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Uniform Convergence to a Left Invariance on Weakly Compact Subsets

Ali Ghaffari^{1*}, Samaneh Javadi², and Ebrahim Tamimi³

ABSTRACT. Let $\{a_\alpha\}_{\alpha \in I}$ be a bounded net in a Banach algebra A and φ a nonzero multiplicative linear functional on A . In this paper, we deal with the problem of when $\|aa_\alpha - \varphi(a)a_\alpha\| \rightarrow 0$ uniformly for all a in weakly compact subsets of A . We show that Banach algebras associated to locally compact groups such as Segal algebras and L^1 -algebras are responsive to this concept. It is also shown that $Wap(A)$ has a left invariant φ -mean if and only if there exists a bounded net $\{a_\alpha\}_{\alpha \in I}$ in $\{a \in A; \varphi(a) = 1\}$ such that $\|aa_\alpha - \varphi(a)a_\alpha\|_{Wap(A)} \rightarrow 0$ uniformly for all a in weakly compact subsets of A . Other results in this direction are also obtained.

1. INTRODUCTION

Let A be an arbitrary Banach algebra and φ a character of A , that is a homomorphism from A onto \mathbb{C} . A is called φ -amenable if there exists a bounded linear functional m on A^* satisfying $\langle m, \varphi \rangle = 1$ and $\langle m, f.a \rangle = \varphi(a)\langle m, f \rangle$ for all $a \in A$ and $f \in A^*$. Approximating m in the weak* topology of A^{**} and then passing to convex combinations, we obtain a bounded net $\{a_\alpha\}_{\alpha \in I}$ in $\{a \in A; \varphi(a) = 1\}$ such that $\|aa_\alpha - \varphi(a)a_\alpha\| \rightarrow 0$ for all a in A [12]. On the other hand, whenever we have a bounded net $\{a_\alpha\}_{\alpha \in I}$ in $\{a \in A; \varphi(a) = 1\}$ such that $\|aa_\alpha - \varphi(a)a_\alpha\| \rightarrow 0$, then each of its weak* accumulation points in A^{**} is a left invariant φ -mean on A^* . For more details on φ -amenability of a Banach algebra the interested reader is referred to [9, 12, 15]. This concept considerably generalizes the notion of left amenability for Lau algebras. Recently

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the notion of α -amenable hypergroups was introduced and studied in [1, 2, 6]. It is clearly that the net $\{a_\alpha\}_{\alpha \in I}$ can be chosen in such a way that $\|aa_\alpha - \varphi(a)a_\alpha\| \rightarrow 0$ uniformly for all a in compact subsets of A . The present paper grew out the attempt to extend the uniform convergence to weakly compact subsets of A .

We shall investigate that this problem is true over a Segal algebra. It has motivated large parts of this paper. In particular, we shall consider the special case $S(G) = L^1(G)$ and it is shown that this problem is equivalent to the amenability of G . Although we are not able to answer for general, we show $Wap(A)$ has a left invariant φ -mean if and only if there exists a bounded net $\{a_\alpha\}_{\alpha \in I}$ in $\{a \in A; \varphi(a) = 1\}$ such that $\|aa_\alpha - \varphi(a)a_\alpha\|_{Wap(A)} \rightarrow 0$ uniformly for all a in weakly compact subsets of A .

2. NOTATION AND PRELIMINARY

In this paper, the second dual A^{**} of a Banach algebra A will always be equipped with the first Arens product which is defined as follows. For $a, b \in A$, $f \in A^*$ and $m, n \in A^{**}$, the elements $f.a$ and $m.f$ of A^* and $mn \in A^{**}$ are defined by

$$\langle f.a, b \rangle = \langle f, ab \rangle, \quad \langle n.f, a \rangle = \langle n, f.a \rangle, \quad \langle mn, f \rangle = \langle m, n.f \rangle,$$

respectively. With this multiplication, A^{**} is a Banach algebra and A is a subalgebra of A^{**} [3]. A functional $f \in A^*$ for which $\{f.a; \|a\| \leq 1\}$ is relatively compact in the weak topology of A^* is said to be weakly almost periodic. The set of weakly almost periodic functionals on A is denoted by $Wap(A)$ (see [4, 8]).

Recall that a Segal algebra $S(G)$ on a locally compact group G , is a dense left ideal of $L^1(G)$ that satisfies the following conditions:

- (i) $S(G)$ is a Banach space with respect to a norm $\|\cdot\|_S$, called a Segal norm, satisfying $\|\psi\|_1 \leq \|\psi\|_S$ for $\psi \in S(G)$, where $\|\cdot\|_1$ denotes the L^1 -norm.
- (ii) For $\psi \in S(G)$ and $y \in G$, $L_y\psi \in S(G)$, where L_y is the left translation operator defined by $L_y\psi(x) = \psi(y^{-1}x)$, $x \in G$. Moreover, the left translation $L_y\psi$, $y \in G$, is continuous in y for each $\psi \in S(G)$.
- (iii) The equality $\|L_y\psi\|_S = \|\psi\|_S$ holds for $\psi \in S(G)$, $y \in G$.

Equipped with the norm $\|\cdot\|_S$ and the convolution product, denoted by $*$, $S(G)$ is a Banach algebra. The inequality $\|h * \psi\|_S \leq \|h\|_1 \|\psi\|_S$ holds for all $h \in L^1(G)$, and $\psi \in S(G)$. The structure of the Segal algebra has been studied in [17].

Finally, we say that an element a of A is φ -maximal if it satisfies $\|a\| = \varphi(a) = 1$. Let $P_1(A, \varphi)$ denote the collection of all φ -maximal

elements of A [11]. When A is an Lau algebra and φ is the identity of the von Neumann algebra A^* , the φ -maximal elements are precisely the positive linear functionals of norm 1 on A^* and hence span A . Let $X(A, \varphi)$ denote the closed linear span of $P_1(A, \varphi)$. Throughout the paper, $\Delta(A)$ will denote the set of all homomorphisms from A onto \mathbb{C} .

3. MAIN RESULTS

Let A be a Banach algebra and let X be a closed subspace of A^* . We say that X is invariant if $f.a \in X$ whenever $f \in X$ and $a \in A$.

Definition 3.1. Let A be a Banach algebra and let X be a closed subspace of A^* with $\varphi \in X$ that is invariant. A continuous functional m on X is called a left invariant φ -mean on X if the following properties hold:

$$\langle m, \varphi \rangle = 1, \quad \langle m, f.a \rangle = \varphi(a) \langle m, f \rangle, \quad (f \in X, a \in A)$$

Definition 3.2. A net $\{a_\alpha\}_{\alpha \in I}$ in $\{a \in A; \varphi(a) = 1\}$ is said to converges strongly to a left invariance uniformly on weakly compact subsets of A if for every weakly compact set $C \subseteq A$, $\|aa_\alpha - \varphi(a)a_\alpha\| \rightarrow 0$ uniformly for all $a \in C$.

In the following, $P_1((S(G), \|\cdot\|_1), 1)$ denotes the collection of all 1-maximal elements of a Segal algebra $S(G)$ with respect to L^1 -norm.

Theorem 3.3. *Let G be a locally compact group. Then the following statements are equivalent:*

- (i) *There is a net $\psi_\alpha \in P_1((S(G), \|\cdot\|_1), 1)$ such that $\|\psi * \psi_\alpha - \psi_\alpha\|_S \rightarrow 0$ for each $\psi \in P_1((S(G), \|\cdot\|_1), 1)$.*
- (ii) *There is a net $\psi_\alpha \in P_1((S(G), \|\cdot\|_1), 1)$ such that for each weakly compact subset $C \subseteq P_1((S(G), \|\cdot\|_1), 1)$, $\|\psi * \psi_\alpha - \psi_\alpha\|_S \rightarrow 0$ uniformly for all $\psi \in C$.*

Proof. (ii) implies (i): This is because the finite subsets in $P_1((S(G), \|\cdot\|_1), 1)$ are weakly compact.

(i) implies (ii): Let $\{\psi_\alpha\}_{\alpha \in I} \subseteq P_1((S(G), \|\cdot\|_1), 1)$ be as in (i). By definition $\|\psi\|_1 \leq \|\psi\|_S$ for all $\psi \in S(G)$, and so $\|\psi * \psi_\alpha - \psi_\alpha\|_1 \rightarrow 0$ for each $\psi \in P_1((S(G), \|\cdot\|_1), 1)$. We can assume that ψ_α is left equicontinuous (that is, given $\epsilon > 0$, there is some neighborhood U of the identity in G such that $\|\delta_x * \psi_\alpha - \psi_\alpha\|_1 < \epsilon$ for any α and $x \in U$) otherwise replace ψ_α by $\psi * \psi_\alpha$ where ψ is a fixed element in $P_1((S(G), \|\cdot\|_1), 1)$. We claim that for every weakly compact subset C of $P_1((S(G), \|\cdot\|_1), 1)$ and $\epsilon \in (0, 1)$, there exists α_0 such that $\|\psi * \psi_\alpha - \psi_\alpha\|_1 < \epsilon$ for all $\alpha \succeq \alpha_0$ and $\psi \in C$. Let ψ_0 be a fixed element in $P_1((S(G), \|\cdot\|_1), 1)$. For

the forward implication, note that the weak topology on $S(G)$ is finer than the relative weak topology on $S(G)$ inherited from $L^1(G)$. By Theorem 4.21.2 in [5], there exists a compact set K in G such that $\int_{G \setminus K} \psi(x) dx < \frac{\epsilon}{4\|\psi_0\|_S}$ for all $\psi \in C$. By the above argument, there exists $\alpha_0 \in I$ such that $\|\delta_x * \psi_\alpha - \psi_\alpha\|_1 < \frac{\epsilon}{2\|\psi_0\|_S}$ for all $\alpha \succeq \alpha_0$ and $x \in K$ (see Proposition 6.7 in [16]). For each $\alpha \succeq \alpha_0$ and $\psi \in C$, we have

$$\begin{aligned} \|\psi * \psi_\alpha - \psi_\alpha\|_1 &= \int \left| \int \psi_\alpha(y^{-1}x) \psi(y) dy - \psi_\alpha(x) \right| dx \\ &\leq \int \left| \int_K (\psi_\alpha(y^{-1}x) - \psi_\alpha(x)) \psi(y) dy \right| dx \\ &\quad + \int \int_{G \setminus K} |\psi_\alpha(y^{-1}x) - \psi_\alpha(x)| \psi(y) dy dx \\ &< \frac{\epsilon \int_K \psi(y) dy}{2\|\psi_0\|_S} + 2 \int_{G \setminus K} \psi(y) dy \int \psi_\alpha(x) dx \\ &< \frac{\epsilon}{\|\psi_0\|_S}. \end{aligned}$$

Let us define $\phi_\alpha = \psi_\alpha * \psi_0$. For each $\alpha \succeq \alpha_0$ and $\psi \in C$, we have

$$\begin{aligned} \|\psi * \phi_\alpha - \phi_\alpha\|_S &= \|\psi * \psi_\alpha * \psi_0 - \psi_\alpha * \psi_0\|_S \\ &\leq \|\psi * \psi_\alpha - \psi_\alpha\|_1 \|\psi_0\|_S \\ &< \epsilon. \end{aligned} \quad \square$$

Let G be a locally compact group with left Haar measure and consider the convolution algebra $L^1(G)$ [7]. Note that the group algebra $L^1(G)$ is amenable with respect to the trivial character 1 precisely when G is amenable [10]. The preceding proposition shows that if G is an amenable locally compact group, then $L^1(G)$ has a bounded net which converges strongly to a left invariance uniformly on weakly compact subsets of $L^1(G)$.

As a straightforward application of our main result, we have the following result:

Corollary 3.4. *Let G be a locally compact group. Then the following statements are equivalent:*

- (i) *There is a net $\psi_\alpha \in P_1(L^1(G), 1)$ such that $\|\psi * \psi_\alpha - \psi_\alpha\|_1 \rightarrow 0$ for each $\psi \in P_1(L^1(G), 1)$, i.e. G is amenable;*
- (ii) *There is a net $\psi_\alpha \in P_1(L^1(G), 1)$ such that for each weakly compact subset $C \subseteq L^1(G)$, $\|\psi * \psi_\alpha - \int \psi(x) dx \psi_\alpha\|_1 \rightarrow 0$ uniformly for all $\psi \in C$.*

Proof. As $L^1(G)$ is a Segal algebra, this is just a re-statement of Theorem 3.3. \square

Let A be an arbitrary Banach algebra. It remains an open question, to the author's knowledge, whether the existence of a bounded net $\{a_\alpha\}_{\alpha \in I}$ in A which converges strongly to a left invariance uniformly on weakly compact subsets of A is equivalent to $\|aa_\alpha - \varphi(a)a_\alpha\| \rightarrow 0$ for each $a \in A$. We show this is the case for,

$$\|a\|_{Wap(A)} = \sup \{ |\langle f, a \rangle| : f \in Wap(A), \|f\| \leq 1 \}, \quad (a \in A)$$

The reason why we are interested in $Wap(A)$ is the following:

Theorem 3.5. *Let A be a Banach algebra with a bounded approximate identity and $\varphi \in \Delta(A)$. Then the following statements are equivalent:*

- (i) *There exists a bounded net $\{a_\alpha\}_{\alpha \in I}$ in $\{a \in A; \varphi(a) = 1\}$ such that $\|aa_\alpha - \varphi(a)a_\alpha\|_{Wap(A)} \rightarrow 0$ for each $a \in A$;*
- (ii) *There exists a bounded net $\{a_\alpha\}_{\alpha \in I}$ in $\{a \in A; \varphi(a) = 1\}$ such that for each weakly compact subset $C \subseteq A$, $\|aa_\alpha - \varphi(a)a_\alpha\|_{Wap(A)} \rightarrow 0$ uniformly for all $a \in C$.*

Proof. By the Banach Alaoghlu's Theorem [18], without loss of generality we may assume that $a_\alpha \rightarrow m$ in the weak* topology of A^{**} . Then $\langle m, f.a \rangle = \varphi(a)\langle m, f \rangle$, for all $f \in Wap(A)$, $a \in A$ [12]. Let $T_f : A \rightarrow A^*$ be a bounded linear mapping specified by $T_f(a) = f.a$. Define the map $\kappa_A : A^* \rightarrow B(A, A^*)$ by $\kappa_A(f) = T_f$. Take $f \in Wap(A)$ and consider $\{a_\alpha f\}_{\alpha \in I}$. The corresponding net $\{T_{a_\alpha f}\}_{\alpha \in I}$ converges to T_{mf} in the weak operator topology. This is immediate from the fact that the weak topology and weak* topology coincide on weak closure $\overline{\{a.f : \|a\| \leq \|m\|\}}$ of $\{a.f : \|a\| \leq \|m\|\}$. The equicontinuity of $\{T_{a_\alpha f}\}_{\alpha \in I}$ is now an exercise in functional analysis. Let C be any weakly compact subset of A . C is weakly bounded, and so C is norm bounded (see Theorem 3.18 in [18]). Let $M = \sup \{\|c\| : c \in C\}$. The net $\{T_{a_\alpha f}\}_{\alpha \in I}$ converges uniformly to $T_{m.f}$ in the weak operator topology on C . This latter fact is crucial for our argument, so we give a proof.

Let W be a weak neighborhood of zero in A^* . Choose a weak neighborhood V of zero in A^* such that $V + V + V \subseteq W$ and a symmetric weak neighborhood U of zero in A such that $T_{a_\alpha f}(U) \subseteq V$ for all $\alpha \in I$ and $T_{mf}(U) \subseteq V$. C is weakly compact, and therefore $C \subseteq S_0 + U$ for some finite set $S_0 = \{a_1, a_2, \dots, a_n\}$. It is a routine matter to see that there exists $\alpha_0 \in I$ such that $T_{a_\alpha f}(a_i) - T_{mf}(a_i) \in V$ for all $\alpha \succeq \alpha_0$ and $a_i \in S_0$. For $\alpha \succeq \alpha_0$ and $a \in C$, we have

$$(T_{a_\alpha f} - T_{mf})(a) \in \bigcup_{i=1}^n (T_{a_\alpha f} - T_{mf})(a_i) + (T_{a_\alpha f} - T_{mf})(U)$$

$$\begin{aligned} &\subseteq \bigcup_{i=1}^n (T_{a_\alpha f} - T_{mf})(a_i) + T_{a_\alpha f}(U) - T_{mf}(U) \\ &\subseteq V + V + V \subseteq W. \end{aligned}$$

By the above argument, for any given $\epsilon > 0$ and any $n \in A^{**}$, there exists $\alpha_0 \in I$ such that

$$|\langle n, T_{a_\alpha f}(a) - T_{mf}(a) \rangle| < \frac{\epsilon}{2}.$$

for all $\alpha \succeq \alpha_0$ and $a \in C$. On the other hands, A has an approximate identity $\{e_\alpha\}_{\alpha \in I}$. Any weak*-lim E of $\{e_\alpha\}_{\alpha \in I}$ is a right identity of Banach algebra A^{**} . Hence for all $\alpha \succeq \alpha_0$ and $a \in C$,

$$\begin{aligned} |\langle aa_\alpha - am, f \rangle| &= |\langle a_\alpha f - mf, a \rangle| \\ &= |\langle E, a_\alpha f.a - mf.a \rangle| \\ &< \frac{\epsilon}{2}. \end{aligned}$$

We also have $|\langle a_\alpha, f \rangle - \langle m, f \rangle| < \frac{\epsilon}{2M}$ for all $\alpha \succeq \alpha_0$. Consequently

$$\begin{aligned} |\langle aa_\alpha - \varphi(a)a_\alpha, f \rangle| &\leq |\langle aa_\alpha - am, f \rangle| + |\varphi(a)| |\langle m, f \rangle - \langle a_\alpha, f \rangle| \\ &< \epsilon. \end{aligned}$$

This means that $aa_\alpha - \varphi(a)a_\alpha \rightarrow 0$ uniformly in the weak topology of $Wap(A)$ for all $a \in C$. An argument similar to that in the proof of Theorem 1.2 in [12] shows that we can find a bounded net $\{u_\alpha\}_{\alpha \in I}$ consisting of convex combination of elements in $\{a_\alpha\}_{\alpha \in I}$ such that $\|au_\alpha - \varphi(a)u_\alpha\|_{Wap(A)} \rightarrow 0$ uniformly for all $a \in C$. \square

A special interesting case is that there exists a left invariant φ -mean on A^* . We obtain:

Theorem 3.6. *Let $\{a_\alpha\}_{\alpha \in I}$ be a bounded net in $\{a \in A; \varphi(a) = 1\}$ which converges strongly to a left invariance uniformly on weakly compact subsets of A and let m be a left invariant φ -mean on A^* . Then there is a net $\{b_\beta\}_{\beta \in J}$ in $\{a \in A; \varphi(a) = 1\}$ such that $b_\beta \rightarrow m$ in the weak* topology and $\{b_\beta\}_{\beta \in J}$ converges strongly to a left invariance uniformly on weakly compact subsets of A .*

Proof. Let such a net $\{a_\alpha\}_{\alpha \in I}$ exists. Choose a net $\{b_\beta\}_{\beta \in J}$ in A with the property that $b_\beta \rightarrow m$ in the weak* topology of A^{**} and $\|b_\beta\| \leq \|m\|$ for all $\beta \in J$ [18]. Since $\langle b_\beta, \varphi \rangle \rightarrow \langle m, \varphi \rangle = 1$, after passing to a subnet and replacing b_β by $\frac{1}{\varphi(b_\beta)} b_\beta$, we can assume that $\varphi(b_\beta) = 1$ and $\|b_\beta\| \leq \|m\| + 1$ for all $\beta \in J$. For each (α, f) in the product directed set $I \times \prod \{J; \alpha \in I\}$, we define $R(\alpha, f) = (\alpha, f(\alpha))$, $\alpha \in I$, $f \in \prod \{J; \alpha \in I\}$ and let $S(\alpha, \beta) = a_\alpha b_\beta$. The iterated limit $\lim_\alpha \lim_\beta a_\alpha b_\beta$

(in the weak* topology of A^{**}) exists and is equal to m . Indeed, for $f \in A^*$

$$\begin{aligned} \lim_{\beta} \langle f, a_{\alpha} b_{\beta} \rangle &= \lim_{\beta} \langle f a_{\alpha}, b_{\beta} \rangle \\ &= \lim_{\beta} \langle b_{\beta}, f a_{\alpha} \rangle \\ &= \langle m, f a_{\alpha} \rangle \\ &= \langle m, f \rangle. \end{aligned}$$

By the Iterated Limit Theorem, see p.69 in [13],

$$\begin{aligned} \lim_{(\alpha, f)} SoR(\alpha, f) &= \lim_{(\alpha, f)} a_{\alpha} b_{f(\alpha)} \\ &= m \end{aligned}$$

in the weak* topology of A^{**} (with respect to (α, f)). It remains to show that $SoR(\alpha, f)$ converges strongly to a left invariance uniformly on weakly compact subsets C of A . Let $\epsilon > 0$ be given. For every weakly compact subset C of A , there exists $\alpha_0 \in I$ such that $\|a a_{\alpha} - \varphi(a) a_{\alpha}\| < \frac{\epsilon}{\|m\|+1}$ for all $\alpha \succeq \alpha_0$ and $a \in C$. If $\alpha \succeq \alpha_0$ and $a \in C$, then

$$\begin{aligned} \|a SoR(\alpha, f) - \varphi(a) SoR(\alpha, f)\| &= \|a a_{\alpha} b_{f(\alpha)} - \varphi(a) a_{\alpha} b_{f(\alpha)}\| \\ &\leq \|a a_{\alpha} - \varphi(a) a_{\alpha}\| (\|m\| + 1) \\ &< \epsilon. \end{aligned}$$

This completes the proof. \square

Proposition 3.7. *Let A be a Banach algebra and $\varphi \in \Delta(A)$. Then the following statements are equivalent:*

- (i) *There exists a net $\{a_{\alpha}\}_{\alpha \in I}$ in $\{a \in A; \varphi(a) = 1\}$ such that $\{a_{\alpha}\}_{\alpha \in I}$ converges to some left invariant φ -mean m with $\|m\| = 1$ in the weak* topology and $\{a_{\alpha}\}_{\alpha \in I}$ converges strongly to a left invariance uniformly on weakly compact subsets of A ;*
- (ii) *For every weakly compact subset C of A and $\epsilon > 0$,*

$$\inf \{ \sup \{ \|ca\|; c \in C \}, \varphi(a) = 1, \|a\| \leq 1 + \epsilon \} \leq (1 + \epsilon) \sup \{ |\varphi(c)|; c \in C \};$$

- (iii) *There exists a net $\{a_{\alpha}\}_{\alpha \in I}$ in A with the following properties: $\varphi(a_{\alpha}) = 1$ for all $\alpha \in I$, $\|a_{\alpha}\| \rightarrow 1$ and $\lim_{\alpha} \|a a_{\alpha}\| = |\varphi(a)|$ uniformly on weakly compact subsets of A .*

Proof. (i) implies (ii): Let C be a weakly compact subset of A , $\epsilon > 0$ and let $\delta > 0$ be given. By hypothesis there exists $\alpha_0 \in I$ such that $\|c a_{\alpha} - \varphi(c) a_{\alpha}\| < \delta$, $\|a_{\alpha}\| \leq 1 + \epsilon$ for all $\alpha \succeq \alpha_0$ and $c \in C$. Thus for every $c \in C$,

$$\begin{aligned} \|c a_{\alpha_0}\| &\leq |\varphi(c)| \|a_{\alpha_0}\| + \delta \\ &< (1 + \epsilon) |\varphi(c)| + \delta. \end{aligned}$$

Since $\delta > 0$ may be chosen arbitrarily, the property holds.

- (ii) implies (i): We claim that for every weakly compact subset C of A and $\epsilon > 0$, there exists $a_{C,\epsilon}$ such that $\varphi(a_{C,\epsilon}) = 1$, $\|a_{C,\epsilon}\| \leq 1 + \epsilon$ and $\|ca_{C,\epsilon} - \varphi(c)a_{C,\epsilon}\| < \epsilon$ for all $c \in C$. Choose $\delta > 0$ such that $(1 + \delta)^2 < 1 + \epsilon$. Take $b_{C,\epsilon} \in A$ such that $\varphi(b_{C,\epsilon}) = 1$ and $\|b_{C,\epsilon}\| \leq 1 + \delta$. Obviously

$$\{c - \varphi(c)b_{C,\epsilon}; c \in C\} \cup \{cb_{C,\epsilon} - c; c \in C\}$$

is weakly compact and also $\varphi(c - \varphi(c)b_{C,\epsilon}) = \varphi(cb_{C,\epsilon} - c) = 0$ for all $c \in C$. By assumption, there exists $a_{C,\epsilon}' \in A$ with $\|a_{C,\epsilon}'\| \leq 1 + \delta$, $\varphi(a_{C,\epsilon}') = 1$ such that $\|(c - \varphi(c)b_{C,\epsilon})a_{C,\epsilon}'\| < \frac{\epsilon}{2}$ and $\|cb_{C,\epsilon}a_{C,\epsilon}' - ca_{C,\epsilon}'\| < \frac{\epsilon}{2}$ for all $c \in C$. Put $a_{C,\epsilon} = b_{C,\epsilon}a_{C,\epsilon}'$. Thus $\|a_{C,\epsilon}\| = \|b_{C,\epsilon}a_{C,\epsilon}'\| \leq (1 + \delta)^2 \leq 1 + \epsilon$ and $\varphi(a_{C,\epsilon}) = 1$. For every $c \in C$, we have

$$\begin{aligned} \|ca_{C,\epsilon} - \varphi(c)a_{C,\epsilon}\| &= \|cb_{C,\epsilon}a_{C,\epsilon}' - \varphi(c)b_{C,\epsilon}a_{C,\epsilon}'\| \\ &\leq \|cb_{C,\epsilon}a_{C,\epsilon}' - ca_{C,\epsilon}'\| + \|ca_{C,\epsilon}' - \varphi(c)b_{C,\epsilon}a_{C,\epsilon}'\| \\ &< \epsilon. \end{aligned}$$

Now, order the pairs (C, ϵ) , $C \subseteq A$ weakly compact, $\epsilon > 0$, in the obvious manner, and let m be a weak* cluster point of the net $\{a_{C,\epsilon}\}$ in A . Then $\|m\| \leq 1$, $\langle m, \varphi \rangle = 1$ and hence $\|m\| = 1$. So $\{a_{C,\epsilon}\}_{C,\epsilon}$ is the required net.

- (iii) implies (ii): Let $\epsilon > 0$ and let C be a weakly compact subset of A . For every $\delta > 0$, there exists $\alpha_0 \in I$ such that $\|ca_\alpha\| - |\varphi(c)| < \delta$ and $\|a_\alpha\| \leq 1 + \epsilon$ for every $\alpha \succeq \alpha_0$ and $c \in C$. Then

$$\begin{aligned} &\inf \{\sup \{\|ca\|; c \in C\}, \varphi(a) = 1, \|a\| \leq 1 + \epsilon\} \\ &\leq \inf \{\sup \{\|ca_\alpha\|; c \in C\}, \alpha \in I\} \\ &\leq (1 + \epsilon) \sup \{|\varphi(c)|; c \in C\} + \delta. \end{aligned}$$

Since $\delta > 0$ may be chosen arbitrarily, the property holds.

- (i) implies (iii): By hypothesis there exists a net $\{a_\alpha\}_{\alpha \in I}$ in A such that $\varphi(a_\alpha) = 1$ for all $\alpha \in I$, $\|a_\alpha\| \rightarrow 1$ and $\|aa_\alpha - \varphi(a)a_\alpha\| \rightarrow 0$ uniformly on weakly compact subsets of A . Let $\epsilon > 0$ and let C be a weakly compact subset of A . Since C is a weakly compact subset of A , C is weakly bounded and so $\{|\varphi(c)|; c \in C\}$ is bounded [18]. Let $k = \sup \{|\varphi(c)|; c \in C\}$. For every $\alpha \in I$ and $c \in C$, we have

$$\begin{aligned} \|aa_\alpha\| - |\varphi(a)| &\leq \|aa_\alpha\| - |\varphi(a)||a_\alpha| + |\varphi(a)|||a_\alpha| - 1| \\ &\leq \|aa_\alpha - \varphi(a)a_\alpha\| + k||a_\alpha| - 1|. \end{aligned}$$

This shows that $\lim_\alpha \|aa_\alpha\| = |\varphi(a)|$ uniformly on weakly compact subsets of A .

□

Let A be a Lau algebra. The identity of A^* will be denoted by e . Also $P(A)$ will denote the cone of all positive functionals in A and $P_1(A)$ will denote the set of all $f \in P(A)$ such that $f(e) = 1$. Lau in [14] proved that A is left amenable if and only if there exists a net $f_\alpha \in P_1(A)$ such that $\lim_\alpha \|f \cdot f_\alpha\| = |f(e)|$ for each $f \in A$. Note that a Banach algebra A^{**} has a left invariant φ -mean if any one of the conditions in Proposition 1 hold.

Definition 3.8. Let A be a Banach algebra and let Z be a compact convex subset of a locally convex Hausdorff topological vector space E . The pair (A, Z) is called a *flow*, if;

- (i) There exists a map $\rho : A \times E \rightarrow E$ such that for each $z \in Z$, the map $\rho(-, z) : A \rightarrow E$ is continuous and linear when A has the weak topology;
- (ii) For any $a, b \in A$ and $z \in Z$, $\rho(a, \rho(b, z)) = \rho(ab, z)$.

If $\varphi \in \Delta(A)$, we say that Z is $P_1(A, \varphi)$ -invariant under ρ if $\rho(a, z) \in Z$ for any $a \in P_1(A, \varphi)$ and $z \in Z$. In this case ρ induces a map $\rho : P_1(A, \varphi) \times Z \rightarrow Z$ of $P_1(A, \varphi)$ on the compact convex subset Z (as affine maps now).

Theorem 3.9. Let A be a Banach algebra and $\varphi \in \Delta(A)$. Among the following two properties, the implication (i) \rightarrow (ii) hold. If $X(A, \varphi) = A$, then (ii) \rightarrow (i).

- (i) There exists a left invariant φ -mean m in $\overline{P_1(A, \varphi)}^{w^*}$;
- (ii) Every flow (A, Z) admits a $P_1(A, \varphi)$ -invariant element $z \in Z$, that is, for all $a \in P_1(A, \varphi)$, $\rho(a, z) = z$.

Proof. Assume that A^{**} has a left invariant φ -mean $m \in \overline{P_1(A, \varphi)}^{w^*}$. Let Z be a compact convex subset of a locally convex Hausdorff topological vector space E and let (A, Z) be a flow. If $f \in E^*$ and $z \in Z$, we may define a functional f^z on A by putting $\langle f^z, a \rangle = \langle f, \rho(a, z) \rangle$, $a \in A$. Since the map $a \mapsto \rho(a, z)$ is continuous, we have $f^z \in A^*$. We embed E into the algebraic dual $(E^*)'$ of E^* with the topology $\sigma((E^*)', E^*)$. If Λ is a $\sigma((E^*)', E^*)$ -cluster point of Z , then there exists a net $\{z_\alpha\}_{\alpha \in I}$ in Z such that $z_\alpha \rightarrow \Lambda$ in the $\sigma((E^*)', E^*)$ -topology. Since Z is compact in E , without loss of generality, we may assume that $z_\alpha \rightarrow z$ for some $z \in Z$. For every $f \in E^*$, we have $\langle z_\alpha, f \rangle \rightarrow \langle \Lambda, f \rangle$ and also $\langle f, z_\alpha \rangle \rightarrow \langle f, z \rangle$. We conclude that $\Lambda = z \in Z$, and so Z is a closed subset in $(E^*)'$.

Let z_0 be a fixed element in Z and let $n \in \overline{P_1(A, \varphi)}^{w^*}$. Define $\Lambda_n : E^* \rightarrow \mathbb{C}$ by $\Lambda_n(f) = \langle n, f^{z_0} \rangle$. It is easily checked that Λ_n is linear, and so $\Lambda_n \in (E^*)'$. Define $\Lambda : \overline{P_1(A, \varphi)}^{w^*} \rightarrow (E^*)'$ by $\Lambda(n) = \Lambda_n$. The

mapping Λ from $\overline{P_1(A, \varphi)}^{w^*}$ equipped with the weak* topology into $(E^*)'$ equipped with the $\sigma((E^*)', E^*)$ -topology is continuous. In particular, if $a \in P_1(A, \varphi)$, $P_1(A, \varphi)$ -invariance of Z imply that $\Lambda(a) = \Lambda_a \in Z$. Indeed, $\Lambda_a = \rho(a, z_0)$. Since $P_1(A, \varphi)$ is weak* dense in $\overline{P_1(A, \varphi)}^{w^*}$ and Z is closed in $(E^*)'$, we conclude that $\Lambda_m \in Z$. We shall show that Λ_m is the required fixed point. Let $a \in P_1(A, \varphi)$ and $f \in E^*$. We consider the mapping $\rho_a : Z \rightarrow Z$ defined by $\rho_a(z) = \rho(a, z)$. We have

$$\begin{aligned} \langle f, \Lambda_m \rangle &= \langle m, f^{z_0} \rangle \\ &= \langle m, f^{z_0} a \rangle \\ &= \langle m, (f \circ \rho_a)^{z_0} \rangle \\ &= \langle f \circ \rho_a, \Lambda_m \rangle \\ &= \langle f, \rho(a, \Lambda_m) \rangle. \end{aligned}$$

This shows that $\rho(a, \Lambda_m) = \Lambda_m$, that is, Λ_m is a fixed point under the map ρ .

Conversely, assume (ii). Let $E = A^{**}$ with weak* topology and $Z = \overline{P_1(A, \varphi)}^{w^*}$. By the Banach-Alaoglu's theorem [18], Z is weak* compact. Define a map ρ of $A \times A^{**}$ into A^{**} by $\rho(a, p) = ap$ for each $a \in A$ and $p \in A^{**}$. Let p be a fixed element in A^{**} and let $\{a_\alpha\}_{\alpha \in I}$ be a net in A converging to $a \in A$ in the weak topology of A . Then, for $f \in A^*$,

$$\begin{aligned} \lim_{\alpha} \langle a_\alpha p, f \rangle &= \lim_{\alpha} \langle a_\alpha, pf \rangle \\ &= \lim_{\alpha} \langle pf, a_\alpha \rangle \\ &= \langle pf, a \rangle \\ &= \langle ap, f \rangle. \end{aligned}$$

This shows that the mapping $a \mapsto \rho(a, p)$ is continuous. By hypothesis there exists $m \in Z = \overline{P_1(A, \varphi)}^{w^*}$ that is fixed under the map ρ , that is, for every $a \in P_1(A, \varphi)$, $am = m$. Hence m is a left invariant φ -mean. \square

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