Case 2. Suppose that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$\|x_{n_j} - p\| < \|x_{n_j+1} - p\|,$$

for all $j \in \mathbb{N}$. Then, Lemma 2.8 implies that there exists a nondecreasing sequence $\{\delta_k\} \subset \mathbb{N}$ such that $\delta_k \to \infty$ and

$$\|x_{\delta_k} - p\| \leq \|x_{\delta_k+1} - p\|, \qquad \|x_k - p\| \leq \|x_{\delta_k+1} - p\|, \tag{3.35}$$

for all $k \in \mathbb{N}$. Thus, from (3.13), (3.23), (3.19), (3.28), (3.35) and the fact that $\beta_n \to 0$, we find that

$$\|z_{\delta_k,m} - x_{\delta_k}\| \to 0,$$
$$\|y_{\delta_k} - v_{\delta_k}\| \to 0,$$
$$\|z_{\delta_k,m} - \beta_{\delta_k,m}\| \to 0,$$
$$\|y_{\delta_k} - x_{\delta_k}\| \to 0,$$

as $k \to \infty$. Therefore, since $q = P_\Theta(u)$, using the procedures similar to that in Case 1, we acquire that

$$\limsup_{k\to\infty} \langle u - q, x_{\delta_k+1} - q\rangle \leq 0. \tag{3.36}$$

Next, as $q \in \Theta$, from (3.33), we get that

$$\|x_{\delta_{k+1}} - q\|^2 \leq (1 - \beta_{\delta_k})\|x_{\delta_k} - q\|^2 + 2\beta_{\delta_k} \left\langle u - q, x_{\delta_{k+1}} - q\right\rangle. \tag{3.37}$$

It follows from 3.35 and (3.36) that

$$\|x_{\delta_k} - q\|^2 \leq 2 \left\langle u - q, x_{\delta_{k+1}} - q\right\rangle. \tag{3.38}$$

and $\|x_{\delta_k} - q\| \to 0$ as $k \to \infty$. This implies from (3.37) that $\left\|x_{\delta_{k+1}} - q\right\| \to 0$ as $k \to \infty$ and hence, since from (3.35) we have $\|x_k - q\| \leq \left\|x_{\delta_{k+1}} - q\right\|$ as $k \to \infty$ we obtain that

$$x_k \to q \text{ as } k \to \infty.$$

Therefore, from the above two cases, we conclude that the sequence $\{x_n\}$ generated by (3.1) converges strongly to a point $q \in \Theta$, where $q = P_\Theta(u)$. The proof is completed. $\square$

If, in Theorem 3.2, we assume that S is a single-valued Lipschitz hemicontractive mapping, then we obtain the following result:

Corollary 3.3. *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A_m : C \longrightarrow H_1$ be a continuous monotone mapping and $B_m : H_1 \longrightarrow H_2$ be a bounded linear operator with its adjoint B_m^* for each $m \in \{1, 2, \ldots, N\}$. Let $F_{1,m} : C \times C \longrightarrow \mathbb{R}$ and $F_{2,m} : Q \times Q \longrightarrow \mathbb{R}$ be bifunctions satisfying Assumption 2.1 for each $m \in \{1, 2, \ldots, N\}$. Let $S : C \longrightarrow C$ be a $L-$Lipschitz hemicontractive such that $(I - S)$ is demiclosed at zero. Assume that $\Theta = \bigcap_{m=1}^{N} \Big(\Omega_m \bigcap VI(C, A_m)\Big) \bigcap F(S)$ is nonempty. Let $\{\beta_n\}$,$\{\alpha_n\}$,$\{\gamma_n\}$,$\{\sigma_n\}$ and $\{\tau_{n,m}\}$ be real sequences in $(0, 1)$ such that*

i. $\beta_n + \gamma_n + \sigma_n = 1$ *and* $0 < \alpha \le \gamma_n, \sigma_n \le \beta < 1$*;*

ii. $\lim_{n\to\infty} \beta_n = 0,\ \sum_{n=0}^{\infty} \beta_n = \infty$*;*

iii. $\sum_{m=1}^{N} \tau_{n,m} = 1$ *and* $0 < \delta \le \tau_{n,m} \le 1$*;*

iv. $\beta_n + \gamma_n \le \alpha_n \le \gamma < \frac{1}{\sqrt{1+L^2}+1}$.

Let $x_0, u \in C$ be arbitrary and let $\{x_n\}$ be a sequence in C generated by

$$
\begin{cases}
z_{n,m} = T_\sigma^{F_{1,m}} \left(I - \lambda_m B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m\right) x_n, \\
e_{n,m} = J_\mu^m z_{n,m}, \\
y_n = \sum_{m=1}^{N} \tau_{n,m} e_{n,m}, \\
u_n = (1 - \alpha_n) y_n + \alpha_n S y_n, \\
x_{n+1} = \beta_n u + \gamma_n S u_n + \sigma_n y_n,
\end{cases}
$$

for all $n \ge 0$, where $\sigma, \tau, \mu > 0$, $\lambda_m \in \left(0, \frac{1}{\eta_m}\right)$, for $\eta_m = \|B_m\|^2$. Then, the sequence $\{x_n\}$ converges strongly to $q = P_\Theta(u)$.

If, in Theorem 3.2, we assume that $A_m \equiv 0$ for each $m \in \{1, 2, \ldots, N\}$, then we get the following result on finite family of split equilibrium problems and fixed point problem for Lipschitz hemicontractive-type multi-valued mapping.

Corollary 3.4. *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $B_m : H_1 \longrightarrow H_2$ be a bounded linear operator with its adjoint B_m^* for each $m \in \{1, 2, \ldots, N\}$. Let $F_{1,m} : C \times C \longrightarrow \mathbb{R}$ and $F_{2,m} : Q \times Q \longrightarrow \mathbb{R}$ be bifunctions satisfying Assumption 2.1 for each $m \in \{1, 2, \ldots, N\}$. Let $S : C \longrightarrow CB(C)$ be a $L-$Lipschitz hemicontractive-type multi-valued mapping such that $(I - S)$ is demiclosed at zero. Assume that $\Theta = \bigcap_{m=1}^{N} \Omega_m \bigcap F(S)$ is nonempty convex and $Sp = \{p\}$ for all $p \in \Theta$. Let $\{\beta_n\}$,$\{\alpha_n\}$,$\{\gamma_n\}$,$\{\sigma_n\}$ and $\{\tau_{n,m}\}$ be real sequences in $(0, 1)$ such that*

i. $\beta_n + \gamma_n + \sigma_n = 1$ *and* $0 < \alpha \leq \gamma_n, \sigma_n \leq \beta < 1$*;*

ii. $\lim_{n\to\infty} \beta_n = 0,\ \sum_{n=0}^{\infty} \beta_n = \infty$*;*

iii. $\sum_{m=1}^{N} \tau_{n,m} = 1$ *and* $0 < \delta \leq \tau_{n,m} \leq 1$*;*

iv. $\beta_n + \gamma_n \leq \alpha_n \leq \gamma < \frac{1}{\sqrt{1+4L^2}+1}$.

Let $\{x_n\}$ be a sequence in C generated by $x_0, u \in C$ by

$$\begin{cases} z_{n,m} = T_\sigma^{F_{1,m}} \left(I - \lambda_m B_m^* \left(I - T_\tau^{F_{2,m}}\right) B_m\right) x_n, \\ y_n = \sum_{m=1}^{N} \tau_{n,m} z_{n,m}, \\ u_n = (1 - \alpha_n) y_n + \alpha_n v_n, \\ x_{n+1} = \beta_n u + \gamma_n w_n + \sigma_n y_n, \end{cases}$$

for all $n \geq 0$, where $v_n \in Sy_n$ and $w_n \in Su_n$ such that $\|v_n - w_n\| \leq 2D(Sy_n, Su_n)$, $\sigma, \tau > 0$, $\lambda_m \in \left(0, \frac{1}{\eta_m}\right)$, for $\eta_m = \|B_m\|^2$. Then, the sequence $\{x_n\}$ converges strongly to a point $q = P_\Theta(u)$.

Remark 3.5. If, in Theorem 3.2, we assume that $H_1 = H_2$, $C = Q$, $B_m \equiv I$ and $F_{2,m} \equiv 0$ for each $m \in \{1, 2, \ldots, N\}$, then we obtain a result on a finite family of equilibrium and variational inequality problems and fixed point problem for Lipschitz hemicontractive-type multi-valued mapping.

Remark 3.6. We remark that

- It is known that the class of hemicontractive-type mappings contains the classes of quasi-nonexpansive and demicontractive mappings. Thus, the results obtained in this paper also hold for these classes of mappings provided that the indicated conditions are satisfied.

- Since every pseudocontractive multi-valued mapping S with $F(S) \neq \emptyset$ and $S(p) = \{p\}, \quad \forall p \in F(S)$, is a hemicontractive-type multi-valued mapping, our results can be applied for this class of mappings and hence for nonexpansive and $k-$strictly pseudocontractive multi-valued mappings provided that the specified assumptions are satisfied because every nonexpansive and $k-$strictly pseudocontractive mappings are pseudocontractive mapping.
- It is also well-known that the class of continuous monotone mappings includes the classes of Lipschitz monotone and $\alpha-$inverse strongly monotone mappings. Hence, our results hold for these classes of mappings provided that the stated assumptions are guaranteed.

Remark 3.7. Our results extend, improve and unify several recent results in the existing literature (e.g., [7, 8, 10, 11, 15, 16, 19, 28, 30, 31, 41] etc) in the sense that our iterative algorithm provides strong convergence to a common solution of a finite family of split equilibrium problems, a finite family of variational inequality problems and a fixed point problem for Lipschitz hemicontractive-type multi-valued mapping in Hilbert space settings. In particular, Theorem 3.2 extends the results of

(i) Okeke and Mewomo [19] from the class of quasi-nonexpansive multi-valued mappings to more general class of Lipschitz hemicontractive-type multi-valued mappings; and from the class of inverse strongly monotone mappings to more general class of continuous monotone mappings.
(ii) Ugwunnadi and Ali [28] from the class of single-valued pseudocontractive mappings to the class of Lipschitz hemicontractive-type multi-valued mappings.
(iii) Meche et al [15] from finite family of equilibrium problems to a finite family of split equilibrium problems.

Moreover, in our results a restriction of upper semi-continuity on the bifunctions is not required.

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