# First and Second Module Cohomology Groups for Non Commutative Semigroup Algebras


Ebrahim Nasrabadi



**Sahand Communications in Mathematical Analysis**

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 17
Number: 4
Pages: 39-47

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2020.119494.733


Sahand Communications in Mathematical Analysis (SCMA) Vol. 17 No. 4 (2020), 39-47
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2020.119494.733

# First and Second Module Cohomology Groups for Non Commutative Semigroup Algebras

Ebrahim Nasrabadi

Abstract. Let $S$ be a (not necessarily commutative) Clifford semigroup with idempotent set $E$. In this paper, we show that the first (second) Hochschild cohomology group and the first (second) module cohomology group of semigroup algbera $\ell^1(S)$ with coefficients in $\ell^\infty(S)$ (and also $\ell^1(S)^{(2n-1)}$ for $n \in \mathbb{N}$) are equal.

## 1. Introduction

It is well known that, for every discrete group $G$, the group algebra $\ell^1(G)$ is always weakly amenable [9] (See [5]). It is also amenable if and only if $G$ is amenable [8]. Both of these facts fail for discrete semigroups (for more details see [2]). Therefore, the difference between amenability and weak amenability concepts is important for group algebras and semigroup algebras. Amini in [1] and [2] by introducing concepts of module and weak module amenability for Banach algebras, tried to make these differences clearer. These concepts which are Banach module over another Banach algebra with compatible actions, was introduced by Amini in [1] and [2]. He showed that, inverse semigroup $S$ with subsemigroup $E$ of idempotent elements is amenable if and only if semigroup algebra $\ell^1(S)$ is $\ell^1(E)$-module amenable, when $\ell^1(E)$ acts on $\ell^1(S)$ via

$$\delta_e \cdot \delta_s = \delta_s, \qquad \delta_s \cdot \delta_e = \delta_s * \delta_e = \delta_{se}, \quad (e \in E, s \in S), \tag{1.1}$$

where $\delta_s$ and $\delta_e$ are the point masses at $s \in S$ and $e \in E$, respectively.

After that, Amini and Bagha in [2], introduced the concept of weak module amenability and showed that, for every commutative inverse

2020 *Mathematics Subject Classification.* 43A07, 46H20, 46H25,
*Key words and phrases.* Clifford semigroup, Weak amenability, Weak module amenability, Cohomology group, Module cohomology group.
Received: 02 January 2020, Accepted: 09 June 2020.

semigroup $S$ with idempodent set $E$, $\ell^1(S)$ is always weakly $\ell^1(E)$-module amenable, where module actions $\ell^1(E)$ on $\ell^1(S)$ are different actions

$$\delta_e \cdot \delta_s = \delta_s \cdot \delta_e = \delta_s * \delta_e = \delta_{se}, \quad (e \in E, s \in S). \tag{1.2}$$

Then this idea has been expanded by author of the current paper along with Pourabbas. They in [11] and [12], after introducing the concept of module cohomology group for Banach algebras extended this result. They showed that the first and second $\ell^1(E)$-module cohomology groups of $\ell^1(S)$ with coefficients in $\ell^1(S)^{(2n-1)}$ $(n \in \mathbb{N})$, are zero and Banach space, respectively, when $\ell^1(S)$ is a Banach $\ell^1(E)$-bimodule with actions (1.2).

Also, Shirinkalam, Pourabbas and Amini in [13] showed that first (second) $\ell^1(E)$-module cohomology group of $\ell^1(S)$ and first (second) Hochschild cohomology group of $\ell^1(G_S)$ are equal, where $G_S$ is the maximal group homomorphic image of $S$ and $\ell^1(S)$ considered to be a Banach $\ell^1(E)$-bimodule (not necessary commutative) with actions (1.1).

On the other hand, the author in [10] proved that for every commutative inverse semigroup $S$ with idempotent set $E$, there is no difference between Hochschild and $\ell^1(E)$-module cohomology groups of $\ell^1(S)$ with coefficients in $\ell^\infty(S)$, when $\ell^1(S)$ is a Banach $\ell^1(E)$-bimodule with actions (1.2).

In this paper, we show that for every Clifford semigroup $S$ (which is not necessarily commutative) with idempotent set $E$, the first (second) Hochschild cohomology group and the first (second) $\ell^1(E)$-module cohomology group of $\ell^1(S)$ with coefficients in $\ell^\infty(S)$ (and also $\ell^1(S)^{(2n-1)}$ for $n \in \mathbb{N}$) are equal, when $\ell^1(E)$ acts on $\ell^1(S)$ by multiplication from right and left. Indeed we prove that

$$\mathcal{H}^k(\ell^1(S), \ell^1(S)^{(2n-1)}) \simeq \mathcal{H}^k_{\ell^1(E)}(\ell^1(S), \ell^1(S)^{(2n-1)}) \quad (k = 1, 2; n \in \mathbb{N}),$$

when $\ell^1(S)$ is a Banach $\ell^1(E)$-bimodule with actions (1.2).

Bowling and Duncan in [3] and Gourdeau, Pourabbas and White in [6] show that for every Clifford semigroup $S$, the first and second Hochschild cohomology group of $\ell^1(S)$ with coefficients in $\ell^\infty(S)$ (and also $\ell^1(S)^{(2n-1)}$ for $n \in \mathbb{N}$) are zero and Banach space, respectively. Their results are along with our findings, improve Theorem 3.1 of [2], Theorem 2.2 of [11] and Theorem 2.3 of [12].

## 2. PRELIMINARY

Let $\mathfrak{A}$ and $A$ be Banach algebras such that $A$ is a Banach $\mathfrak{A}$-bimodule and let $X$ be a commutative Banach $\mathfrak{A}$-$A$-module with compatible actions (for more details see [1], [2], [10],[11] and [12]). An $n$-$\mathfrak{A}$-module map is a bounded mapping $\phi : A^n \to X$ with the following properties;

$$\begin{aligned}
&\phi(a_1, a_2, \dots, a_{i-1}, b \pm c, a_{i+1}, \dots, a_n) \\
&\qquad = \phi(a_1, a_2, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) \\
&\qquad\quad \pm \phi(a_1, a_2, \dots, a_{i-1}, c, a_{i+1}, \dots, a_n), \\
&\phi(\alpha \cdot a_1, a_2, \dots, a_n) = \alpha \cdot \phi(a_1, a_2, \dots, a_n), \\
&\phi(a_1, a_2, \dots, a_n \cdot \alpha) = \phi(a_1, a_2, \dots, a_n) \cdot \alpha,
\end{aligned}$$

and

$$\phi(a_1, a_2, \dots, a_i \cdot \alpha, a_{i+1}, \dots, a_n) = \phi(a_1, a_2, \dots, a_i, \alpha \cdot a_{i+1}, \dots, a_n),$$

where $a_1, \dots, a_n, b, c \in A$ and $\alpha \in \mathfrak{A}$. Note that, $\phi$ is not necessarily $n$-linear but still its boundedness implies its norm continuity (since $\phi$ preserves subtraction). We use the notation $\mathcal{C}^n_{\mathfrak{A}}(A, X)$ for the set of all bounded (continuous) $n$-$\mathfrak{A}$-module maps from $A^n$ to $X$.

Consider the $\mathfrak{A}$-module complex cochain

$$0 \longrightarrow X \xrightarrow{\delta^0} \mathcal{C}^1_{\mathfrak{A}}(A, X) \xrightarrow{\delta^1} \mathcal{C}^2_{\mathfrak{A}}(A, X) \xrightarrow{\delta^2} \cdots ,$$

where the map $\delta^0 : X \longrightarrow \mathcal{C}^1_{\mathfrak{A}}(A, X)$ is given by $\delta^0(x)(a) = a \cdot x - x \cdot a$ and for $n \in \mathbb{N}$, the $n$-coboundary operator $\delta^n : \mathcal{C}^n_{\mathfrak{A}}(A, X) \longrightarrow \mathcal{C}^{n+1}_{\mathfrak{A}}(A, X)$ is given by

$$\begin{aligned}
(\delta^n \phi)(a_1, \dots, a_{n+1}) &= a_1 \cdot \phi(a_2, \dots, a_{n+1}) \\
&\quad + \sum_{i=1}^{n} (-1)^i \phi(a_1, \dots, a_i a_{i+1}, \dots a_{n+1}) \\
&\quad + (-1)^{n+1} \phi(a_1, \dots, a_n) \cdot a_{n+1},
\end{aligned} \tag{2.1}$$

for $\phi \in \mathcal{C}^n_{\mathfrak{A}}(A, X)$ and $a_1, \dots, a_{n+1} \in A$. It is easy to see that $\delta^{n+1} \circ \delta^n = 0$, for every $n \in \mathbb{Z}^+$. The space $\ker \delta^n$ of all bounded $n$-$\mathfrak{A}$-module cocycles is denoted by $\mathcal{Z}^n_{\mathfrak{A}}(A, X)$ and the space $\operatorname{Im} \delta^{n-1}$ of all bounded $n$-$\mathfrak{A}$-module coboundaries is denoted by $\mathcal{B}^n_{\mathfrak{A}}(A, X)$. We also recall that $\mathcal{B}^n_{\mathfrak{A}}(A, X)$ is included in $\mathcal{Z}^n_{\mathfrak{A}}(A, X)$ and that the $n$-th $\mathfrak{A}$-module cohomology group $\mathcal{H}^n_{\mathfrak{A}}(A, X)$ is defined by the quotient

$$\mathcal{H}^n_{\mathfrak{A}} A, X) = \frac{\mathcal{Z}^n_{\mathfrak{A}}(A, X)}{\mathcal{B}^n_{\mathfrak{A}}(A, X)}.$$

**Remark 2.1.** In the above notations, if $\mathfrak{A} = \mathbb{C}$ and module action is scaler multiplication, the space $\ker \delta^n$ of all bounded $n$-linear cocycles is denoted by $\mathcal{Z}^n(A, X)$ and the space $\operatorname{Im} \delta^{n-1}$ of all bounded $n$-linear coboundaries is denoted by $\mathcal{B}^n(A, X)$. The $n$-th Hochschild cohomology group $\mathcal{H}^n(A,X)$ is defined by the quotient

$$\mathcal{H}^n(A,X) = \frac{\mathcal{Z}^n(A,X)}{\mathcal{B}^n(A,X)}.$$

**Definition 2.2.** A commutative Banach $\mathfrak{A}$-module algebra $A$ is called weak $\mathfrak{A}$-module amenable ($(n)$-weak $\mathfrak{A}$-module amenable) if $\mathcal{H}^1_{\mathfrak{A}}(A, A^*) = 0$ $\big(\mathcal{H}^1_{\mathfrak{A}}(A, A^{(n)}) = 0\big)$.

**Definition 2.3.** In the previous definition, if $\mathfrak{A} = \mathbb{C}$ and module action is scaler multiplication, then $A$ is called weak amenable and $(n)$-weak amenable, respectively.

## 3. The First Module and Hochschild Cohomology Group

In this section, it is assumed that $S$ is a (not necessarily commutative) Clifford semigroup with idempotent set $E$. We recall that $S$ is a Clifford semigroup if it is an inverse semigroup with each idempotent central, or equivalently, if it is a strong semilattice of group. So we can write our Clifford semigroup as $S = \cup\{G_e : e \in E\}$ where $G_e$ is a group with identity $e$, and $G_eG_{e'} \subseteq G_{ee'}$ for every $e, e' \in E$. We know that $\ell^1(S)$ is a commutative Banach $\ell^1(E)$-bimodule with actions $\delta_s \cdot \delta_e = \delta_e \cdot \delta_s = \delta_{se}$ $(e \in E, s \in S)$, where $\delta_e$ and $\delta_s$ are the point masses at $e$ and $s$, respectively. From now on, we remove the dot for simplicity.

**Remark 3.1.**
(i) Let $X$ be a Banach $\ell^1(S)$-module. As usual, we identify the element of $S$ with point masses in $\ell^1(S)$. Indeed, for every $\lambda \in \mathbb{C}$ and $s, t \in S$, phrases $\lambda s$, $s + t$ and $st$ as elements of $\ell^1(S)$ means that $\lambda\delta_s$, $\delta_s + \delta_t$ and $\delta_s \cdot \delta_t$. There is an obvious one-one corresponding between $\mathcal{C}^1(\ell^1(S), X)$ $\big(\mathcal{C}^2(\ell^1(S), X)\big)$ and the space of bounded functions from $S$ $(S \times S)$ into $X$. Thus we use the same notation for $\psi \in \mathcal{C}^1(\ell^1(S), X)$ $\big(\phi \in \mathcal{C}^2(\ell^1(S), X)\big)$ and $\psi$ as a function on $S$ ($\phi$ as a function on $S \times S$).
(ii) We know that functions of finite support are dense in semigroup algebra $\ell^1(S)$ , and on the other hand, every function of finite support is a linear combination of point masses. Therefore, because every $\ell^1(E)$-module map on $\ell^1(S)$ is additive and continuous, throughout this paper, we consider elements of $S$ as representative of elements of $\ell^1(S)$.

**Lemma 3.2.** *Every continuous $\ell^1(E)$-module map $\psi : \ell^1(S) \to \ell^\infty(S)$ is linear. In particular, $\mathcal{Z}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) \subseteq \mathcal{Z}^1(\ell^1(S), \ell^\infty(S))$.*

*Proof.* Let $\lambda \in \mathbb{C}, g \in G_e \subseteq S$ and $\phi \in \mathcal{C}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$. Since $\lambda e \in \ell^1(E)$, we have

$$\phi(\lambda g) = \phi(\lambda e g)$$

$$= \lambda e\phi(g)$$
$$= \lambda\phi(eg)$$
$$= \lambda\phi(g).$$

Thus, the result directly follows from Remark 3.1. □

**Lemma 3.3.** *Suppose $\psi \in \mathcal{C}^1(\ell^1(S), \ell^\infty(S))$ such that $\delta^1\psi(g,h) = 0$, if $g$ or $h$ lies in $E$. Then $\psi \in \mathcal{C}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$. In particular, $\mathcal{Z}^1(\ell^1(S), \ell^\infty(S)) \subseteq \mathcal{Z}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$.*

*Proof.* Let $e \in E$, we have

$$\begin{aligned} 0 &= \delta^1\psi(e,e) \\ &= e\psi(e) - \psi(e) + \psi(e)e \\ &= 2e\psi(e) - \psi(e). \end{aligned}$$

So we get $2e\psi(e) = \psi(e)$. But $e\psi(e) = e(2e\psi(e)) = 2e\psi(e)$. This shows that $e\psi(e) = 0$ and so $\psi(e) = 0$. Now for every $e \in E$ and $g \in S$,

$$\begin{aligned} 0 &= \delta^1\psi(e,g) \\ &= e\psi(g) - \psi(eg) + \psi(e)g \\ &= e\psi(g) - \psi(eg), \end{aligned}$$

that shows $\psi(eg) = e\psi(g)$. Similarly we can show that $\psi(ge) = \psi(g)e$ and the proof is complete. □

**Theorem 3.4.** *Let $S$ be a Clifford semigroup with idempotent set $E$. Then*

$$\mathcal{H}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) = \mathcal{H}^1(\ell^1(S), \ell^\infty(S)).$$

*Proof.* By Lemmas 3.2 and 3.3, we get

$$\mathcal{Z}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) = \mathcal{Z}^1(\ell^1(S), \ell^\infty(S)).$$

Therefore,

$$\begin{aligned} \mathcal{H}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) &= \frac{\mathcal{Z}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))}{\mathcal{B}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))} \\ &= \frac{\mathcal{Z}^1(\ell^1(S), \ell^\infty(S))}{\mathcal{B}^1(\ell^1(S), \ell^\infty(S))} \\ &= \mathcal{H}^1(\ell^1(S), \ell^\infty(S)). \end{aligned}$$

□

The next Corollary improves [2, Theorem 3.1] and [11, Theorem 2.4].

**Corollary 3.5.** *Let $S$ be a Clifford semigroup with idempotent set $E$. Then $\ell^1(S)$ is $(2n-1)$-weak module amenable (as an $\ell^1(E)$-module).*

*Proof.* Apply Theorem 3.4 and [3, Theorem 2.1]. ☐

## 4. The Second Module and Hochschild Cohomology Group

In this section, similar to the previous section, it is assumed that $S$ is a Clifford semigroup (not necessarily commutative) with idempotent set $E$. By (2.1), we note that the coboundary operators $\delta^1$ and $\delta^2$ are given by

$$\begin{aligned}(\delta^1\psi)(a,b) &= a\psi(b) - \psi(ab) + \psi(a)b,\\ (\delta^2\phi)(a,b,c) &= a\phi(b,c) - \phi(ab,c) + \phi(a,bc) - \phi(a,b)c,\end{aligned} \tag{4.1}$$

for every $a,b,c \in \ell^1(S)$, $\psi \in \mathcal{C}^1(\ell^1(S),\ell^\infty(S))$ and $\phi \in \mathcal{C}^2(\ell^1(S),\ell^\infty(S))$.

**Lemma 4.1.** *Every continuous $\ell^1(E)$-module map $\phi : \ell^1(S)\times\ell^1(S) \to \ell^\infty(S)$ is 2-linear. In particular, $\mathcal{Z}^2_{\ell^1(E)}(\ell^1(S),\ell^\infty(S)) \subseteq \mathcal{Z}^2(\ell^1(S),\ell^\infty(S))$.*

*Proof.* Proof of Lemma 3.2 can be applied to any component of a bilinear map. ☐

**Lemma 4.2.** *Suppose $\phi \in \mathcal{C}^2(\ell^1(S),\ell^\infty(S))$ such that $(\delta^2\phi)(g,h,k) = 0$, if any one of $g,h,k$ lies in $E$ and $\phi(g,h) = 0$ if $g$, $h$ or $k$ lies in $E$ and $\phi(g,h) = 0$ if $g$ or $h$ lies in $E$. Then $\phi \in \mathcal{C}^2_{\ell^1(E)}(\ell^1(S),\ell^\infty(S))$.*

*Proof.* Let $e \in E$ and $g,h \in S$,

$$\begin{aligned}0 &= (\delta^2\phi)(e,g,h)\\ &= e\phi(g,h) - \phi(eg,h) + \phi(e,gh) - \phi(e,g)h\\ &= e\phi(g,h) - \phi(eg,h).\end{aligned}$$

This implies that $e\phi(g,h) = \phi(eg,h)$. Similarly by applying the 2-cocycle equations $(\delta^2\phi)(g,e,h) = 0$ and $(\delta^2\phi)(g,h,e) = 0$, we can show that $\phi(ge,h) = \phi(g,eh)$ and $\phi(g,he) = \phi(g,h)e$, respectively. ☐

**Lemma 4.3.** *Let $\phi \in \mathcal{Z}^2(\ell^1(S),\ell^\infty(S))$. Then $\phi(g,h) = 0$ if $g$ or $h$ lies in $E$. Moreover, $\phi \in \mathcal{Z}^2_{\ell^1(E)}(\ell^1(S),\ell^\infty(S))$.*

*Proof.* Let $\phi \in \mathcal{Z}^2(\ell^1(S),\ell^\infty(S))$, by Theorem 2.5 of [4], $\phi = \delta\psi_1$ on $E$ for some $\psi_1 \in \mathcal{C}^1(\ell^1(S),\ell^\infty(S))$. So if we define $\phi_1 := \phi - \delta^1\psi_1$, then $\phi_1(e_1,e_2) = 0$ for every $e_1,e_2 \in E$. By Lemma 4.2 of [6], there exists a $\psi_2 \in \mathcal{C}^1(\ell^1(S),\ell^\infty(S))$ such that $(\phi_1 - \delta^1\psi_2)(g,e) = 0$ for every $g \in S$ and $e \in E$ with $eg = g$. So if we define $\phi_2 = \phi_1 - \delta^1\psi_2$ and applying the 2-cocycle equation $\delta^2\phi_2(e,g,h)(k) = 0$ for $e \in E$ and $g,h,k \in S$ with $ge = g$, we obtain (using $\phi_2(g,e) = 0$)

$$\phi_2(g,h)(k) = \phi_2(g,h)(ek).$$

Similarly $\phi_2(g,h)(k) = \phi_2(g,h)(ek)$, whenever $he = h$. Now for $g \in G_{e_1}$ and $h \in G_{e_2}$ define $\psi_3(g)(h) = \phi_2(g,e')(h)$, where $e' = e_1e_2$. Then

Lemma 4.4 of [6] shows that $(\phi_2 - \delta\psi_3)(g,e)(h) = 0$ for every $g, h \in S$ and $e \in E$. This means that $(\phi_2 - \delta\psi_3)|_{\ell^1(S)\times\ell^1(E)} = 0$. Therefore,

$$(\phi - \delta\psi)(e,g) = 0,$$

where $\psi = \psi_1 + \psi_2 + \psi_3$. Now by replacing $\phi$ by $\phi - \delta\psi$, we can assume that $\phi(g,e) = 0$, whenever $e \in E$ and $g \in S$. Now with the same argument above we have $\phi(g,h) = 0$, if $g$ or $h$ lies in $E$. Finally, the map $\phi$ satisfies the assumption of Lemma 4.2 and so $\phi \in \mathcal{Z}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$. □

**An alternate proof for Lemma 4.3** Let $\phi \in \mathcal{Z}^2(\ell^1(S), \ell^\infty(S))$. We can simply check that the space $\mathcal{Z}^1(\ell^1(S), \ell^\infty(S))$ is a commutative Banach $\ell^1(E)$-bimodule with the actions

$$(e \bullet \varphi)(g) := e\varphi(g), \qquad (\varphi \bullet e)(g) := \varphi(eg) - \varphi(e)g, \tag{4.2}$$

where $e \in E$, $g \in S$ and $\varphi \in \mathcal{Z}^1(\ell^1(S), \ell^\infty(S))$.

Define $\psi \in \mathcal{C}^1(\ell^1(E), \mathcal{Z}^1(\ell^1(S), \ell^\infty(S)))$ by:

$$\psi(e)(g) := \phi(e,g), \quad (e \in E,\ g \in S). \tag{4.3}$$

By using of the 2-coboundary operator $\delta^2$ in (4.1) and actions (4.2), for each $e, e' \in E$ and $g \in S$, we obtain

$$\begin{aligned} 0 &= (\delta^2\phi)(e,e',g) \\ &= e\phi(e',g) - \phi(ee',g) + \phi(e,e'g) - \phi(e,e')g \\ &= e\psi(e')(g) - \psi(ee')(g) + \psi(e)(e'g) - \psi(e)(e')g \\ &= [e \bullet \psi(e') - \psi(ee') + \psi(e) \bullet e'](g). \end{aligned}$$

This shows that $\psi \in \mathcal{Z}^1(\ell^1(E), \mathcal{Z}^1(\ell^1(S), \ell^\infty(S)))$. Since $\ell^1(E)$ is commutative and weak amenable (by [3, Lemma 1.1]), we have $\psi = 0$. That means $\phi(e,g) = 0$ for every $e \in E$ and $g \in S$. Similarly we can show that $\phi(g,e) = 0$, which proves our claim. Finally, the map $\phi$ satisfies the assumption of Lemma 4.2, and so $\phi \in \mathcal{Z}^1_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$.

**Lemma 4.4.** $\mathcal{B}^2_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)) = \mathcal{B}^2(\ell^1(S), \ell^\infty(S))$.

*Proof.* Let $\phi \in \mathcal{B}^2(\ell^1(S), \ell^\infty(S))$, so $\phi \in \mathcal{Z}^2(\ell^1(S), \ell^\infty(S))$ and by Lemma 4.3 we get

$$\phi(g,h) = 0,$$

if $g$ or $h$ lies in $E$. On the other hand, by assumption there exists $\psi \in \mathcal{C}^1(\ell^1(S), \ell^\infty(S))$ such that $\phi = \delta^1\psi$. But the map $\psi$ satisfies the assumption of Lemma 3.3 and so $\psi$ is a $\ell^1(E)$-module map. This implies $\phi \in \mathcal{B}^2_{\ell^1(E)}(\ell^1(S), \ell^\infty(S))$ and so

$$\mathcal{B}^2(\ell^1(S), \ell^\infty(S)) \subseteq \mathcal{B}^2_{\ell^1(E)}(\ell^1(S), \ell^\infty(S)).$$

The other inclusion is clear by Lemma 3.2 and the observation that

$$\begin{aligned}\mathcal{B}^2_{\ell^1(E)}(\ell^1(S),\ell^\infty(S)) &= \delta^1(\mathcal{C}^1_{\ell^1(E)}(\ell^1(S),\ell^\infty(S)))\\ &\subseteq \delta^1(\mathcal{C}^1(\ell^1(S),\ell^\infty(S)))\\ &= \mathcal{B}^2(\ell^1(S),\ell^\infty(S)).\end{aligned}$$

□

**Theorem 4.5.** *Let $S$ be a Clifford semigroup with idempotent set $E$. Then*

$$\mathcal{H}^2_{\ell^1(E)}(\ell^1(S),\ell^\infty(S)) \simeq \mathcal{H}^2(\ell^1(S),\ell^\infty(S)).$$

*Proof.* We define morphism

$$\begin{aligned}\Gamma:\ \mathcal{H}^2_{\ell^1(E)}(\ell^1(S),\ell^\infty(S)) &\to \mathcal{H}^2(\ell^1(S),\ell^\infty(S))\\ \phi+\mathcal{B}^2_{\ell^1(E)}(\ell^1(S),\ell^\infty(S)) &\mapsto \phi+\mathcal{B}^2(\ell^1(S),\ell^\infty(S)).\end{aligned}$$

The morphism $\Gamma$ is well-defined by Lemma 4.1, surjective by Lemma 4.3, and injective by Lemma 4.4. Hence, by Lemma 0.5.9 of [7], $\Gamma$ is a topological isomorphism. □

We know that every commutative inverse semigroup is a Clifford semigroup, so from [6], we have the following corollary which improves [12, Theorem 2.3].

**Corollary 4.6.** *Let $S$ be a Clifford semigroup with idempotent set $E$. Then $\mathcal{H}^2_{\ell^1(E)}(\ell^1(S),\ell^\infty(S))$ $(\mathcal{H}^2_{\ell^1(E)}(\ell^1(S),\ell^1(S)^{(2n-1)}))$ is a Banach space.*

**Acknowledgment.** The author would like to express his deep gratitude to the referees for their careful reading of the earlier version of the manuscript and several insightful comments.

## References

1. M. Amini, *Module amenability for semigroup algebras*, Semigroup Forum., 69 (2004), pp. 243-254.
2. M. Amini and D. E. Bagha, *Weak module amenability for semigroup algebras*, Semigroup Forum., 71 (2005), pp. 18-26.
3. S. Bowling and J. Duncan, *First order cohomology of Banach semigroup algebras*, Semigroup Forum., 56 (1998), pp. 130-145.
4. H.G. Dales and J. Duncan, *Second order cohomology in groups of some semigroup algebras*, (Banach Algebra 97 (Blaubeuren) Walter de Gruyter, Berlin), (1998), pp. 101-117.
5. M. Despi´c and F. Ghahramani, *Weak amenability of group algebras of locally compact groups*, Canad. Math. Bull, 37 (1994), pp. 165-167.

6. F. Gourdeau, A.R. Pourabbas and M.C. White, *Simplicial cohomology of some semigroup algebras*, Can. Math. Bull, 50 (2007), pp. 56-70.
7. A.Y. Helemskii, *The homology of Banach and topological algebras*, Moscow Univ. Press. Moscow (1986); English transl, Kluwer Academic Publishers, Dordrecht, (1989)
8. B.E. Johnson, *Cohomology in Banach algebras*, Mem. Am. Math. Soc, 127, 96 (1972)
9. B.E. Johnson, *Weak amenability of group algebras*, Bull. London Math. Soc, 23 , (1991), pp. 281-284.
10. E. Nasrabadi, *The equality of Hochschild cohomology group and module cohomology group for semigroup algebras*, Bol. Soc. Paran. Mat., (2020), In press, DOI:10.5269/bspm.44931.
11. E. Nasrabadi and A. Pourabbas, *Module cohomology group of inverse semigroup algebras*, Bull. Iranian Math. Soc., 37 , (2011), pp. 157-169.
12. E. Nasrabadi and A. Pourabbas, *Second module cohomology group of inverse semigroup algebras*, Semigroup Fourm., 81 , (2010), pp. 269-278.
13. A. Shirinkalam, A.R. Pourabbas and M. Amini, *Module and Hochschild cohomology of certain semigroup algebras.* Funct. Anal. Appl., 49 , (2015), pp. 315-318.

Faculty of Mathematics Science and Statistics, University of Birjand, Birjand, 9717851367, Iran.

*Email address*: nasrabadi@birjand.ac.ir