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On the Spaces of λ_r -almost Convergent and λ_r -almost Bounded Sequences

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ABSTRACT. The aim of the present work is to introduce the concept of λ_r -almost convergence of sequences. We define the spaces $f(\lambda_r)$ and $f_0(\lambda_r)$ of λ_r -almost convergent and λ_r -almost null sequences. We investigate some inclusion relations concerning those spaces with examples and we determine the β - and γ -duals of the space $f(\lambda_r)$. Finally, we give the characterization of some matrix classes.

1. PRELIMINARIES AND BACKGROUND

By w , we denote the space of all real or complex valued sequences. Any vector subspace of w is called sequence space. We write ℓ_∞ , c , c_0 for the classical sequence spaces of all bounded, convergent, null, respectively. Throughout this paper, we simply write $x = (x_k)$ instead of $x = (x_k)_{k=0}^\infty$ and $\mathbb{N} = \{0, 1, 2, \dots\}$.

Let X be a sequence space. If X is a Banach space and

$$\tau_k : X \rightarrow \mathbb{C}, \quad \tau_k(x) = x_k$$

is a continuous for all $k \in \mathbb{N}$, X is called a BK -space. The sequence spaces ℓ_∞ , c and c_0 are BK -spaces with the norm given by $\|x\|_\infty = \sup_k |x_k|$ for all $k \in \mathbb{N}$.

A continuous linear functional ϕ on ℓ_∞ is called a Banach limit if

- (i) $\phi(x) \geq 0$ for $x = (x_k)$ and $x_k \geq 0$ for every k ,
- (ii) $\phi(x_{\sigma(k)}) = \phi(x_k)$, where σ is shift operator which is defined on w by $\sigma(k) = k + 1$ and

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(iii) $\phi(e) = 1$, where $e = (1, 1, 1, \dots)$.

A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent to the generalized limit α if all Banach limits of x are α and denoted by $f\text{-}\lim x = \alpha$. Lorentz [17] introduced that $f\text{-}\lim x_k = \alpha$ uniformly in n if and only if

$$\lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m x_{k+n} \text{ uniformly in } n.$$

We denote the sets of all almost convergent sequences f by

$$f = \left\{ x = (x_k) \in w : \lim_{m \rightarrow \infty} t_{mn}(x) = \alpha \text{ uniformly in } n \right\},$$

where

$$t_{mn}(x) = \sum_{k=0}^m \frac{1}{m+1} x_{k+n}, \quad t_{-1,n} = 0.$$

It is well known that $c \subset f \subset \ell_\infty$ strictly hold. Since these inclusions, norms $\|\cdot\|_f$ and $\|\cdot\|_{\ell_\infty}$ of the spaces f and ℓ_∞ are equivalent, so the sets f and f_0 are BK -spaces with the norm $\|x\|_f = \sup_{m,n} |t_{mn}(x)|$.

A matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} = 0$ for all $n \in \mathbb{N}$. It is trivial that $A(Bx) = (AB)x$ holds for the triangle matrices A, B and a sequence x . Further, a triangle matrix U uniquely has an inverse $U^{-1} = V$ which is also a triangle matrix. Then $x = U(Vx) = V(Ux)$ holds for all $x \in w$.

If A is an infinite matrix with complex entries a_{nk} for $n, k \in \mathbb{N}$, then we write $A = (a_{nk})$ instead of $A = (a_{nk})_{n,k=0}^\infty$. Any sequence in the n^{th} row of A is indicated by A_n , that is $A_n = (a_{nk})_{k=0}^\infty$ for every $n \in \mathbb{N}$. If $x = (x_k) \in w$ then we define the A -transform of x as the sequence $Ax = (A_n(x))_{n=0}^\infty$, where

$$(1.1) \quad A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k,$$

provided the series (1.1) converges for $n \in \mathbb{N}$. $x = (x_k)$ is called A -summable to $a \in \mathbb{C}$ if Ax converges to a which is called A -limit of x . If $x \in X$ implies that $Ax \in Y$, then we say that A defines a matrix mapping from X into Y and denote it by $A : X \rightarrow Y$. By $(X : Y)$ we mean the class of all infinite matrices such that $A : X \rightarrow Y$.

For an arbitrary sequence space X , the matrix domain of an infinite matrix A in X is defined by

$$(1.2) \quad X_A = \{x \in w : Ax \in X\},$$

which is a sequence space. If A is triangle, then one can easily observe that the sequence spaces X_A and X are linearly isomorphic, i.e., $X_A \cong X$.

Constructing a new sequence space X_A generated by the limitation matrix A from a sequence space X is the expansion or the contraction of the original space X . Using domain of a triangle matrix to construct a new sequence spaces was studied by many authors. (for instance [1]-[15])

2. ON THE CONCEPT OF λ_r -SUMMABILITY

Let $\Lambda = \{\lambda_k : k = 0, 1, \dots\}$ be a set which consists of strictly increasing sequence of positive numbers tending to ∞ , that is $0 < \lambda_0 < \lambda_1 < \dots$ and $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$. Throughout this paper, we assume that $r \geq 1$ is an integer. We define the infinite matrix $\Lambda^r = (\lambda_{nk}^r)$ by

$$\lambda_{nk}^r = \begin{cases} \frac{\lambda_k - \lambda_{k-r}}{\lambda_n}; & 0 \leq k \leq n \text{ and } r|n - k, \\ 0; & \text{otherwise} \end{cases}$$

for $n, k \in \mathbb{N}$. It is clear that the matrix Λ^r is a triangle, that is $\lambda_{nn} \neq 0$ and $\lambda_{nk} = 0$ for $k > n, n = 0, 1, 2, \dots$

We note that if we choose $r = 1$ the Λ^r matrix reduces to the matrix Λ which is defined in [12]. Also, if $r = 1$ for the sequence $\lambda_k = k + r$ the Λ^r matrix is coincide with the matrix of Cesàro means given in [13] and [14].

Now, let $x = (x_n) \in w$ and $n \geq 1$. Then, we obtain that

$$\begin{aligned} x_n - \Lambda_n^r(x) &= \frac{1}{\lambda_n} \sum_{\substack{i=0 \\ r|n-i}}^n (\lambda_i - \lambda_{i-r})(x_n - x_i) \\ &= \frac{1}{\lambda_n} \sum_{\substack{i=0 \\ r|n-i}}^n (\lambda_i - \lambda_{i-r}) \sum_{\substack{k=i+r \\ r|n-k}}^n (x_k - x_{k-r}) \\ &= \frac{1}{\lambda_n} \sum_{\substack{k=r \\ r|n-k}}^n (x_k - x_{k-r}) \sum_{\substack{i=0 \\ r|n-i}}^{k-r} (\lambda_i - \lambda_{i-r}) \\ &= \frac{1}{\lambda_n} \sum_{\substack{k=r \\ r|n-k}}^n \lambda_{k-r} (x_k - x_{k-r}). \end{aligned}$$

Hence we have that

$$(2.1) \quad x_n - \Lambda_n^r(x) = S_n^r(x),$$

for $n \in \mathbb{N}$. Here the sequence $S^r(x) = (S_n^r(x))_{n=0}^\infty$ is defined by

$$(2.2) \quad S_0^r(x) = 0 \quad \text{and} \quad S_n^r(x) = \frac{1}{\lambda_n} \sum_{\substack{k=r \\ r|n-k}}^n \lambda_{k-r} (x_k - x_{k-r}).$$

We have the following result from (2.1) and Lemma 2.2.

Theorem 2.1. *For a sequence $x = (x_k) \in w$ with $a \in \mathbb{C}$, let $f - \lim_{n \rightarrow \infty} x_n = a$. Then, $f - \lim_{n \rightarrow \infty} \Lambda_n^r(x) = a$ holds if and only if $S^r(x) \in f_0$.*

Proof. Firstly, we assume that $f - \lim_{n \rightarrow \infty} x_n = f - \lim_{n \rightarrow \infty} \Lambda_n^r(x) = a$. We have that the equality

$$(2.3) \quad \frac{1}{m+1} \sum_{k=0}^m [x_{n+k} - \Lambda_{n+k}^r(x)] = \frac{1}{m+1} \sum_{k=0}^m S_{n+k}^r(x),$$

holds for all $m, n \in \mathbb{N}$. Since (2.3) tends to zero as $m \rightarrow \infty$ uniformly in n , we have

$$\lim_{n \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m S_{n+k}^r(x) = 0 \text{ uniformly in } n.$$

This means that $S^r(x) \in f_0$.

Conversely, assume that $S^r(x) \in f_0$ and let $f - \lim_{n \rightarrow \infty} x_n = a$. We have that

$$\lim_{n \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m [x_{n+k} - \Lambda_{n+k}^r(x)] = 0,$$

from (2.3) as $m \rightarrow \infty$. Consequently, the desired result

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m x_{n+k} &= \lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m \Lambda_{n+k}^r(x) \\ &= a \end{aligned}$$

is obtained. □

3. THE SPACES OF λ_r -ALMOST CONVERGENT AND λ_r -ALMOST NULL SEQUENCES

In this section, we introduce the following spaces as the sets of all λ_r -almost convergent sequences and λ_r -almost null sequences, respectively, that is

$$\begin{aligned} f(\lambda_r) &= \left\{ x \in w : \lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m \Lambda_n^r(x) = l \text{ uniformly in } n \right\}, \\ f_0(\lambda_r) &= \left\{ x \in w : \lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m \Lambda_n^r(x) = 0 \text{ uniformly in } n \right\}. \end{aligned}$$

Using the notation of (1.2) we write again these spaces given above as the matrix domains of the triangle Λ^r in the spaces f and f_0 , respectively, such that

$$f(\lambda_r) = (f)_{\Lambda^r}, \quad f_0(\lambda_r) = (f_0)_{\Lambda^r}.$$

Now we define the sequence $y = (y_n)$ which is connected with the sequence $x = (x_k)$ by the λ_r -transform, i.e.,

$$(3.1) \quad \begin{aligned} y_n &= \Lambda_n^r(x) \\ &= \frac{1}{\lambda_n} \sum_{\substack{k=0 \\ r|n-k}}^n (\lambda_k - \lambda_{k-r}) x_k, \end{aligned}$$

for all $n \in \mathbb{N}$.

Theorem 3.1. *The sequence spaces $f(\lambda_r)$ and $f_0(\lambda_r)$ are BK-spaces with the same norm given by*

$$(3.2) \quad \begin{aligned} \|x\|_{f(\lambda_r)} &= \|\Lambda^r(x)\|_f \\ &= \sup_{n,m \in \mathbb{N}} |t_{mn}(\Lambda^r(x))|, \end{aligned}$$

where

$$\begin{aligned} t_{mn}(\Lambda^r(x)) &= \frac{1}{m+1} \sum_{j=0}^m \Lambda_{n+j}^r(x) \\ &= \frac{1}{m+1} \sum_{j=0}^m \sum_{\substack{k=0 \\ r|n+j-k}}^{n+j} \frac{\lambda_k - \lambda_{k-r}}{\lambda_{n+j}} x_k, \end{aligned}$$

for all $m, n \in \mathbb{N}$.

Proof. It is well known that f and f_0 are BK-spaces with the norm $\|\cdot\|_\infty$. Also the matrix Λ^r is a triangle matrix. Hence $f(\lambda_r)$ and $f_0(\lambda_r)$ are BK-spaces endowed with the norm $\|\cdot\|_{f(\lambda_r)}$ from Theorem 4.3.2 given in [16]. \square

Theorem 3.2. *The sequence spaces $f(\lambda_r)$, $f_0(\lambda_r)$ are norm isomorphic to the spaces f and f_0 , respectively, that is $f(\lambda_r) \cong f$ and $f_0(\lambda_r) \cong f_0$.*

Proof. To prove $f(\lambda_r) \cong f$ we need show the existence of a linear bijection between the spaces $f(\lambda_r)$ and f which preserves the norm. Let define T as (3.1), from $f(\lambda_r)$ to f by $x \rightarrow y = Tx = \Lambda^r(x)$. The linearity of T is clear. Also $x = \theta$ whenever $Tx = T\theta$ and hence T is injective.

Now, let $y = (y_k) \in f$ and $x = (x_k)$ defined by

$$x_k = \begin{cases} -\frac{\lambda_j}{\lambda_k - \lambda_{k-r}}, & j = k - r, \\ \frac{\lambda_j}{\lambda_k - \lambda_{k-r}}, & j = k, \\ 0, & \text{otherwise.} \end{cases}$$

for all $k \in \mathbb{N}$. Then we have

$$\begin{aligned} \sum_{\substack{k=0 \\ r|n+j-k}}^{n+j} \frac{\lambda_k - \lambda_{k-r}}{\lambda_{n+j}} x_k &= \sum_{\substack{k=0 \\ r|n+j-k}}^{n+j} \frac{\lambda_k y_k - \lambda_{k-r} y_{k-r}}{\lambda_{n+j}} \\ &= y_{n+j} \end{aligned}$$

which gives

$$\frac{1}{m+1} \sum_{j=0}^m \sum_{\substack{k=0 \\ r|n+j-k}}^{n+j} \frac{\lambda_k - \lambda_{k-r}}{\lambda_{n+j}} x_k = \frac{1}{m+1} \sum_{j=0}^m y_{n+j}.$$

Hence,

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{j=0}^m \Lambda_{n+j}^r(x) &= \lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{j=0}^m y_{n+j} \\ &= l \text{ uniformly in } n. \end{aligned}$$

This means that $x \in f(\lambda_r)$ and T is surjective. T is also norm preserving from (3.2). The desired result is obtained. \square

We note that absolute property does not hold on the spaces $f(\lambda_r)$ and $f_0(\lambda_r)$, that is $\|x\|_{f(\lambda_r)} \neq \| |x| \|_{f_0(\lambda_r)}$ for at least one sequence x in each of these spaces, where $|x| = (|x_k|)$. Consequently, these spaces are BK -spaces of non-absolute type. Further, $f(\lambda_r)$ and $f_0(\lambda_r)$ has no Schauder basis from Corollary 3.3 in [11] and Theorem 2.3 in [18].

4. SOME INCLUSION RELATIONS

Theorem 4.1. *The inclusions $c(\lambda_r) \subset f(\lambda_r) \subset \ell_\infty(\lambda_r)$ strictly hold.*

Proof. Let $x = (x_k)$ be a sequence in $c(\lambda_r)$. Then, $\Lambda^r(x) \in c$ and we know that the inclusion $c \subset f$ holds. Hence, $\Lambda^r(x) \in f$, that is $x \in f(\lambda_r)$. Now to prove strictness of the inclusion we give the following example.

Example 4.2. Consider the sequence $x = (x_k)$ defined by

$$(4.1) \quad x_k = \begin{cases} (-1)^k, & \text{if } r \text{ is even} \\ -\frac{\lambda_k + \lambda_{k-r}}{\lambda_k - \lambda_{k-r}}, & \text{if } r \text{ is odd} \end{cases}$$

for all $k \in \mathbb{N}$. Then we have

$$(4.2) \quad \Lambda_n^r(x) = (-1)^n$$

for all $n \in \mathbb{N}$. This means that $x \in c(\lambda_r)$. Further, we have

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{j=0}^m \Lambda_{n+j}^r(x) &= \lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{j=0}^m (-1)^{n+j} \\ &= \lim_{m \rightarrow \infty} \frac{(-1)^n}{m+1} \left[\frac{1 + (-1)^m}{2} \right] \\ &= 0. \end{aligned}$$

Hence $x \in f(\lambda_r)$ and the inclusion $c(\lambda_r) \subset f(\lambda_r)$ is strict.

Now we prove the inclusion $f(\lambda_r) \subset \ell_\infty(\lambda_r)$ holds. Let $y = (y_k) \in f(\lambda_r)$. Then, since $\Lambda^r(y) \in f$ and $f \subset \ell_\infty$, we have $\Lambda^r(y) \in \ell_\infty$ and $f(\lambda_r) \subset \ell_\infty(\lambda_r)$ holds. To see strictness of this inclusion consider the sequence z defined in [19] by $y = \Lambda^r(z)$, where

$$y = (0, 0, 0, \dots, 1, \dots, 1, \dots, 0, \dots, 0, \dots),$$

and blocks of 0's are increasing by factors of 100 and the blocks of 1 increasing by factors of 10. This sequence is in ℓ_∞ but not in f . Hence, $z \in \ell_\infty(\lambda_r) \setminus f(\lambda_r)$ and the inclusion $f(\lambda_r) \subset \ell_\infty(\lambda_r)$ is strict. \square

Theorem 4.3. *The inclusion $f_0(\lambda_r) \subset f(\lambda_r)$ strictly holds.*

Proof. Let $x = (x_k)$ be a sequence in $f_0(\lambda_r)$. Then, we have $\Lambda^r(x) \in f_0$. Since $f_0 \subset f$, $\Lambda^r(x) \in f$ and $x \in f(\lambda_r)$. Consequently, $f_0(\lambda_r) \subset f(\lambda_r)$ holds. Now to see strictness of this inclusion consider $x = (x_k)$ defined by $x_k = 1$ for all $k \in \mathbb{N}$. Obviously, $x \in f(\lambda_r)$ and we have

$$\lim_{n \rightarrow \infty} \frac{1}{m+1} \sum_{j=0}^m \sum_{\substack{k=0 \\ r|n+j-k}}^{n+j} \frac{\lambda_k - \lambda_{k-r}}{\lambda_{n+j}} x_k = 1.$$

Hence, $x \notin f_0(\lambda_r)$ and the inclusion $f_0(\lambda_r) \subset f(\lambda_r)$ is strict. \square

By taking into account $\lambda \in \Lambda$, we have $\lambda_{k+r}/\lambda_k > 1$ for all $k \in \mathbb{N}$. Hence, there are only two distinct cases of the sequence λ , either $\liminf_{k \rightarrow \infty} \lambda_{k+r}/\lambda_k = 1$ or $\liminf_{k \rightarrow \infty} \lambda_{k+r}/\lambda_k > 1$. Clearly, we obtain the following result:

Lemma 4.4. (i) $\liminf_{k \rightarrow \infty} \lambda_{k+r}/\lambda_k = 1$ if and only if $\left(\frac{\lambda_k}{\lambda_k - \lambda_{k-r}} \right) \notin \ell_\infty$.
 (ii) $\liminf_{k \rightarrow \infty} \lambda_{k+r}/\lambda_k > 1$ if and only if $\left(\frac{\lambda_k}{\lambda_k - \lambda_{k-r}} \right) \in \ell_\infty$.

Theorem 4.5. (i) *The inclusions $f_0 \subset f_0(\lambda_r)$ and $f \subset f(\lambda_r)$ strictly hold.*

- (ii) *The equalities $f_0 = f_0(\lambda_r)$ and $f = f(\lambda_r)$ hold if and only if $\Lambda^r(x) \in f_0$ for every x in the spaces $f(\lambda_r)$ and $f_0(\lambda_r)$, respectively.*

Proof. (i) Let $x = (x_k) \in c$. We know that $c \subset f$ and Λ^r is regular, hence $x \in f$, $\Lambda^r(x) \in c$. Therefore, we obtain that $x \in f(\lambda_r)$ and the inclusion $f \subset f(\lambda_r)$ holds. To see strictness of this inclusion let define a sequence $x = (x_k)$ by (4.1) and suppose that $\liminf_{k \rightarrow \infty} \frac{\lambda_{k+r}}{\lambda_k} = 1$. Since $x \notin \ell_\infty$, we obtain $x \notin f$. But $\Lambda^r(x) \in f$ and $x \in f(\lambda_r)$. This completes the proof. Similarly, one can prove that the inclusion $f_0 \subset f_0(\lambda_r)$ strictly holds.

- (ii) If we assume that $x \in f(\lambda_r)$, we have $S^r(x) \in f_0$. Hence,

$$\lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m S(x)_{n+k} = 0.$$

Then, we have

$$\lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m [x_{n+k} - \Lambda_{n+k}^r(x)] = 0$$

from (2.2). This means that

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m x_{n+k} &= \lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m \Lambda_{n+k}^r(x) \\ &= l \text{ uniformly in } n. \end{aligned}$$

Hence, $f(\lambda_r) \subset f$. By combining the inclusion $f \subset f(\lambda_r)$ the equality $f(\lambda_r) = f$ is obtained.

Conversely, assume that the equality $f = f(\lambda_r)$ holds. By (2.2), we have $S^r(x) \in f_0$. Following similar way, the results which are concerning to $f_0(\lambda_r)$ will be obtained. □

Theorem 4.6. *Neither of the spaces ℓ_∞ and $f(\lambda_r)$ includes the other.*

Proof. Consider the sequences defined by $\lambda_k = k$ and $x_k = 1/r$. Then, since $\Lambda^r(x) = e \in f$, $x \in f(\lambda_r)$. It is obvious that $x \in \ell_\infty \cap f(\lambda_r)$. Now, consider the sequence x given by (4.1) and suppose that $\liminf_{k \rightarrow \infty} \frac{\lambda_{k+r}}{\lambda_k} = 1$. Then, since $\Lambda^r(x) = (-1)^n \in f$, $x \in f(\lambda_r)$ but $x \notin \ell_\infty$. Further, let take $\lambda_k = k$ and define another sequence

$$y = (0, \dots, 0, 1/r, \dots, 1/r, 0, \dots, 0, 1/r, \dots, 1/r, 0, \dots, 0, \dots),$$

where the block's of 0's are increasing by factors of 100 and the blocks of $1/r$ are increasing by factors of 10. Then,

$$\Lambda^r(y) = (0, \dots, 0, 1, \dots, 1, 0, \dots, 0, 1, \dots, 1, 0, \dots, 0, \dots) \notin f$$

and $y \notin f(\lambda_r)$, where the blocks of 0's are increasing by factors of 100 and the blocks of 1's are increasing by factors of 10, but $y \in \ell_\infty$. This means that $y \in \ell_\infty \setminus f(\lambda_r)$. Hence the spaces ℓ_∞ and $f(\lambda_r)$ overlap, but neither of them include each other. \square

5. THE β - AND γ -DUALS OF THE SET $f(\lambda_r)$ AND SOME CERTAIN MATRIX CLASSES

In this section, we determine the β -, γ -duals of the space $f(\lambda_r)$. The β -, γ - duals of a sequence space μ are defined as followings;

$$\begin{aligned} \mu^\beta &= \{x = (x_k) \in w : xy = (x_k y_k) \in cs \text{ for all } y = (y_k) \in \mu\} \\ \mu^\gamma &= \{x = (x_k) \in w : xy = (x_k y_k) \in bs \text{ for all } y = (y_k) \in \mu\}. \end{aligned}$$

Then, we characterize some matrix transformations between $f(\lambda_r)$ and classical sequence spaces. Now we begin the following lemmas which will be used in the proof of our results.

Lemma 5.1 ([21]). $A = (a_{nk}) \in (f : \ell_\infty)$ if and only if

$$(5.1) \quad \sup_{n \in \mathbb{N}} \sum_k |a_{nk}| < \infty.$$

Lemma 5.2 ([21]). $A = (a_{nk}) \in (f : c)$ if and only if (5.1) holds and there are $\alpha, \alpha_k \in \mathbb{C}$ such that

$$(5.2) \quad \lim_{n \rightarrow \infty} a_{nk} = \alpha_k,$$

$$(5.3) \quad \lim_{n \rightarrow \infty} \sum_k a_{nk} = \alpha,$$

$$(5.4) \quad \lim_{n \rightarrow \infty} \sum_k |\Delta(a_{nk} - \alpha_k)| = 0.$$

Lemma 5.3 ([22]). $A \in (\ell_\infty : f)$ if and only if (5.1) holds and

$$(5.5) \quad f - \lim a_{nk} = \alpha_k \text{ exists for each fixed } k,$$

$$(5.6) \quad \lim_{m \rightarrow \infty} \sum_k |a(n, k, m) - \alpha_k| = 0 \text{ uniformly in } n.$$

Lemma 5.4 ([23]). $A \in (c : f)$ if and only if (5.1), (5.5) hold and

$$(5.7) \quad f - \lim \sum_k a_{nk} = \alpha.$$

Lemma 5.5 ([22]). $A \in (f : f)$ if and only if (5.1), (5.5), (5.7) hold and

$$(5.8) \quad \lim_{m \rightarrow \infty} \sum_k |\Delta[a(n, k, m) - \alpha_k]| = 0 \text{ uniformly in } n.$$

Theorem 5.6. *The γ -dual of the space $f(\lambda_r)$ is the set $e_1 \cap e_2$, where*

$$e_1 = \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} \left| \bar{\Delta}_r \left(\frac{a_k}{\lambda_k - \lambda_{k-r}} \right) \lambda_k \right| < \infty \right\},$$

$$e_2 = \left\{ a = (a_k) \in w : \left(\frac{a_n}{\lambda_n - \lambda_{n-r}} \lambda_n \right) \in \ell_\infty \right\}.$$

Proof.

$$\begin{aligned} \sum_{k=0}^n a_k x_k &= \sum_{k=0}^n a_k \left[-\frac{\lambda_{k-r}}{\lambda_k - \lambda_{k-r}} y_{k-r} + \frac{\lambda_k}{\lambda_k - \lambda_{k-r}} y_k \right] \\ &= \sum_{k=0}^{n-1} \bar{\Delta}_r \left(\frac{a_k}{\lambda_k - \lambda_{k-r}} \right) \lambda_k y_k + \frac{a_n}{\lambda_n - \lambda_{n-r}} \lambda_n y_n \\ &= T_n(y) \end{aligned}$$

for all $n \in \mathbb{N}$. $T = (t_{nk})$ is the matrix defined by

$$(5.9) \quad t_{nk} = \begin{cases} \bar{\Delta}_r \left(\frac{a_k}{\lambda_k - \lambda_{k-r}} \right) \lambda_k, & \text{if } 0 < k < n-1, \\ \frac{\lambda_n}{\lambda_n - \lambda_{n-r}} a_n, & \text{if } k = n, \\ 0, & \text{if } k > n. \end{cases}$$

for all $k, n \in \mathbb{N}$. We deduce from $T_n(y)$ that $ax = (a_k x_k) \in bs$ whenever $x = (x_k) \in f(\lambda_r)$ if and only if $Ty \in \ell_\infty$ whenever $y = (y_k) \in f$, where $T = (t_{nk})$ is defined by (5.9). Therefore, we obtain from Lemma 5.1 that $(f(\lambda_r))^\gamma = e_1 \cap e_2$. \square

Theorem 5.7. *Define the sets e_3 and e_4 by*

$$e_3 = \left\{ a = (a_k) \in w : \left\{ \frac{\lambda_n}{\lambda_n - \lambda_{n-r}} a_n \right\} \in c \right\},$$

$$e_4 = \left\{ a = (a_k) \in w : \lim_{n \rightarrow \infty} \sum_k |\Delta(t_{nk} - \alpha_k)| = 0 \right\},$$

then $\{f(\lambda_r)\}^\beta = e_3 \cap e_4$.

Proof. Take any $a = (a_k) \in w$. It is easily seen from $T_n(y)$ that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in f(\lambda_r)$ if and only if $Ty \in c$ whenever $y = (y_k) \in f$. It is clear that the columns of the matrix T lie in c where $T = (t_{nk})$ defined in (5.9). We have the consequence by Lemma 5.2 that $\{f(\lambda_r)\}^\beta = e_3 \cap e_4$. \square

Theorem 5.8. *Let assume that $A = (a_{nk})$ and $B = (b_{nk})$ are the infinite matrices which are connected with relation*

$$(5.10) \quad \hat{a}_{nk} = b_{nk}$$

$$= \overline{\Delta}_r \left(\frac{a_{nk}}{\lambda_k - \lambda_{k-r}} \right) \lambda_k$$

and μ is any given sequence space. Then, $A \in (f(\lambda_r) : \mu)$ if and only if $B \in (f : \mu)$ and

$$(5.11) \quad \left\{ \frac{\lambda_n}{\lambda_n - \lambda_{n-r}} a_{nk} \right\}_{k \in \mathbb{N}} \in c_0.$$

Proof. Firstly, keep in mind that the sequences f and $f(\lambda_r)$ are norm isomorphic. Then, we assume that $A \in (f(\lambda_r) : \mu)$ and take any $y = (y_k) \in f$. Then, BA^r exists and $\{a_{nk}\}_{k \in \mathbb{N}} \in \{f(\lambda_r)\}^\beta$ which yields that $(b_{nk})_{k \in \mathbb{N}} \in \ell_1$ for each $n \in \mathbb{N}$. Hence By exists for each $y \in f$ and thus letting $m \rightarrow \infty$ in the equality

$$\begin{aligned} \sum_{k=0}^m b_{nk} y_k &= \sum_{k=0}^m \left[\overline{\Delta}_r \left(\frac{a_{nk}}{\lambda_k - \lambda_{k-r}} \right) \lambda_k \right] \cdot \left(\sum_{\substack{i=0 \\ r|k-i}}^k \frac{(\lambda_k - \lambda_{k-r})}{\lambda_k} x_i \right) \\ &= \sum_{k=0}^m a_{nk} x_k, \end{aligned}$$

for all $n, m \in \mathbb{N}$. We have by (5.10) that $By = Ax$ which gives the result $B \in (f : \mu)$.

Conversely, suppose that (5.11) holds for every fixed $k \in \mathbb{N}$ and $B \in (f : \mu)$. Let take any $x = (x_k) \in f(\lambda_r)$. Then Ax exists. Further, we obtain

$$\begin{aligned} \sum_{k=0}^m a_{nk} x_k &= \sum_{k=0}^{m-1} \overline{\Delta}_r \left(\frac{a_{nk}}{\lambda_k - \lambda_{k-r}} \right) \lambda_k y_k + \frac{a_{nm}}{\lambda_m - \lambda_{m-r}} \lambda_m y_m \\ &= \sum_{k=0}^m b_{nk} y_k \end{aligned}$$

for all $n, m \in \mathbb{N}$, as $m \rightarrow \infty$, that $Ax = By$ and this shows that $A \in (f(\lambda_r) : \mu)$. \square

Theorem 5.9. Let μ be any sequence space and assume that $A = (a_{nk})$ and $E = (e_{nk})$ are the infinite matrices which are connected by the relation

$$e_{nk} = \sum_{\substack{j=0 \\ r|n-j}}^n \frac{\lambda_j - \lambda_{j-r}}{\lambda_n} a_{jk},$$

for all $n, k \in \mathbb{N}$. Then $D \in (\mu : f(\lambda_r))$ if and only if $E \in (\mu : f(\lambda_r))$.

Proof. Let $x = (x_k) \in \mu$ and consider the following equality

$$\sum_{\substack{j=0 \\ r|n-j}}^n \frac{\lambda_j - \lambda_{j-r}}{\lambda_n} \sum_{k=0}^j a_{jk} x_k = \sum_{k=0}^j e_{nk} x_k,$$

for $n \in \mathbb{N}$. Further, by letting $j \rightarrow \infty$,

$$\sum_{\substack{j=0 \\ r|n-j}}^n \frac{\lambda_j - \lambda_{j-r}}{\lambda_n} \sum_{k=0}^{\infty} a_{jk} x_k = \sum_{k=0}^{\infty} e_{nk} x_k,$$

for $n \in \mathbb{N}$. Then, we have $\{\Lambda^r(Ax)\}_n = (Ex)_n$ for all $n \in \mathbb{N}$. Since $Ax \in f(\lambda_r)$, $Ex \in f$ whenever $x \in \mu$. This completes the proof. \square

Corollary 5.10. *The following statements hold:*

- (i) $A = (a_{nk}) \in (f(\lambda_r) : \ell_\infty)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in \{f(\lambda_r)\}^\beta$ and (5.1) holds with \hat{a}_{nk} instead of a_{nk} .
- (ii) $A = (a_{nk}) \in (f(\lambda_r) : c)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in \{f(\lambda_r)\}^\beta$ and (5.1), (5.2), (5.3), (5.4) hold with \hat{a}_{nk} instead of a_{nk} .
- (iii) $A = (a_{nk}) \in (\ell_\infty : f(\lambda_r))$ if and only if (5.1), (5.5) and (5.6) hold with e_{nk} instead of a_{nk} .
- (iv) $A = (a_{nk}) \in (f : f(\lambda_r))$ if and only if (5.1), (5.5), (5.7) and (5.8) hold with e_{nk} instead of a_{nk} .
- (v) $A = (a_{nk}) \in (c : f(\lambda_r))$ if and only if (5.1), (5.5), (5.7) hold with e_{nk} instead of a_{nk} .

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