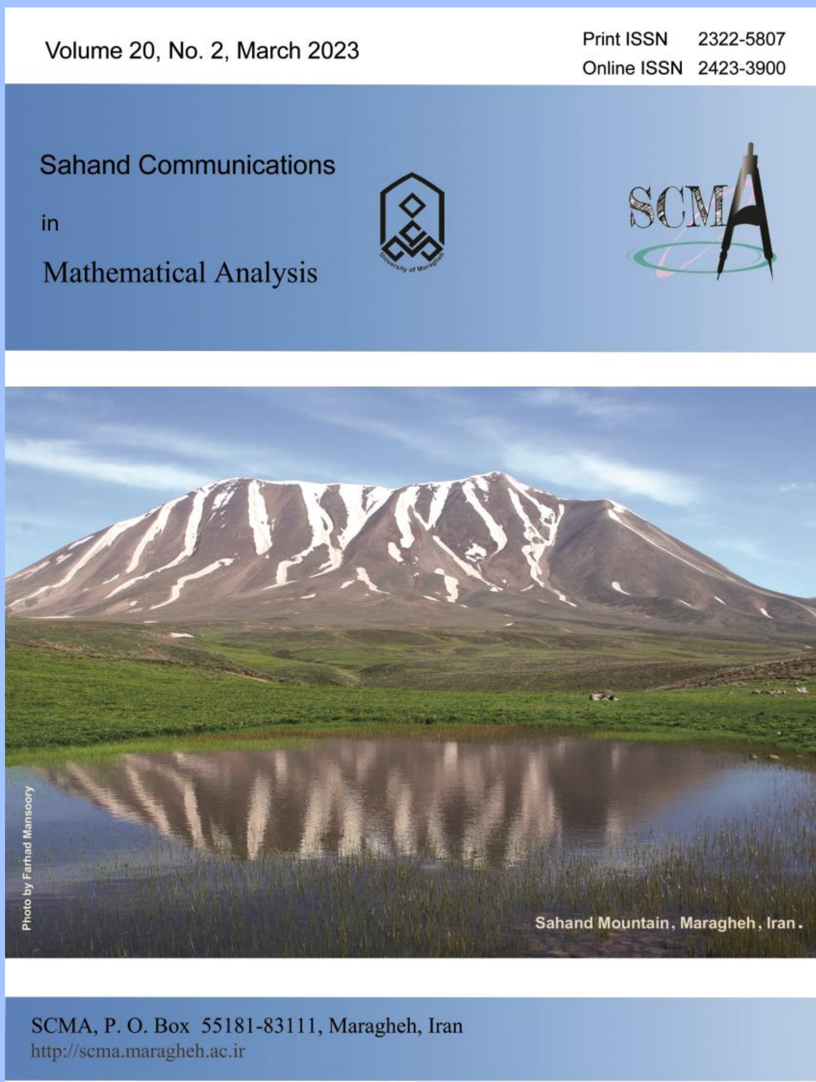
Investigation of the Properties of a New Class of Interpolation Polynomials Based on Fibonacci Numbers

Moosa Ebadi and Sareh Haghkhah

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 20
Number: 2
Pages: 133-146

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2022.544445.1030



Sahand Communications in Mathematical Analysis (SCMA) Vol. 20 No. 2 (2023), 133-146
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2022.544445.1030

Investigation of the Properties of a New Class of Interpolation Polynomials Based on Fibonacci Numbers

Moosa Ebadi[1*] and Sareh Haghkhah[2]

Abstract. In this paper, a class of new polynomials based on Fibonacci sequence using Newton interpolation is introduced. This target is performed once using Newton forward- divided- difference formula and another more using Newton backward- divided- difference formula. Some interesting results are obtained for forward and backward differences. The relationship between forward (and backward) differences and the Khayyam- Pascal's triangle are also examined.

1. Introduction

The Fibonacci number sequence appeared in the solution to the following problem:
"A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month those becomes productive?"
The resulting sequence is

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

The recursive formula for these numbers is:

$$f_1 = f_2 = 1, \qquad f_{n+1} = f_n + f_{n-1}, \quad n \geq 2.$$

Recall that the n^{th} number of this sequence is the sum of the two previous numbers [8].

2020 *Mathematics Subject Classification.* 34B24, 34B27.
Key words and phrases. Fibonacci sequence, Newton interpolation, Forward differences, Backward differences.
Received: 07 December 2021, Accepted: 20 December 2022.
* Corresponding author.

There are some developments about Fibonacci sequence and polynomials generated by Fibonacci sequence. Fibonacci Polynomials (FP) are generated recursively similar to Fibonacci sequence [1] and Fibonacci Coefficient Polynomials (FCP) are generated by putting Fibonacci sequence as coefficient [4]. Also Fibonacci Lagrange interpolation polynomials (FLIP) are generated recursively and implicitly by using Lagrange interpolation [7]. In this work, we propose a novel approach to construct polynomial based on Fibonacci sequence. The idea is by locating Fibonacci point $p_n = (n, f_n)$ as the point from the n^{th} term of Fibonacci sequence as points in coordinate system and then find a polynomial that passes through those points by using Newton interpolation. We perform this target once using Newton forward- divided difference formula and another more using Newton backward- divided- difference formula. In the field of numerical analysis, interpolation is the selection of a function $p(x)$ from a given class of functions in such a way that the graph of $y = p(x)$ passes through a finite set of given data points. The simplest approximation of an interpolation is a polynomial. One of the polynomial interpolation methods is the Newton interpolation. Suppose that we are given a table of different points (x_i, f_i) for $i = 0, 1, \dots, n$:

TABLE 1. The $n+1$ different points (x_i, f_i) for $i = 0, 1, \dots, n$.

x_i	x_0	x_1	x_2	$\cdots$	x_{n-2}	x_{n-1}	x_n
f_i	f_0	f_1	f_2	$\cdots$	f_{n-2}	f_{n-1}	f_n

When the nodes are arranged consecutively with equal spacing, Newton forward- divided- difference formula can be expressed as follow:

$$
\begin{aligned}
p_n(x) = &\sum_{k=0}^{n} \binom{s}{k} \Delta^k f_0 \\
= &f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 \\
&+ \cdots + \frac{s(s-1)(s-2)\cdots(s-k+1)}{k!}\Delta^k f_0 \\
&+ \cdots + \frac{s(s-1)(s-2)\cdots(s-n+1)}{n!}\Delta^n f_0.
\end{aligned} \tag{1.1}
$$

In this formula we have $h = x_{i+1} - x_i$, for each $i = 0, 1, \dots, n-1$ and $x = x_o + sh$. Then $x - x_i = (s - i)h$. Also the forward difference Δf_i is defined by

$$\Delta f_i = f(x_{i+1}) - f(x_i), \quad \text{for } i \geq 0. \tag{1.2}$$

Higher powers of the operator Δ are defined recursively by

$$\Delta^{k+1}(f_i) = \Delta^k(\Delta f_i), \quad \text{for } k \geq 1. \tag{1.3}$$

We also agree that $\Delta^0 f_i$ means the same as f_i.

Newton backward- divided- difference formula can be expressed as follow:

$$\begin{aligned} p_n(x) =& \sum_{k=0}^{n} (-1)^k \binom{-s}{k} \nabla^k f_n \\ =& \sum_{k=0}^{n} \binom{s+k-1}{k} \nabla^k f_n \\ =& f_n + s\nabla f_n + \frac{s(s+1)}{2!}\nabla^2 f_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_n \\ &+ \cdots + \frac{s(s+1)(s+2)\cdots(s+n-1)}{n!}\nabla^n f_n \end{aligned} \tag{1.4}$$

where $x = x_n + sh$ and the backward difference ∇f_i is defined by

$$\nabla f_i = f(x_i) - f(x_{i-1}), \quad \text{for } i \geq 1. \tag{1.5}$$

Higher powers are defined recursively by

$$\nabla^{k+1}(f_i) = \nabla^k(\nabla f_i), \quad \text{for } k \geq 1. \tag{1.6}$$

Note that $\nabla^k f_i$ is defined only for $i \geq k$. We also agree that $\nabla^0 f_i$ means the same as f_i. For some sources on Newton interpolation method see [3] and [6]. We denote $\mathbb{N}$ as the set of natural number and $\mathbb{W} = \{0, 1, 2, \ldots\}$.

2. Interpolating the Fibonacci Sequence Using Newton Forward- Divided- Difference Formula

Let denote Fibonacci point $p_n = (n, f_n)$ as the point from the n^{th} term of Fibonacci sequence as points in coordinate system. For example, the Fibonacci points for $n = 0$ until $n = 4$ are shown in Table 2.

Table 2. The Fibonacci points for $i = 0, \ldots, 4$.

x_i	$x_0 = 0$	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$	$x_4 = 4$
f_i	$f_0 = 1$	$f_1 = 1$	$f_2 = 2$	$f_3 = 3$	$f_4 = 5$
(x_i, f_i)	(0, 1)	(1, 1)	(2, 2)	(3, 3)	(4, 5)

In this paper, we define $FNIP_n(x)$ as the polynomial that generated using Newton interpolation (using forward- divided- difference formula) from p_i for $i = 0, 1, \ldots, n$ (we call it here Fibonacci point $p_i = (i, f_i)$). Since the points are arranged consecutively with equal spacing $h = 1$

and $x_0 = 0$, consequently, we have $x = x_0 + sh = s$ in (1.1) and then we can write

$$FNIP_n(x) = \sum_{k=0}^{n} \binom{x}{k} \Delta^k f_0. \tag{2.1}$$

According to relation (1.2), the generation of the constructing forward differences $\Delta^k f_i$ for $i = 0, 1, \ldots, 10$ is outlined in Table 3.

TABLE 3. The forward differences $\Delta_i^k = \Delta^k f_i$ for $i = 0, 1, \ldots, 10$.

x_i	f_i	Δ_i	Δ_i^2	Δ_i^3	Δ_i^4	Δ_i^5	Δ_i^6	Δ_i^7	Δ_i^8	Δ_i^9	Δ_i^{10}
0	1										
		0									
1	1		1								
		1		-1							
2	2		0		2						
		1		1		-3					
3	3		1		-1		5				
		2		0		2		-8			
4	5		1		1		-3		13		
		3		1		-1		5		-21	
5	8		2		0		2		-8		34
		5		1		1		-3		13	
6	13		3		1		-1		5		
		8		2		0		2			
7	21		5		1		1				
		13		3		1					
8	34		8		2						
		21		5							
9	55		13								
		34									
10	89										

$$FNIP_1(x) = 1$$
$$FNIP_2(x) = \frac{x^2 - x + 2}{2}$$
$$FNIP_3(x) = \frac{-x^3 + 6x^2 - 5x + 6}{6}$$
$$FNIP_4(x) = \frac{x^4 - 8x^3 + 23x^2 - 16x + 12}{12}$$

The graphics of above polynomials are shown in Figure 1. We recall that $FNIP_n(i) = f_i$ for $i = 1, \ldots, n$. Before deriving another formula for $FNIP_n(x)$, we will prove a theorem about $\Delta^k f_i$. Indeed by observing the behavior of $\Delta^k f_i$ in Table 3, we guess the formula stated in Theorem 2.1 and then we will prove it by induction.

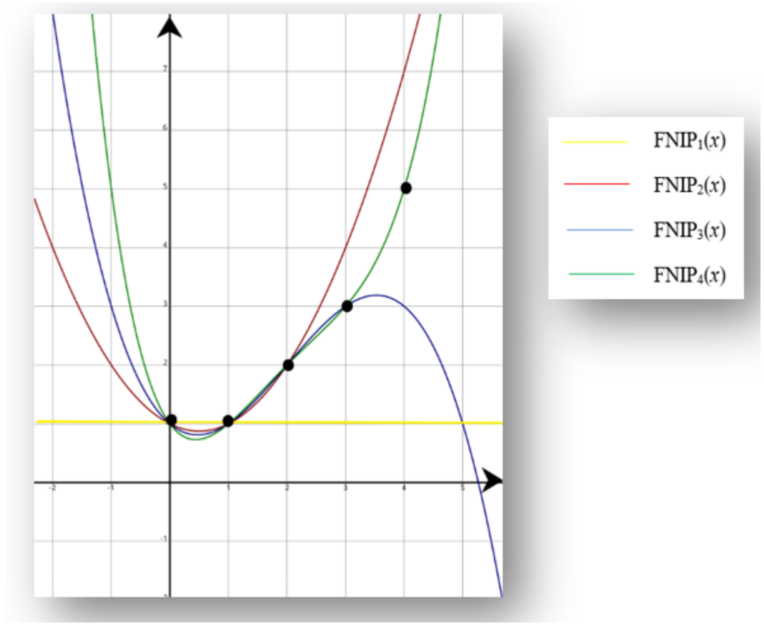


FIGURE 1. $FNIP_i(x)$ for $i = 1, \ldots, 4$.

Theorem 2.1. *Let* (i, f_i) *be the Fibonacci points for* $i \in \mathbb{W}$. *Then the forward differences* $\Delta^k f_i$ *for* $i, k \in \mathbb{W}$, *are obtained by*

$$\Delta^k f_i = \begin{cases} f_{i-k}, & i > k-1, \\ 0, & i = k-1, \\ (-1)^{k-i} f_{k-2-i}, & i < k-1. \end{cases} \tag{2.2}$$

Proof. By induction on k and according to the recursive formula of the operator Δ, that is (1.3), the theorem is easily proved. First we recall that due to the property of the Fibonacci sequence and relation (1.2), we have

$$\Delta f_0 = f_1 - f_0 = 0$$

and

$$\Delta f_i = f_{i+1} - f_i = f_{i-1} \ldots \quad \text{for } i \geq 1.$$

Note that the result is true for $k = 0$. Since $i \geq k = 0$ (or $i > k - 1$), according to the first rule of (2.2) we have

$$\Delta^0 f_i = f_{i-0} = f_i.$$

So the result is true for $k = 0$. Let k be a positive integer for which the statement is true. We will show that the statement is true for $k + 1$;

that is

$$\Delta^{k+1} f_i = \begin{cases} f_{i-k-1}, & i > k, \\ 0, & i = k, \\ (-1)^{k+1-i} f_{k-1-i}, & i < k. \end{cases} \tag{2.3}$$

Note that

$$\begin{aligned} \Delta^{k+1} f_i &= \Delta^k(\Delta f_i) \\ &= \Delta^k \left(\begin{cases} f_{i-1} & i \geq 1 \\ f_1 - f_0 & i = 0 \end{cases} \right). \end{aligned}$$

Now if $i \geq 1$, then we have

$$\begin{aligned} \Delta^k f_{i-1} &= \begin{cases} f_{i-1-k}, & i-1 > k-1, \\ 0, & i-1 = k-1, \\ (-1)^{k-i+1} f_{k-2-i+1}, & i-1 < k-1, \end{cases} \\ &= \begin{cases} f_{i-k-1}, & i > k, \\ 0, & i = k, \\ (-1)^{k+1-i} f_{k-1-i}, & i < k. \end{cases} \end{aligned}$$

So the statement (2.3) is established for this case. And if $i = 0$, then we consider the cases: $k = 0$, $k = 1$, $k = 2$ and $k > 2$.
If $k = 0$, then we have

$$\Delta f_0 = f_1 - f_0 = 0.$$

In this case, since $i = k$, the statement (2.3) is established due to the correctness of the second rule. It's easy to show that if $k = 1$, then

$$\Delta^2 f_0 = \Delta(f_1 - f_0) = \Delta f_1 - \Delta f_0 = f_2 - f_1 - f_1 + f_0 = f_0$$

and for $k = 2$, we have

$$\Delta^3 f_0 = \Delta^2(f_1 - f_0) = -f_1.$$

So for $i = 0$ and $k = 1, 2$ the statement (2.3) is established due to the correctness of the third rule. If $k > 2$, then we have

$$\Delta^k(f_1 - f_0) = \Delta^k f_1 - \Delta^k f_0.$$

According to induction hypothesis, since $k-1 > 1$, due to the third rule we have

$$\begin{aligned} \Delta^k f_1 - \Delta^k f_0 =& (-1)^{k-1} f_{k-2-1} - (-1)^k f_{k-2} \\ =& (-1)^{k-1} f_{k-3} + (-1)^{k-1} f_{k-2} \\ =& (-1)^{k-1}(f_{k-3} + f_{k-2}) \\ =& (-1)^{k-1} f_{k-1}. \end{aligned}$$

The last term is the third rule of (2.3). So the proof is complete. $\square$

Theorem 2.2. *Fibonacci Newton interpolation polynomials constructed by using forward- divided- difference formula, can be express as:*

$$FNIP_n(x) = 1 + \sum_{k=2}^{n} \binom{x}{k} (-1)^k f_{k-2}.$$

Proof. Applying Theorem 2.1 for $i = 0$ and $k > 1$, we have

$$\Delta^k f_0 = (-1)^k f_{k-2}. \tag{2.4}$$

In (2.1) we show that

$$FNIP_n(x) = \sum_{k=0}^{n} \binom{x}{k} \Delta^k f_0.$$

Replacing (2.4) in above formula for $k \geq 2$, we have

$$\begin{aligned} FNIP_n(x) =& f_0 + x\Delta f_0 + \sum_{k=2}^{n} \binom{x}{k} (-1)^k f_{k-2} \\ =& 1 + 0 + \frac{x(x-1)}{2!} + \frac{x(x-1)(x-2)}{3!}(-1) \\ &+ \frac{x(x-1)(x-2)(x-3)}{4!}(2) \\ &\vdots \\ &+ \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!}(-1)^n f_{n-2}. \end{aligned} \tag{2.5}$$

So the proof is complete. □

Corollary 2.3. *The leading coefficient of $FNIP_n(x)$ is $\dfrac{(-1)^n f_{n-1}}{n!}$.*

Proof. It is clear that the coefficient of x^n in the last term of (2.5) is $\dfrac{(-1)^n f_{n-1}}{n!}$. □

Remark that this corollary is proved in [7] for the leading coefficient of Fibonacci Lagrange interpolation polynomials. Since polynomials of degree n passing through $n+1$ points (i, f_i) are unique, this result is not far-fetched. We end this section by showing the relationship between $\Delta^k f_i$ and the Khayyam- Pascal's triangle. Note that

$$\begin{aligned} \Delta^0 f_i =& f_i, \\ \Delta^1 f_i =& f_{i+1} - f_i, \\ \Delta^2 f_i =& f_{i+2} - 2f_{i+1} + f_i, \\ \Delta^3 f_i =& f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i, \end{aligned}$$

$$\Delta^4 f_i = f_{i+4} - 4f_{i+3} + 6f_{i+2} - 4f_{i+1} + f_i.$$

By induction on $k \geq 0$ we can show that

$$\Delta^k f_i = \sum_{j=0}^{k} \binom{k}{j} f_{i+k-j}(-1)^j. \tag{2.6}$$

First we note that the result is true for $k = 0$ and $k = 1$. If $k = 0$, then

$$\Delta^0 f_i = \sum_{j=0}^{0} \binom{0}{j} f_{i-j}(-1)^j = f_i$$

and for $k = 1$, we have

$$\Delta f_i = \sum_{j=0}^{1} \binom{1}{j} f_{i+1-j}(-1)^j = f_{i+1} - f_i.$$

Now we assume (2.6) holds for $k > 1$. We will show that the statement, is true for $k + 1$; that is

$$\Delta^{k+1} f_i = \sum_{j=0}^{k+1} \binom{k+1}{j} f_{i+k+1-j}(-1)^j.$$

We will need to use Pascal's identity in the form

$$\binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j}.$$

We have

$$\begin{aligned}
\Delta^{k+1} f_i =& \Delta^k(\Delta f_i) \\
=& \Delta^k(f_{i+1} - f_i) = \Delta^k f_{i+1} - \Delta^k f_i \\
=& \sum_{j=0}^{k} \binom{k}{j} f_{i+1+k-j}(-1)^j - \sum_{j=0}^{k} \binom{k}{j} f_{i+k-j}(-1)^j \\
=& f_{i+1+k} + \sum_{j=1}^{k} \binom{k}{j} f_{i+1+k-j}(-1)^j - \sum_{j=0}^{k-1} \binom{k}{j} f_{i+k-j}(-1)^j \\
& - f_i(-1)^k \\
=& f_{i+1+k} + \sum_{j=1}^{k} \binom{k}{j} f_{i+1+k-j}(-1)^j
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{j=1}^{k} \binom{k}{j-1} f_{i+k-j+1}(-1)^{j-1} - f_i(-1)^k \\
=& f_{i+1+k} + \sum_{j=1}^{k} \left[\binom{k}{j} + \binom{k}{j-1} \right] f_{i+k-j+1}(-1)^j + f_i(-1)^{k+1}.
\end{aligned}
$$

From Pascal's identity, it follows that

$$
\begin{aligned}
\Delta^{k+1} f_i =& f_{i+1+k} + \sum_{j=1}^{k} \binom{k+1}{j} f_{i+k-j+1}(-1)^j + f_i(-1)^{k+1} \\
=& \sum_{j=0}^{k+1} \binom{k+1}{j} f_{i+k-j+1}(-1)^j.
\end{aligned}
$$

Hence the result is true for $k+1$ and by induction, (2.6) is true for all $k \geq 0$. According to (2.2) and (2.6), the following theorem can be expressed.

Theorem 2.4. *Let $\{f_n\}$ be the Fibonacci sequence. Then*

$$
\sum_{j=0}^{k} \binom{k}{j} f_{i+k-j}(-1)^j = \begin{cases} f_{i-k}, & i > k-1, \\ 0, & i = k-1, \\ (-1)^{k-i} f_{k-2-i}, & i < k-1. \end{cases}
$$

For example, for $i = 2$, the following triangle can be drawn with respect to the Khayyam-Pascal's triangle.

TABLE 4. The relationship between $\Delta^k f_i$ and the Khayyam- Pascal's triangle.

$i = 2$	$\sum_{j=0}^{k} \binom{j}{k} f_{i+k-j}(-1)^j$	$\Delta^k f_i$
$k = 0$	$1f_2$	f_2
$k = 1$	$1f_3 \quad - \quad 1f_2$	f_1
$k = 2$	$1f_4 \quad - \quad 2f_3 \quad + \quad 1f_2$	f_0
$k = 3$	$1f_5 \quad - \quad 3f_4 \quad + \quad 3f_3 \quad - \quad 1f_2$	0
$k = 4$	$1f_6 \quad - \quad 4f_5 \quad + \quad 6f_4 \quad - \quad 4f_3 \quad + \quad 1f_2$	f_0

3. Interpolating the Fibonacci Sequence Using Newton Backwar-Divided-Differences Formula

In this section, we will consider the Newton interpolation polynomials passing through the points $p_i = (i, f_i)$, for $i = 0, 1, \ldots, n$ and generated by using backward-divided-difference. For simplicity, we denote these polynomials by $P_n(x)$. Since the points are arranged consecutively with equal spacing $h = 1$ and $x_n = n$, consequently, in (1.4) we have $x = n+s$ or $s = x - n$ and then we can write

$$P_n(x) = \sum_{k=0}^{n} \binom{x-n+k-1}{k} \nabla^k f_n. \tag{3.1}$$

The backward differences $\nabla^k f_i$ for $i = 0, 1, \ldots, 10$ can also be seen in Table 3. Before deriving another formula for $P_n(x)$, we will prove a theorem about $\nabla^k f_i$. Indeed by observing the behavior of $\nabla^k f_i$ in Table 3, we guess below formula and then we prove it by induction.

Theorem 3.1. *Let (i, f_i) be the Fibonacci points for $i \in \mathbb{W}$. Then the backward differences $\nabla^k f_i$ for $i \geq k \in \mathbb{W}$, are obtained by*

$$\nabla^k f_i = \begin{cases} (-1)^i f_{2k-2-i}, & k \leq i < 2k-1, \\ 0, & i = 2k-1, \\ f_{i-2k}, & i > 2k-1. \end{cases} \tag{3.2}$$

Proof. By induction on k and according to the recursive formula of the operator ∇, that is (1.6), the theorem similar to the Theorem 2.1, is easily proved. We only note that due to the property of the Fibonacci sequence and relation (1.5), we have

$$\nabla f_i = f_i - f_{i-1} = f_{i-2}, \quad \text{for } i \geq 2. \qquad \Box$$

Theorem 3.2. *Fibonacci Newton interpolation polynomials constructed by using backward- divided- difference formula, can be express as:*

$$P_n(x) = \sum_{k=0}^{\frac{n-1}{2}} \binom{x-n+k-1}{k} f_{n-2k} - \sum_{k=\frac{n+3}{2}}^{n} \binom{x-n+k-1}{k} f_{2k-2-n}$$

if n is odd, and

$$P_n(x) = \sum_{k=0}^{\frac{n}{2}} \binom{x-n+k-1}{k} f_{n-2k} + \sum_{k=\frac{n}{2}+1}^{n} \binom{x-n+k-1}{k} f_{2k-2-n}$$

if n is even.

Proof. In (3.1) we show that

$$P_n(x) = \sum_{k=0}^{n} \binom{x-n+k-1}{k} \nabla^k f_n.$$

Applying above formula and Theorem 3.1 for $i = n$ and $k \leq i$, we have

$$\begin{aligned}
P_n(x) = &\sum_{k=0}^{\frac{n-1}{2}} \binom{x-n+k-1}{k} \nabla^k f_n + \binom{x-n+k-1}{k} \nabla^{\frac{n+1}{2}} f_n \\
&+ \sum_{k=\frac{n+3}{2}}^{n} \binom{x-n+k-1}{k} \nabla^k f_n \\
= &\sum_{k=0}^{\frac{n-1}{2}} \binom{x-n+k-1}{k} f_{n-2k} - \sum_{k=\frac{n+3}{2}}^{n} \binom{x-n+k-1}{k} f_{2k-2-n}
\end{aligned}$$

if n is odd, and

$$\begin{aligned}
P_n(x) = &\sum_{k=0}^{\frac{n}{2}} \binom{x-n+k-1}{k} \nabla^k f_n + \sum_{k=\frac{n}{2}+1}^{n} \binom{x-n+k-1}{k} \nabla^k f_n \\
= &\sum_{k=0}^{\frac{n}{2}} \binom{x-n+k-1}{k} f_{n-2k} + \sum_{k=\frac{n}{2}+1}^{n} \binom{x-n+k-1}{k} f_{2k-2-n}
\end{aligned}$$

if n is even. So the proof is complete. □

We end this section by showing the relationship between $\nabla^k f_i$ and the Khayyam- Pascal's triangle. Note that

$$\begin{aligned}
\nabla^0 f_i =& f_i, \\
\nabla^1 f_i =& f_i - f_{i-1}, \\
\nabla^2 f_i =& f_i - 2f_{i-1} + f_{i-2}, \\
\nabla^3 f_i =& f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3}, \\
\nabla^4 f_i =& f_i - 4f_{i-1} + 6f_{i-2} - 4f_{i-3} + f_{i-4}.
\end{aligned}$$

Similar to the proof of Equation (2.6) by induction on k it can be shown that

$$\nabla^k f_i = \sum_{j=0}^{k} \binom{k}{j} f_{i-j} (-1)^j. \tag{3.3}$$

According to (3.2) and (3.3), the following theorem can be expressed.

Theorem 3.3. *Let $\{f_n\}$ be the Fibonacci sequence. Then*

$$\sum_{j=0}^{k} \binom{k}{j} f_{i-j}(-1)^j = \begin{cases} (-1)^i f_{2k-2-i}, & k \leq i < 2k-1, \\ 0, & i = 2k-1, \\ f_{i-2k}, & i > 2k-1. \end{cases}$$

For example, for $i = 5$, the following triangle can be drawn with respect to the Khayyam-Pascal's triangle.

TABLE 5. The relationship between $\nabla^k f_i$ and the Khayyam- Pascal's triangle.

$i=5$	$\sum_{j=0}^{k} \binom{j}{k} f_{i-j}(-1)^j$	$\nabla^k f_i$
$k=0$	$1f_5$	f_5
$k=1$	$1f_5 \quad - \quad 1f_4$	f_3
$k=2$	$1f_5 \quad - \quad 2f_4 \quad + \quad 1f_3$	f_1
$k=3$	$1f_5 \quad - \quad 3f_4 \quad + \quad 3f_3 \quad - \quad 1f_2$	0
$k=4$	$1f_5 \quad - \quad 4f_4 \quad + \quad 6f_3 \quad - \quad 4f_2 \quad + \quad 1f_1$	$-f_1$

4. FUTURE WORK

There are certain polynomials that are used in the numerical solution of differential and integral equations. For example, we can mention the use of Bernoulli polynomials in solving nonlinear two-dimensional integral equations [2] and the convergence of Nyström methods for solving Fredholm$-$integral equations of the second kind with the smooth kernel can be analyzed using an interpolatory projection based on Legendre polynomials of degree $\leq n$ [5]. For the numerical approximation of the definite integral in the following form

$$\int_{-1}^{1} f(x)dx \tag{4.1}$$

by Gauss$-$Legendre method, the goal is to find coefficients such as ω_i, $i = 1, \ldots, n$ and nodes such as x_i, $i = 1, \ldots, n$, which are the roots of Legendre's orthogonal polynomials, so the rule is stated as

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} \omega_i f(x_i). \tag{4.2}$$

Now the question is whether there are a new class of orthogonal polynomials whose non-zero roots are functions of the numbers of the Fibonacci sequence and can apply to relation (4.2)? For example, the following two Legendre polynomials apply in the above conditions:

$$p(x) = \frac{1}{2}(3x^2 - 1), \qquad q(x) = \frac{1}{2}(5x^3 - 3x). \tag{4.3}$$

5. Conclusion

In this paper a new method to construct polynomials based on Fibonacci sequence using Newton interpolation is introduced. By observing the behavior of $\Delta^k f_i$ and $\nabla^k f_i$ in Table 3, we obtained some interesting results.

As mentioned, in the forward differences $\Delta^k f_0$ for $k \geq 2$, the Fibonacci numbers with alternating signs are appeared. And in the backward differences $\nabla^k f_n$ for $k = 0, \ldots, n$, every other number appears to be decreasing, until we reach to f_0 or f_1 in the integral part of $[\frac{n+1}{2}]$. If n is odd, for $k = \frac{n+1}{2}$ we have $\nabla^k f_n = 0$ and for $k > \frac{n+1}{2}$, opposite of the odd index Fibonacci numbers are appeared. And if n is even, for $k > \frac{n+1}{2}$, the Fibonacci sequence sentences are appeared with even index.

For example, for $n = 9$, the backward differences $\nabla^k f_n$ for $k = 0, 1, \ldots, 9$ are

$$f_9,\ f_7,\ f_5,\ f_3,\ f_1,\ 0,\ -f_1,\ -f_3,\ -f_5,\ -f_7$$

respectively and for $n = 10$, the backward differences $\nabla^k f_n$ for $k = 0, 1, \ldots, 10$ are

$$f_{10},\ f_8,\ f_6,\ f_4,\ f_2,\ f_0,\ f_0,\ f_2,\ f_4,\ f_6,\ f_8$$

respectively. We also showed the relationship between forward (and backward) differences and the Khayyam- Pascal's triangle.

Acknowledgment. The authors wish to thank referees for their useful comments.

References

1. T. Amdeberhan, X. Chen, V. Moll and B. Sagan, *Generalized Fibonacci polynomials and fibonomial coefficients*, Ann. Comb., 18 (2014), pp. 541-562.
2. S. Bazm, *Numerical solution of a class of nonlinear two-dimensional integral equations using Bernoulli polynomials*, Sahand Commun. Math. Anal., 3 (1) (2016), pp. 37-51
3. R.L. Burden, J.D. Faires and A.M. Burden, *Numerical Analysis*, Tenth edition, Cengage Learning, Boston, Massachusetts, 2016.
4. D. Garth, D. Mills and P. Mitchell, *Polynomials generated by the Fibonacci sequence*, J. Integer Seq., 10 (2007), Article 07.6.8.

5. B. Hamza, A. Chafik and T. Mohamed, *Legendre Superconvergent Degenerate Kernel and Nystrom Methods for Fredholm Integral Equations*, Sahand Commun. Math. Anal., 20 (1) (2023), pp. 45-60
6. D. Kincaid and W. Cheney, *Numerical Analysis Mathematics of Scientific Computing*, Third Edition., AMS., 2017.
7. M.S. Mufid, T. Asfihani and L. Hanaf,*On The Lagrange Interpolation of Fibonacci Sequence*, IJCSAM., 2 (2016), pp. 38-40.
8. T. Scott and P. Marketos,*On the origin of Fibonacci sequence*, MacTutor History of Mathematics, University of St Andrews, Scotland, 2014.

[1] Department of Mathematics, University of Farhangian, Tehran, Iran.
Email address: moosa.ebadi@yahoo.com

[2] Department of Mathematics, University of Farhangian, Tehran, Iran.
Email address: sareh_haghkhah@cfu.ac.ir