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Abstract. In this work systems of sines $\sin(n+\beta)t$, $n=1,2,\ldots,$ and cosines $\cos(n-\beta)t$, $n=0,1,2,\ldots,$ are considered, where $\beta\in R-$is a real parameter. The subspace $M^{p,\alpha}(0,\pi)$ of the Morrey space $L^{p,\alpha}(0,\pi)$ in which continuous functions are dense is considered. Criterion for the completeness, minimality and basicity of these systems with respect to the parameter β in the subspace $M^{p,\alpha}(0,\pi)$, $1<p<+\infty$, are found.

1. Introduction

When solving many partial differential equations of mixed or elliptic types, the Fourier method yields spectral problems whose eigenfunctions are systems of sines

$$\sin(n-\beta)t, \quad n\in N, \tag{1.1}$$

and cosines

$$\cos(n-\beta)t, \quad n\in Z_+ \ (Z_+=\{0\}\cup N), \tag{1.2}$$

where $\beta\in R$ is some real parameter (for more details see [22–24, 28, 29]). When substantiating a formal solution, it is required to study the basis properties (completeness, minimality, basicity) of these systems in the corresponding Banach spaces of functions on the segment $[0,\pi]$. The basis properties of systems (1.1) and (1.2) are closely related to the corresponding properties of systems of exponents

$$E_\beta=\left\{e^{i(n-\beta\text{sign})\,t}\right\}_{n\in Z}, \tag{1.3}$$

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$$1 \cup \left\{ e^{i(n-\beta \text{sign})\, t} \right\}_{n \neq 0} , \tag{1.4}$$

in function spaces on the segment $[-\pi, \pi]$. The study of the basis properties of systems (1.3) and (1.4) in Lebesgue spaces of functions started apparently with the work of Paley and Wiener [27] and N. Levinson [19]. The final results on the basicity of system (1.3) belong to M.I. Kadets [16] (the Riesz basicity in L_2) and A. M. Sedletskii [30] (the basicity in L_p). Criterion regarding the basis properties of systems (1.1) and (1.2) are obtained in the works of E.I. Moiseev [20, 21]. This direction was developed in the works of B.T. Bilalov [2–5] regarding systems of a more general form.

Recently, in connection with specific problems, interests in studying various problems of modern mathematics in non-standard function spaces has significantly increased. For more information on related issues, see monographs [1, 13–15, 17, 18]. The desire to study the differential equations describing the above mentioned problems in non-standard Sobolev spaces required the study of the basis properties of systems (1.1) and (1.2) in the corresponding non-standard Lebesgue spaces of functions. In [6–12, 25, 26], the basis properties of systems (1.3) and (1.4) were established in Lebesgue spaces of functions with a variable summability exponent and in Morrey-type spaces. In the work of B.T. Bilalov [6] criteria regarding the basis properties of system (1.3) in Morrey-type spaces are established. In [26], similar results were obtained for the system (1.4) in the same spaces.

In this paper, criteria for the basis properties of the system of sines (1.1) and cosines (1.2) in Morrey-type spaces are established. In this case, similar results regarding the systems of exponents (1.3) and (1.4) in spaces $M^{p,\alpha}(-\pi, \pi)$ are used. The proposed approach differs from the method used in the works of E.I. Moiseev [20, 21].

2. Needful Information

Let us first accept some standard notations. $N-$is the set of natural numbers; $Z-$are integers; $[x]-$is the integer part of number x.

Let's define the Morrey space $L^{p,\alpha}(a,b)$. It is a Banach space of all measurable functions over (a,b) with the finite norm

$$\|f\|_{L^{p,\alpha}(a,b)} = \sup_{I \subset (a,b)} \left(|I|^{\alpha-1} \int_I |f(t)|^p \, dt \right)^{\frac{1}{p}},$$

where sup is taken over all intervals $I \subset (a,b)$. For $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ the following continuous embedding holds: $L^{p,\alpha_1} \subset L^{p,\alpha_2}$. It is easy to notice that $L^{p,1}(a,b) = L_p(a,b)$ and $L^{p,0}(a,b) = L_\infty(a,b)$ are true. Moreover, $L^{p,\alpha}(a,b) \subset L_1(a,b)$, $\forall p > 1$, $\forall \alpha \in [0,1]$. It is known that

$L^{p,\alpha}(a,b)$, $1 \le p < +\infty$, $\alpha \in (0,1)$, is not separable and $C[a,b]$ is not dense in it. Let

$$M^{p,\alpha}(a,b) = \left\{ f \in L^{p,\alpha}(a,b) : \|f(\cdot+\delta) - f(\cdot)\|_{L^{p,\alpha}(a,b)} \to 0, \, \delta \to 0 \right\}.$$

As shown in [6], $M^{p,\alpha}(a,b)$, for $1 \le p < +\infty$, $0 \le \alpha < 1$, is a separable Banach space and $C_0^\infty(a,b)$ (infinitely differentiable and finite supported functions over (a,b)) is dense in it. When defining the space $M^{p,\alpha}(a,b)$, the function $f(\cdot)$ is assumed to be extended outside the interval (a,b) by zero.

We will also need the following well-known lemma

Lemma 2.1. *Let $f \in M^{p,\alpha}(a,b)$, $1 \le p < +\infty$, $0 \le \alpha < 1$, be an arbitrary function. Then $\|f\chi_E\|_{L^{p,\alpha}(a,b)} \to 0$, as $|E| \to 0$, where $E \subset (a,b)$ is an arbitrary interval and $|E|$ is its length.*

We will essentially use the results of [6, 11, 26]. Let us represent these results.

Theorem 2.2 ([6])**.** *Let $2\mathrm{Re}\beta + \frac{\alpha}{p} \notin Z$, $1 < p < +\infty$, $0 < \alpha < 1$. Then the system of exponents* (1.3) *forms a basis for $M^{p,\alpha}(-\pi,\pi)$ if and only if $d(E_\beta) = \left[2\mathrm{Re}\beta + \frac{\alpha}{p}\right] = 0$. For $d(E_\beta) < 0$ it is not complete, but is minimal; and for $d(E_\beta) > 0$ it is complete but is not minimal in $M^{p,\alpha}(-\pi,\pi)$.*

We will need the following

Definition 2.3. A sequence $\{x_n\}_{n\in N} \subset X$ in a Banach space X is called a defect system if after eliminating from it or attaching to it a finite number of elements, it becomes complete and minimal in X and we call this number the defect of the system and denote it $d\left(\{x_n\}_{n\in N}\right)$.

From this definition and the proof of [6, Theorem 2.2], it follows that the defect of the system E_β is equal to $|d(E_\beta)|$ and more precisely, for $d(E_\beta) > 0$ it contains $d(E_\beta)$ superfluous elements and for $d(E_\beta) < 0$ it is necessary to add $|d(E_\beta)|$ elements to it.

A similar result holds for the system of exponents (1.4).

Theorem 2.4 ([26])**.** *Let $2\mathrm{Re}\beta + \frac{\alpha}{p} \notin Z$, $1 < p < +\infty$, $0 < \alpha < 1$. System of exponents* (1.4) *forms a basis for $M^{p,\alpha}(-\pi,\pi)$ if and only if $d(E_\beta) = 0$. For $d(E_\beta) < 0$ it is not complete, but is minimal; for $d(E_\beta) > 0$ it is complete, but is not minimal in it.*

Multiply each member of the system E_β by a function E_β $e^{\frac{i}{2}t}$ and transform

$$e^{\frac{i}{2}t}E_\beta = e^{\frac{i}{2}t}\left\{e^{-i\beta t}e^{int};\, e^{i\beta t}e^{-ikt}\right\}_{n\ge 0; k\ge 1}$$

$$\equiv \left\{e^{i\left(-\beta-\frac{1}{2}\right)t}e^{ikt};\ e^{-i\left(-\beta-\frac{1}{2}\right)t}e^{-ikt}\right\}_{k\geq 1}$$
$$\equiv \left\{e^{i\left(n-\tilde{\beta}\,\mathrm{sign}\,n\right)t}\right\}_{n\neq 0}$$
$$\equiv e_{\tilde{\beta}},$$

where $\tilde{\beta}=\beta+\frac{1}{2}$. Following Definition 2.3, we have

$$d\left(e_{\tilde{\beta}}\right)=d\left(E_{\beta}\right)$$
$$=\left[2\mathrm{Re}\beta+\frac{\alpha}{p}\right]$$
$$=\left[2\mathrm{Re}\tilde{\beta}+\frac{\alpha}{p}-1\right]$$
$$=\left[2\mathrm{Re}\tilde{\beta}+\frac{\alpha}{p}\right]-1.$$

Thus, the defect of the system $e_{\tilde{\beta}}$ is equal to $\left|d\left(e_{\tilde{\beta}}\right)\right|$, where

$$d\left(e_{\tilde{\beta}}\right)=\left[2\mathrm{Re}\tilde{\beta}+\frac{\alpha}{p}\right]-1.$$

As a result, we obtain the following:

Corollary 2.5. *Let* $2\mathrm{Re}\beta+\frac{\alpha}{p}\notin Z$, $1<p<+\infty$, $0<\alpha<1$. *Then the system of exponent* $\left\{e^{i(n-\beta \mathrm{signn})t}\right\}_{n\neq 0}$, *forms a basis for* $M^{p,\alpha}\left(-\pi,\pi\right)$ *if and only if* $d_{\beta}=\left[2\mathrm{Re}\beta+\frac{\alpha}{p}\right]=1$. *For* $d_{\beta}<1$ *it is not complete, but is minimal; and for* $d_{\beta}>1$ *it is complete, but is not minimal in it .*

In order to obtain the main results, we essentially use the following propositions, which are proved similar to the results in [25].

Consider the following unitary system of functions of the form

$$v_n^{\pm}\left(t\right)\equiv a\left(t\right)\omega_n^{+}\left(t\right)\pm b\left(t\right)\omega_n^{-}\left(t\right),\quad n\in N,$$

where $a\left(t\right),b\left(t\right),\omega_n^{+}\left(t\right),\omega_n^{-}\left(t\right)$ are some complex-valued functions given on $(0,\pi)$ and the associated with it double system

$$\left\{A\left(t\right)\ W_n\left(t\right);\ A\left(-t\right)\ W_n\left(-t\right)\right\}_{n\in N},$$

where

$$A\left(t\right)=\begin{cases} a\left(t\right), & t\in\left[0,a\right],\\ b\left(-t\right), & t\in\left[-a,0\right),\end{cases}$$
$$W_n\left(t\right)=\begin{cases} \omega_n^{+}\left(t\right), & t\in\left[0,a\right],\\ \omega_n^{-}\left(-t\right), & t\in\left[-a,0\right).\end{cases}$$

Let us consider the following double system

$$V_{n;m}\equiv\left(A\left(t\right)\ W_n\left(t\right);A\left(-t\right)\ W_m\left(-t\right)\right),\quad n,m\in N.$$

The following proposition is true.

Proposition 2.6. *The system* $\{V_{n;n}\}_{n\in N}$ $\left(1\cup\{V_{n;n}\}_{n\in N}\right)$ *forms a basis for* $M^{p,\alpha}(-\pi,\pi)$, $1<p<+\infty$, $0<\alpha\le 1$, *if and only if each of the systems* $\{v_n^+\}_{n\in N}$ *and* $\{v_n^-\}_{n\in N}$ *(systems* $1\cup\{v_n^+\}_{n\in N}$ *and* $\{v_n^-\}_{n\in N}$*) forms a basis for* $M^{p,\alpha}(0,\pi)$.

From this proposition we directly obtain the following:

Corollary 2.7. *System of exponents*

$$E_\beta^0\equiv\left\{e^{i(n-\beta\mathrm{signn})t}\right\}_{n\neq 0},\quad \left(1\cup E_\beta^0\right),$$

forms a basis for $M^{p,\alpha}(-\pi,\pi)$, $1<p<+\infty$, $0<\alpha\le 1$, *if and only if systems of sines* (1.1) *and cosines* $\{\cos(n-\beta)t\}_{n\in N}$ *(systems of sines* (1.1) *and cosines* $1\cup\{\cos(n-\beta)t\}_{n\in N}$*) form bases for* $M^{p,\alpha}(0,\pi)$.

3. Main Results

We will study the basis properties of systems of sines (1.1) and cosines (1.2) in spaces $M^{p,\alpha}(0,\pi)$. Let

$$2\mathrm{Re}\beta+\frac{\alpha}{p}\notin Z, 1<p<+\infty,\quad 0<\alpha<1.$$

Assume that $d(E_\beta)\equiv\left[2\mathrm{Re}\beta+\frac{\alpha}{p}\right]=0$, i.e. the inequalities

$$0<2\mathrm{Re}\beta+\frac{\alpha}{p}<1,$$

hold. In this case, by [26, Theorem 2.2], the system of exponents (1.4) forms a basis for $M^{p,\alpha}(-\pi,\pi)$. Then, as follows from Corollary 2.7, the system of sines (1.1) forms a basis for $M^{p,\alpha}(0,\pi)$. And now, let the equality $d(E_\beta)=1$ hold, i.e. the inequalities

$$1<2\mathrm{Re}\beta+\frac{\alpha}{p}<2,$$

are fulfilled. In this case, by Corollary 2.5, the system of exponents $\left\{e^{i(n-\beta\,\mathrm{signn})\,t}\right\}_{n\neq 0}$ forms a basis for $M^{p,\alpha}(-\pi,\pi)$ and as a result, by the results of Corollary 2.7, the system of sines (1.1) forms a basis for $M^{p,\alpha}(0,\pi)$.

Therefore, under the condition

$$0<2\mathrm{Re}\beta+\frac{\alpha}{p}<2,$$

the system (1.1) forms a basis for $M^{p,\alpha}(0,\pi)$. Consider the case when $2\mathrm{Re}\beta+\frac{\alpha}{p}<0$, let, for example, the inequalities

$$-2<2\mathrm{Re}\beta+\frac{\alpha}{p}<0\quad\Leftrightarrow\quad 0<2\mathrm{Re}(\beta+1)+\frac{\alpha}{p}<2,$$

hold. Consider the system $\{\sin (n-\beta)\ t\}_{n\geq 0}$ and transform it

$$\{\sin (n-(\beta+1)+1)\, t\}_{n\geq 0} \equiv \{\sin (n-\beta_1)\ t\}_{n\geq 1}\,,$$

where $\beta_1 = \beta + 1$. Consequently, the following inequalities

$$0 < 2\text{Re}\beta_1 + \frac{\alpha}{p} < 2,$$

hold and as a result the system $\{\sin (n-\beta)\ t\}_{n\geq 0}$ forms a basis, and therefore the system (1.1) is minimal, but is not complete in $M^{p,\alpha}(0,\pi)$. Continuing this process, we obtain that for $2\text{Re}\beta + \frac{\alpha}{p} < 0$ the system (1.1) is minimal, but is not complete in $M^{p,\alpha}(0,\pi)$.

With similar reasoning, it is proved that for $2\text{Re}\beta + \frac{\alpha}{p} > 2$, the system (1.1) is complete, but is not minimal in $M^{p,\alpha}(0,\pi)$. So, the following theorem is proved

Theorem 3.1. *Let* $2\text{Re}\beta + \frac{\alpha}{p} \notin Z$, $1 < p < +\infty$, $0 < \alpha < 1$. *System of sines* (1.1) *forms a basis for* $M^{p,\alpha}(0,\pi)$ *if and only if* $\left[\text{Re}\beta + \frac{\alpha}{2p}\right] = 0$. *Moreover, for* $\left[Re\beta + \frac{\alpha}{2p}\right] < 0$ *it is not complete, but is minimal; and for* $\left[\text{Re}\beta + \frac{\alpha}{2p}\right] > 0$ *it is complete, but is not minimal in* $M^{p,\alpha}(0,\pi)$.

We proceed to study the basis properties of the system of cosines

$$\{\cos (n-\beta)\ t\}_{n\in N}\,. \tag{3.1}$$

Let $2\text{Re}\beta + \frac{\alpha}{p} \notin Z$ and consider the case $1 < 2\text{Re}\beta + \frac{\alpha}{p} < 2$. In this case, by Corollary 2.5, the system E_β^0 forms a basis for $M^{p,\alpha}(-\pi,\pi)$, $1 < p < +\infty$, $0 < \alpha < 1$. Then, according to the results of Corollary 2.7, the system of cosines (3.1) forms a basis for $M^{p,\alpha}(0,\pi)$. Consider the case when the condition

$$2 < 2\text{Re}\beta + \frac{\alpha}{p} < 3 \quad \Rightarrow \quad 0 < 2\text{Re}\,(\beta-1) + \frac{\alpha}{p} < 1,$$

holds. Then, as follows from Theorem 2.2, the system of exponents $E_{\beta-1}$ forms a basis for $M^{p,\alpha}(-\pi,\pi)$, $1 < p < +\infty$, $0 < \alpha < 1$. At the same time, as follows from Theorem 2.4 the system $1 \cup E_{\beta-1}^0$ forms a basis for $M^{p,\alpha}(-\pi,\pi)$. As a result the system of cosines $1 \cup \{\cos (n-(\beta-1))\ t\}_{n\geq 1}$ forms a basis for $M^{p,\alpha}(0,\pi)$ and therefore the system $\{\cos (n-\beta)\ t\}_{n\geq 2}$ is minimal, but is not complete in $M^{p,\alpha}(0,\pi)$. On the other hand it is obvious that an arbitrary function from $M^{p,\alpha}(-\pi,\pi)$ can be expanded in an exponential system

$$\left\{e^{-i\tilde{\beta}t}\, e^{int};\, e^{i\tilde{\beta}t}\, e^{-ikt}\right\}_{n;k\geq 0},$$

in the same space, where $\tilde{\beta} = \beta - 1$. Using this fact, it is easy to prove that an arbitrary function from $M^{p,\alpha}(0,\pi)$ can also be expanded in a system of cosines

$$\left\{\cos\left(n-\tilde{\beta}\right)\ t\right\}_{n\geq 0} \equiv \left\{\cos\left(n-\beta\right)\ t\right\}_{n\geq 1}.$$

Since the system $\{\cos(n-\beta)\ t\}_{n\geq 2}$ is minimal, but incomplete in $M^{p,\alpha}(0,\pi)$, it follows that the system $\{\cos(n-\beta)\ t\}_{n\geq 1}$ is complete and minimal and at the same time forms a basis for $M^{p,\alpha}(0,\pi)$. Thus, if the inequalities

$$1 < 2\mathrm{Re}\beta + \frac{\alpha}{p} < 3,$$

hold, the system $\{\cos(n-\beta)\ t\}_{n\geq 1}$ forms a basis for $M^{p,\alpha}(0,\pi)$. Similarly to the case of a system of sines, it is established that for $2\mathrm{Re}\beta + \frac{\alpha}{p} < 1$ it is not complete, but is minimal; and for $2\mathrm{Re}\beta + \frac{\alpha}{p} > 3$ is complete, but is not minimal in $M^{p,\alpha}(0,\pi)$. So, the following theorem is true.

Theorem 3.2. *Let* $2\mathrm{Re}\beta + \frac{\alpha}{p} \notin Z$, $1 < p < +\infty$, $0 < \alpha < 1$. *System of cosines* $\{\cos(n-\beta)\,t\}_{n\geq 1}$ *forms a basis in* $M^{p,\alpha}(0,\pi)$ *if and only if* $\left[\mathrm{Re}\beta + \frac{\alpha}{2p} - \frac{1}{2}\right] = 0$. *For* $\left[\mathrm{Re}\beta + \frac{\alpha}{2p} - \frac{1}{2}\right] < 0$ *it is not complete, but it is minimal; for* $\left[\mathrm{Re}\beta + \frac{\alpha}{2p} - \frac{1}{2}\right] > 0$ *it is complete, but is not minimal in* $M^{p,\alpha}(0,\pi)$.

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References

1. D.R. Adams, *Morrey spaces*, Switzherland, Springer, 2016.
2. B.T. Bilalov, *A system of exponential functions with shift and the Kostyuchenko problem*, Sib. Math. J., 50(2) (2009), pp. 223-230.
3. B.T. Bilalov, *Bases of a system of exponentials in* L_p, Dokl. Akad. Nauk, 392(5) (2003), pp. 583-585.
4. B.T. Bilalov, *Basis Properties of Some Systems of Exponents, Cosines and Sines*, Sib. Math. J., 45 (2004), pp. 264-273.
5. B.T. Bilalov, *On solution of the Kostyuchenko problem*, Sib. Math. J., 53(3) (2012), pp. 404-418
6. B.T. Bilalov, *The basis property of a perturbed system of exponentials in Morrey-type spaces*, Sib. Math. J., 60:2 (2019), pp. 249-271.

7. B.T. Bilalov, T.B. Gasymov, A.A. Guliyeva, *On solvability of Riemann boundary value problem in Morrey-Hardy classes*, Turkish J. Math., 40(50) (2016), 1085-1101.
8. B.T. Bilalov, A.A. Guliyeva, *On basicity of the perturbed systems of exponents in Morrey-Lebesgue space*, Internat. J. Math., 25:1450054, (2014), pp. 1-10.
9. B.T. Bilalov, Z.G. Guseynov, *Basicity criterion for perturbed systems of exponents in Lebesgue spaces with variable summability*, Dokl. Akad. Nauk, 436(5) (2011), pp. 586-589.
10. B.T. Bilalov, Z.G. Guseynov, *Basicity of a system of exponents with a piece-wise linear phase in variable spaces*, Mediterr. J. Math, 9(3) (2012), pp. 487-498.
11. B.T. Bilalov, A.A. Huseynli, S.R. El-Shabrawy, *Basis Properties of Trigonometric Systems in Weighted Morrey Spaces*, Azerb. J. Math., 9(2) (2019), pp. 200-226
12. B.T. Bilalov, F.Sh. Seyidova, *Basicity of a system of exponents with a piecewise linear phase in Morrey-type spaces*, Turkish J. Math. 43 (2019), pp. 1850 - 1866
13. R.E. Casillo, H. Rafeiro, *An introductory course in Lebesgue spaces*, Springer, 2016.
14. D.V. Cruz-Vrible, A. Fiorenza, *Variable Lebesgue spaces*, Springer-Verlag, Basel, 2013.
15. L. Diening, P. Harjuleto, P. Hasto, M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*, Springer, 2011, 509 p.
16. M.I. Kadets, *The exact value of the Paley-Wiener constant*, Dokl. Akad. Nauk, 155 (1964), pp. 1253-1254.
17. V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral Operators in Non-Standard Function Spaces*, Volume 1: Variable Exponent Lebesgue and Amalgam Spaces, Springer, 2016.
18. V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral Operators in Non-Standard Function Spaces*, Volume 2: Variable Exponent Hölder, Morrey-Campanato and Grand Spaces, Springer, 2016.
19. B.Y. Levin, *Distribution of Roots of Entire Functions*, Moscow, GITL, 1956 (*in Russian*).
20. E.I. Moiseev, *On basicity of a system of sines*, Differ. Equ. 23(1) (1987), pp. 177-179.
21. E.I. Moiseev, *On basicity of systems of sines and cosines*, Dokl. Akad. Nauk, 275(4) (1984), pp. 794-798.
22. E.I. Moiseev, *On solution of Frankle's problem in special domain*, Differ. Equ., 28(4), (1992) pp. 682-692.
23. E.I. Moiseev, *On some boundary value problems for mixed type equations*, Differ. Equ., 28(1), (1992) pp. 123-132.

24. E.I. Moiseev, N.O. Taranov, *Solution of a Gellerstedt problem for the Lavrent'ev-Bitsadze equation*, Differ. Equ., 45 (2009), pp. 543-548.
25. T.R. Muradov, *On basicity of double and unitary systems in generalized Lebesgue spaces*, Math. Æterna, 5(4) (2015), pp. 657-666.
26. N.P. Nasibova, F.Sh. Seyidova, *On the basicity of a perturbed system of exponents with a unit in Morrey-type spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics, 39(1) (2019), pp. 151-161.
27. R. Paley, N. Wiener, *Fourier Transforms in the Complex Domain*, Amer. Math. Soc. Colloq. Publ., 19 (Amer. Math. Soc., Providence, RI, 1934)
28. S.M. Ponomarev, *On one eigenvalue problem*, Dokl. Akad. Nauk,, 249 (1979), pp. 1068-1070. (*in Russian*)
29. S.M. Ponomarev, *On the theory of boundary value problems for equations of mixed type in three-dimensional domains*, Dokl. Akad. Nauk,, 246 (6) (1979) pp. 303-1304.
30. A.M. Sedletskii, *Biorthogonal expansions in series of exponents on intervals of real axis*, Uspekhi Mat. Nauk, 37(5) (1982), pp. 51-95.

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