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Jawad Ettayb

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Abstract. In this paper, we define the notions of semi-regular operator, analytical core, surjectivity modulus and the injectivity modulus of bounded linear operators on non-Archimedean Banach spaces over $\mathbb{K}$. We give a necessary and sufficient condition on the range of bounded linear operators to be closed. Moreover, many results are proved.

1. Introduction

In the classical setting, Saphar [14] has introduced the concept of algebraic core and the analytic core has been introduced by Vrbová [1] these notions palyed an important role in spectral theory of bounded linear operators on complex Banach spaces. Another important notions in perturbation theory and spectral theory are the reduced minimum modulus, surjectivity modulus and the injectivity modulus of bounded linear operators on classical Banach spaces. The perturbation theory has been used in a large number of different settings in physics and applied mathematics such as the perturbation method for sound radiation by a vibrating plate in a light fluid and quantum mechanics, for more details, we refer to [5] and [8].

In non-Archimedean operators theory, the problem of perturbations of bounded linear operators studied firstly by Serre [16] who dealt with a compact perturbations of the identity on non-Archimedean Banach spaces having an orthogonal base. Gurson [6] lifted this restriction by working on perturbations of the identity on general non-Archimedean Banach spaces. Moreover a complete study of perturbations of the identity was finally achieved by Schikhof [15]. This perturbations had a

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useful tool to study the theory of non-Archimedean differential operators, for more details, we refer to [11] and [12]. Recently, Araujo, Perez and Vega [2] aimed at a general theory of the perturbations of continuous linear operators between non-Archimedean Banach spaces by compact operators. This work is motivated by the pioneer work of Kiyosawa [9] who studied the perturbations of bounded linear operators on non-Archimedean Banach spaces.

Throughout this paper, X and Y are non-Archimedean (n.a) Banach spaces over a non trivially complete valued field $\mathbb{K}$ with valuation $|\cdot|$, $B(X,Y)$ denotes the set of all bounded linear operators from X into Y. If $X = Y$, we write $B(X,X) = B(X)$. If $A \in B(X)$, $N(A)$ and $R(A)$ denote the kernel and the range of A respectively. A non-Archimedean Banach space X over $\mathbb{K}$ is said to be a free non-Archimedean Banach space if there exists a family $(x_i)_{i\in I}$ of elements of X indexed by a set I such that each $x \in X$ can be written uniquely like a pointwise convergent series defined by $x = \sum\limits_{i\in I} \lambda_i x_i$ and $\|x\| = \sup\limits_{i\in I} |\lambda_i|\|x_i\|$. The family $(x_i)_{i\in I}$ is then called an orthogonal basis for X. If, for all $i \in I$, $\|x_i\| = 1$, then $(x_i)_{i\in I}$ is called an orthonormal basis of X. Recall that every bounded linear operator A on a free non-Archimedean Banach space X can be written in a unique fashion as a pointwise convergent series, that is, there exists an infinite matrix $(a_{i,j})_{(i,j)\in I\times I}$ with coefficients in $\mathbb{K}$ such that

$$A = \sum_{i,j\in I} a_{ij} e'_j \otimes e_i \quad \text{and} \quad \forall j \in I, \quad \lim_{i\in I} |a_{ij}|\|e_i\| = 0,$$

where $(\forall i \in I)$ $e'_i(u) = u_i$ (e'_i is the linear form associated with e_i). Moreover, for each $j \in I$, $Ae_j = \sum\limits_{i\in I} a_{ij} e_i$ and its norm is defined by

$$\| A \| = \sup_{i,j\in I} \frac{|a_{ij}|\|e_i\|}{\|e_j\|}.$$

2. Basic Definitions and Needed Results

We continue by recalling some preliminaries.

Definition 2.1 ([3]). Let $\omega = (\omega_i)_i$ be a sequence of non-zero elements of $\mathbb{K}$. We define $\mathbb{E}_\omega$ by

$$\mathbb{E}_\omega = \left\{ x = (x_i)_i : \ \forall i \in \mathbb{N},\ x_i \in \mathbb{K} \text{ and } \lim_{i\to\infty} |\omega_i|^{\frac{1}{2}}|x_i| = 0 \right\},$$

and it is equipped with the norm

$$(\forall x \in \mathbb{E}_\omega) : x = (x_i)_i \subset \mathbb{E}_\omega, \qquad \|x\| = \sup_{i\in\mathbb{N}} \left(|\omega_i|^{\frac{1}{2}} |x_i| \right).$$

Remark 2.2 ([3]). (i) The space $(\mathbb{E}_\omega,\ \|\cdot\|)$ is a non-Archimedean Banach space.

(ii) For $x=(x_i)_i$ and $y=(y_i)_i$, the inner product is defined by

$$\begin{aligned}\langle\cdot,\cdot\rangle:\mathbb{E}_\omega\times\mathbb{E}_\omega &\longrightarrow \mathbb{K}\\ (x,y) &\mapsto \sum_{i=0}^{\infty} x_i y_i \omega_i.\end{aligned}$$

Hence, the space $(\mathbb{E}_\omega,\|\cdot\|,\langle\cdot,\cdot\rangle)$ is called a p-adic (or non-Archimedean) Hilbert space.

(iii) The orthogonal basis $\{e_i,\ i\in\mathbb{N}\}$ is called the canonical basis of $\mathbb{E}_\omega$.

For $X=\mathbb{E}_\omega$, we have the following proposition.

Proposition 2.3 ([4]). *Let $A=\sum_{i,j\in\mathbb{N}} a_{ij}e'_j\otimes e_i\in B(\mathbb{E}_\omega)$. Then A has an adjoint $A^*\in B(\mathbb{E}_\omega)$ if and only if for all $i\in\mathbb{N}$, $\lim_{j\to\infty}|\omega_j|^{\frac{-1}{2}}|a_{ij}|=0$.*

We denote by $B_0(X)$ the set of all bounded linear operators on X having an adjoint.

Remark 2.4 ([13]). Let X be a non-Archimedean Banach space over $\mathbb{K}$. If $A\in B(X)$, we define two norms of A by

$$\|A\|=\sup_{x\in X\setminus\{0\}}\frac{\|Ax\|}{\|x\|}\quad\text{and}\quad \|A\|_0=\sup_{x\in X,\ \|x\|=1}\frac{\|Ax\|}{\|x\|}. \tag{2.1}$$

These two norms are always equivalent and are equal if $\|X\|\subseteq|\mathbb{K}|$ or if the valuation of $\mathbb{K}$ is dense.

For more details on non-Archimedean operators theory, we refer to [3] and [13].

Definition 2.5 ([10]). A non-Archimedean normed space X is of countable type if it contains a countable set whose linear hull is dense in X.

A subspace D of X is called complemented if there exists a continuous projection $P\in B(X)$ (not necessarily an orthoprojection) with $R(P)=D$. Then $D=R(P)$ and $D_1=N(P)$ are closed subspaces (such a D_1 is called a complement of D) and X is the algebraic direct sum of D and D_1. Conversely, if X is a non-Archimedean Banach space and D, D_1 are closed subspaces such that the algebraic direct sum of them is X, then D is complemented and D_1 is a complement of D, for more details, we refer to [10]. We have the following theorem.

Theorem 2.6 ([10]). *Let X be a Banach space of countable type over $\mathbb{K}$. Then each closed subspace D of X is complemented. In fact, for* *every $\varepsilon > 0$ there exists a continuous projection $P \in B(X)$ onto D with $\|P\| \leq 1+\varepsilon$.*

We have the following definition.

Definition 2.7 ([10])**.** A non-Archimedean field $\mathbb{K}$ is called spherically complete if each decreasing sequence of balls in $\mathbb{K}$ has a non-empty intersection.

Theorem 2.8 ([10])**.** *Let $\mathbb{K}$ be spherically complete. Then each normed space of countable type has an orthogonal base.*

$X' = B(X, \mathbb{K})$ (dual space of X) is the set of all linear functional from X into $\mathbb{K}$. We have the following theorem.

Theorem 2.9 ([13])**.** *Suppose that $\mathbb{K}$ is spherically complete. Let X be a non-Archimedean Banach space over $\mathbb{K}$. For any non zero $x \in X\backslash\{0\}$, there exists $f \in X'$ such that $f(x) = 1$ and $\|f\| = \|x\|^{-1}$.*

As the classical setting, we define the graph of an unbounded linear operators on non-Archimedean Banach spaces. The set of all unbounded linear operators on X is denoted by $U(X)$. We extend the following results of [3] from $\mathbb{E}_\omega$ to general non-Archimedean Banach spaces as follows.

Definition 2.10. Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A \in U(X)$, the graph of A is denoted by $G(A)$ and is given by

$$G(A) = \{(x, Ax) \in X \times X : x \in D(A)\}.$$

Definition 2.11. Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A \in U(X)$, A is said to be closed if its graph is a closed subspace of $X \times X$ and A is said to be closable if it has a closed extention. The collection of closed linear operators is denoted by $\mathcal{C}(X)$.

As in the classical theory of an unbounded linear operators, we characterize the closedness of an operator $A \in U(X)$ as follows: for all $(x_n)_{n\in\mathbb{N}} \subset D(A)$ such that $\lim\limits_{n\to\infty} \|x_n - x\| = 0$ and $\lim\limits_{n\to\infty} \|Ax_n - y\| = 0$ for some $(x, y \in X)$, then $x \in D(A)$ and $Ax = y$.

Remark 2.12. Note that if $A \in B(X)$, then it is closed. In fact: since A is bounded, then $D(A) = X$. Moreover, if $(x_n)_{n\in\mathbb{N}} \subset X$ such that $x_n \to x$ in X as $n \to \infty$, then by the boundedness of A, it follows that $Ax_n \to Ax$ as $n \to \infty$, that is, $(x_n, Ax_n) \to (x, Ax)$ in $X \times X$ as $n \to \infty$, hence $G(A)$ is closed.

Obviously, $B(X) \subset \mathcal{C}(X)$.

Remark 2.13. As in the classical setting, it is not difficult to see that if $A \in \mathcal{C}(X)$ and $B \in B(X)$, then $A + B \in \mathcal{C}(X)$.

Let $A : D(A) \subset X \to X$ be an unbounded linear operator. Note that $D(A)$ endowed by the non-Archimedean graph norm defined by: for all $x \in D(A)$,

$$\|x\|_{D(A)} = \max\{\|x\|, \|Ax\|\}.$$

We have the following proposition.

Proposition 2.14. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A \in U(X)$. Then $(D(A), \|\cdot\|_{D(A)})$ is non-Archimedean Banach space if and only if A is closed.*

We have the following theorem.

Theorem 2.15 ([1])**.** *Let X be vector space, let A be linear operator on X. The following statements are equivalent:*

(i) *For all $m \in \mathbb{N}$, $N(A) \subseteq R(A^m)$.*
(ii) *For all $n \in \mathbb{N}$, $N(A^n) \subseteq R(A)$.*
(iii) *For all $n \in \mathbb{N}$ and for each $m \in \mathbb{N}$, $N(A^n) \subseteq R(A^m)$.*
(iv) *For all $n, m \in \mathbb{N}$, $N(A^n) = A^m(N(A^{n+m}))$.*

If $\|X\| \subseteq |\mathbb{K}|$ and from Definition 2 of section 4 of [9], we have the following definition.

Definition 2.16 ([9])**.** Let X be a non-Archimedean Banach space over $\mathbb{K}$ such that $\|X\| \subseteq |\mathbb{K}|$ and let M, N be two linear subspaces of X. Define

$$\delta(M,N) = \begin{cases} \sup\limits_{x \in M, \|x\|=1} d(x,N), & \text{if } M \neq \{0\}, \\ 0, & \text{otherwise} \end{cases}$$

and $g(M,N) = \max\{\delta(M,N), \delta(N,M)\}$ where $d(x,N) = \inf\limits_{y \in N} \|x - y\|$, $g(M,N)$ is called the gap between the subspaces M and N.

Definition 2.17 ([1])**.** Let A be a linear operator on a vector space X. The algebraic core $C(A)$ is defined to be the greatest subspace M of X for which $A(M) = M$.

Remark 2.18 ([1])**.** (i) If A is surjective, then $C(A) = X$.
(ii) For all $n \in \mathbb{N}, C(A) = A^n(C(A)) \subseteq R^n(X)$, thus $C(A) \subseteq R^\infty(A)$ where $R^\infty(A) = \bigcap\limits_{n \in \mathbb{N}} R(A^n)$.

The next result gives a precise description of the subspace $C(A)$ in terms of sequences.

Theorem 2.19 ([1])**.** *For a linear operator A on a vector space X, the following statements are equivalent:*

(i) $x \in C(A)$.

(ii) *There exists a sequence* $(u_n)_{n\in\mathbb{N}} \subseteq X$ *such that* $x = u_0$ *and for all* $n \in \mathbb{N}^*$, $Au_{n+1} = u_n$.

As the classical setting, we have.

Definition 2.20 ([9]). Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A \in B(X)$, the reduced minimum modulus of A is defined by

$$\gamma(A) = \inf_{x \notin N(A)} \frac{\|Ax\|}{d(x, N(A))}.$$

By convention $\gamma(0) = \infty$.

Lemma 2.21 ([9]). *Let* X *be a non-Archimedean Banach space over* $\mathbb{K}$, *let* $A \in B(X)$, *we have:*

$$\gamma(A) > 0 \text{ if and only if } R(A) \text{ is closed.}$$

3. Main Results

We have the following proposition.

Proposition 3.1. *Let* X *be a non-Archimedean Banach space over* $\mathbb{K}$, *let* $A \in B(X)$ *be invertible. Then* $\gamma(A) = \|A^{-1}\|^{-1}$.

Proof. Since $A \in B(X)$ is invertible, then $N(A) = \{0\}$ and A is surjective, hence for each $x \in X\backslash\{0\}$, we have

$$d(x, N(A)) = d(x, \{0\}) = \|x\|.$$

Set $y = Ax$, we have

$$\begin{aligned}
\gamma(A) &= \inf_{x\in X\backslash\{0\}} \frac{\|Ax\|}{\|x\|} \\
&= \inf_{x\in X\backslash\{0\}} \left(\frac{1}{\frac{\|x\|}{\|Ax\|}} \right) \\
&= \left(\sup_{x\in X\backslash\{0\}} \frac{\|x\|}{\|Ax\|} \right)^{-1} \\
&= \left(\sup_{y\in X\backslash\{0\}} \frac{\|A^{-1}y\|}{\|y\|} \right)^{-1} \\
&= \left\|A^{-1}\right\|^{-1}.
\end{aligned}$$

□

We have the following theorem.

Theorem 3.2. *Let* X *be a non-Archimedean Banach space over* $\mathbb{K}$, *let* $A, B \in B(X)$, *the following statements hold:*

(i) *If* $AB = \lambda BA$, *then* $\gamma(AB) = |\lambda|\gamma(BA)$.

(ii) *If A^{-1} exists, then* $\gamma(AB) \leq \|A\|\gamma(B)$.

Proof. Let $A, B \in B(X)$, we have:

(i) If $AB = \lambda BA$, then $N(AB) = N(BA)$. Hence

$$\begin{aligned}\gamma(AB) &= \inf_{x \notin N(AB)} \frac{\|ABx\|}{\|x\|} \\ &= |\lambda| \inf_{x \notin N(BA)} \frac{\|BAx\|}{\|x\|} \\ &= |\lambda|\gamma(BA).\end{aligned}$$

(ii) Since A is injective, then $N(AB) = N(B)$ and

$$\begin{aligned}\gamma(AB) &= \inf_{x \notin N(AB)} \frac{\|ABx\|}{\|x\|} \\ &= \inf_{x \notin N(B)} \frac{\|ABx\|}{\|x\|} \\ &\leq \|A\| \inf_{x \notin N(B)} \frac{\|Bx\|}{\|x\|} \\ &= \|A\|\gamma(B).\end{aligned}$$

□

The following statement holds.

Theorem 3.3. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A \in B(X)$ and assume that there is a closed subspace M of X such that $R(A) \cap M = \{0\}$ and $R(A) \oplus M$ is closed. Then $R(A)$ is closed.*

Proof. The space $X \times M$ under the norm

$$(\forall (x, y) \in X \times M)\ \|(x, y)\| = \max\{\|x\|, \|y\|\},$$

is a non-Archimedean Banach space. Consider the map $S : X \times M \to X$ defined by for all $x \in X,\ y \in M,\ S(x, y) = Ax + y$. Hence $S(X \times M) = R(A) \oplus M$ which is closed. From Lemma 2.21, $\gamma(S) > 0$ if and only if $R(A) \oplus M$ is closed. Since

$$\begin{aligned}\gamma(S) &= \inf_{(x,y) \notin N(S)} \frac{\|S(x, y)\|}{d((x, y), N(S))} \\ &> 0.\end{aligned}$$

Clearly, $N(S) = N(A) \times \{0\}$, if $x \notin N(A)$, then $(x, 0) \notin N(S)$, thus

$$\begin{aligned}d((x, 0), N(S))\gamma(S) &\leq \|S(x, 0)\| \\ &= \|Ax\|.\end{aligned}$$

Hence

$$\gamma(A) \geq \gamma(S)$$

$$> 0.$$

By Lemma 2.21, we conclude that $R(A)$ is closed. □

We have the following:

Definition 3.4. Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A \in B(X)$, A is said to be semi-regular if $R(A)$ is closed and one of the equivalent conditions of Theorem 2.15 holds.

We give some examples of semi-regular operators.

Example 3.5. (i) Surjective operators are semi-regulars.
(ii) Injective operator with closed range is semi-regular.

The following statements hold.

Theorem 3.6. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let A be semi-regular operator, then for all $n \in \mathbb{N}$, $(\gamma(A))^n \leq \gamma(A^n)$.*

Proof. As the classical setting. □

Proposition 3.7. *Let X be a non-Archimedean Banach space over $\mathbb{K}$ such that $\|X\| \subseteq |\mathbb{K}|$, let $A \in \mathcal{C}(X)$ and $\lambda, \mu \in \mathbb{K}$, we have:*

$$\gamma(\lambda I - A)\delta(N(\mu I - A), N(\lambda I - A)) \leq |\lambda - \mu|.$$

Proof. Let $\lambda, \mu \in \mathbb{K}$. If $\lambda = \mu$, the proof is trivial. Let $\lambda \neq \mu$, if $u \in N(\mu I - A)$, then $u \notin N(\lambda I - A)$ (because $N(\lambda I - A) \cap N(\mu I - A) = \{0\}$). Since

$$\begin{aligned} \gamma(\lambda I - A)d(u, N(\lambda I - A)) &\leq \|(A - \lambda I)u - (A - \mu I)u\| \\ &= |\lambda - \mu|\|u\|. \end{aligned}$$

Thus $\gamma(\lambda I - A)d(N(\mu I - A), N(\lambda I - A)) \leq |\lambda - \mu|$. □

Definition 3.8. Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A \in B(X)$. The analytical core of A is the set $K(A)$ of all $x \in X$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset X$ and a constant $\delta > 0$ such that:

(i) $x = u_0$ and for all $n \in \mathbb{N}^*$, $Au_{n+1} = u_n$;
(ii) For all $n \in \mathbb{N}$, $\|u_n\| \leq \delta^n \|x\|$.

Theorem 3.9. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A \in B(X)$. Then:*

(i) *$K(A)$ is a subspace of X.*
(ii) *$A(K(A)) = K(A)$.*
(iii) *$K(A) \subseteq C(A)$.*
(iv) *For all $n \in \mathbb{N}$, $K(A) \subseteq R(A^n)$.*
(v) *For all $\lambda \in \mathbb{K}\backslash\{0\}$, $N(\lambda I - A) \subseteq K(A)$.*

Proof. (i) Let $x, y \in K(A)$ and $\lambda \in \mathbb{K}$, we have $\lambda x \in K(A)$. Since $x \in K(A)$, then there is a sequence $(u_n)_{n\in\mathbb{N}} \subset X$ and a constant $\delta_1 > 0$ such that $x = u_0$ and for all $n \in \mathbb{N}^*$, $Au_{n+1} = u_n$ and $(\forall n \in \mathbb{N})\ \|u_n\| \leq \delta_1^n \|x\|$.
Since $y \in K(A)$, there exists $(v_n)_{n\in\mathbb{N}} \subset X$ and a constant $\delta_2 > 0$ such that $y = v_0$ and for all $n \in \mathbb{N}^*$, $Av_{n+1} = v_n$ and for all $n \in \mathbb{N}$, $\|v_n\| \leq \delta_2^n \|y\|$. Set: $\delta = \max\{\delta_1, \delta_2\}$. Then

$$\begin{aligned}\|u_n + v_n\| &\leq \max\{\|u_n\|, \|v_n\|\}\\ &\leq \max\{\delta_1^n\|x\|, \delta_2^n\|y\|\}\\ &\leq \delta^n \max\{\|x\|, \|y\|\}.\end{aligned}$$

If $x + y = 0$, there is nothing to demonstrate since $0 \in K(A)$. Assume that $x + y \neq 0$. Set

$$\mu = \frac{\max\{\|x\|, \|y\|\}}{\|x + y\|}.$$

It is easy to see that $\mu \geq 1$, then $\mu \leq \mu^n$. Consequently,

$$\begin{aligned}(\forall n \in \mathbb{N})\ \|u_n + v_n\| &\leq \delta^n \mu \|x + y\|\\ &\leq (\delta\mu)^n \|x + y\|.\end{aligned}$$

Hence (ii) of Definition 3.8 holds. Thus $x + y \in K(A)$, then $K(A)$ is a subspace of X.
(ii) As the classical setting.
(iii) By the definition of $C(A)$ and (ii).
(iv) Obvious.
(v) Let $x \in N(\lambda I - A)$, put $\delta = |\lambda|^{-1}$. For all $n \in \mathbb{N}$, $x_n = \lambda^{-n}x$ satisfies the definition of $K(A)$. □

3.1. **Question.** Are $C(A)$ and $K(A)$ closed?

Theorem 3.10. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A \in B(X)$, then*

(i) *If F is closed subspace of X such that $A(F) = F$, then $F \subseteq K(A)$.*
(ii) *If $C(A)$ is closed, then $C(A) = K(A)$.*

Proof. (i) Let $A_0 : F \to F$ denote the restriction of A on F. By assumption F is a non-Archimedean Banach space and $A(F) = F$, so, by the open mapping theorem, A_0 is open. This means that there exists a constant $\delta > 0$ such that for all $x \in F$, there is $u \in F$ such that $Au = x$ and $\|u\| \leq \delta\|x\|$. Now, if $x \in F$, set $u_0 = x$ and consider $u_1 \in F$ such that $Au_1 = u_0$ and

$\|u_1\| \leq \delta\|u_0\|$. By repeating this procedure, for all $n \in \mathbb{N}$, we can find $u_n \in F$ such that

$$Au_n = u_{n-1} \quad \text{and} \quad \|u_n\| \leq \delta\|u_{n-1}\|.$$

We conclude that for all $n \in \mathbb{N}$, $\|u_n\| \leq \delta^n\|x\|$, thus $x \in K(A)$. Consequently $F \subseteq K(A)$.

(ii) Suppose that $C(A)$ is closed. Since $A(C(A)) = C(A)$, from (i), we have $C(A) \subseteq K(A)$ and by (iii) of Theorem 3.9, we have $K(A) \subseteq C(A)$, hence $C(A) = K(A)$. □

Set $\alpha(A) = \dim N(A)$ (if it exists), we have the following definition.

Lemma 3.11. *Let X be a non-Archimedean Banach space over $\mathbb{K}$ and let $A, B \in B(X)$ such that $\|B\| \leq 1$. If $\alpha(A) = 0$, $R(A) = X$ and $|\lambda| < \gamma(A)$. Then,*

(i) $\alpha(A - \lambda B) = \alpha(A) = 0$.
(ii) $\gamma(A - \lambda B) \geq \gamma(A) - |\lambda|$.

Proof. (i) From $\alpha(A) = 0$ and $R(A)$ is surjective, thus A is invertible and A^{-1} exists. By Proposition 3.1, $\gamma(A) = \|A^{-1}\|^{-1}$. Since $|\lambda| < \gamma(A)$, then $|\lambda|\|B\|\|A^{-1}\| < 1$, hence $C = I - \lambda BA^{-1}$ is invertible, thus $CA = A - \lambda B$ is invertible. Then $N(A - \lambda B) = \{0\}$, hence $\alpha(A - \lambda B) = \alpha(A) = 0$.

(ii) Since

$$|\|Ax\| - |\lambda|\|Bx\|| \leq \|(A - \lambda B)x\|.$$

Consequently

$$\gamma(A) - |\lambda| \leq \gamma(A - \lambda B). \qquad \square$$

Let $r > 0$, put $B(0, r) = \{\lambda \in \mathbb{K} : |\lambda| < r\}$, we have the following theorems.

Theorem 3.12. *Suppose that $\mathbb{K}$ is spherically complete, let X be a non-Archimedean Banach space of countable type over $\mathbb{K}$, let $A, B \in B(X)$ such that $\|B\| \leq 1$ and $R(A) = X$. Suppose that there is M closed subspace of X such that $X = N(A) \oplus M$. Then there is $\rho > 0$ such that for all $\lambda \in B(0, \rho)$,*

(i) $R(A - \lambda B) = R(A)$.
(ii) $X = N(A - \lambda B) \oplus M$.

Proof. Let A_0 and B_0 be the restrictions of A and B to M. Thus A_0 is a one-to-one bounded linear operator from M into X, then $0 < \gamma(A_0) < \infty$. Moreover, B_0 is a bounded linear operator from M into X and $\|B_0\| \leq 1$. Consequently for $|\lambda| < \gamma(A_0)$. By Lemma 3.11, $\alpha(A_0 - \lambda B_0) = \alpha(A_0) = 0$ and $R(A_0 - \lambda B_0) = X$. Set $\rho = \gamma(A_0)$, then for all $\lambda \in B(0, \rho)$, we have:

(i)

$$\begin{aligned} X &= R(A_0 - \lambda B_0) \\ &\subset R(A - \lambda B) \\ &\subset X, \end{aligned}$$

thus $R(A - \lambda B) = X$. Consequently $R(A - \lambda B) = R(A)$.

(ii) Let $x \in X,\ (A-\lambda B)x \in X = R(A_0-\lambda B_0)$, then there is $u \in M$ such that

$$\begin{aligned} (A - \lambda B)x &= (A_0 - \lambda B_0)u \\ &= (A - \lambda B)u. \end{aligned}$$

Thus $x-u \in N(A-\lambda B)$ and $u \in M$. Hence $x = x-u+u$, then $X = N(A - \lambda B) + M$ and

$$\begin{aligned} N(A - \lambda B) \cap M &= N(A_0 - \lambda B_0) \\ &= \{0\}. \end{aligned}$$

We conclude that $X = N(A - \lambda B) \oplus M$. □

Theorem 3.13. *Let X, Y be two non-Archimedean Banach spaces over $\mathbb{K}$, let $A \in B(X, Y)$. If $R(A)$ is closed in Y, then there is a constant $M > 0$ such that, for given $x \in X$, there exists $y \in X$ such that $Ax = Ay$ and $\|y\| \leq M\|Ax\|$.*

Proof. Consider the operator $\hat{A} : X/N(A) \to R(A)$ by for all $x \in X$, $\hat{A}\hat{x} = Ax$. It is easy to see that $\hat{A}$ is well defined and one-to-one and continuous linear operator from $\hat{X}$ into $R(A)$. By the open mapping theorem, there is a $M_1 > 0$ such that for all $x \in X$,

$$\begin{aligned} \|\hat{x}\| &= \|x + N(A)\| \\ &\leq M_1\|Ax\|. \end{aligned}$$

Let $x \in X$, if $Ax \neq 0$, then there exists $u \in N(A)$ such that

$$\begin{aligned} \|x + u\| &\leq \|x + N(A)\| + \|Ax\| \\ &\leq M_1\|Ax\| + \|Ax\| \\ &= (M_1 + 1)\|Ax\|. \end{aligned}$$

Set $M = M_1 + 1$, in this case, we put $y = x + u$, then $\|y\| \leq M\|Ax\|$. If $Ax = 0$, hence we take $y = 0$, therefore $\|y\| \leq M\|Ax\|$. So for $x \in X$ given there exists $y \in X$ such that $Ax = Ay$ and $\|y\| \leq M\|Ax\|$. □

We have the following definition.

Definition 3.14. Let X, Y be a non-Archimedean Banach spaces over $\mathbb{K}$ and let $A \in B(X, Y)$. The injectivity modulus of A is defined by

$$j(A) = \inf_{x \in X \setminus \{0\}} \frac{\|Ax\|}{\|x\|}.$$

The surjectivity modulus of A is given by

$$k(A) = \sup\{r \geq 0 : rB_Y \subset A(B_X)\},$$

where $B_X = \{x \in X : \|x\| < 1\}$ and $B_Y = \{y \in Y : \|y\| < 1\}$.

Remark 3.15. Obviously, if $A \in B(X)$, then $j(A) \leq \|A\|$.

The following statements hold.

Proposition 3.16. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A, B \in B(X)$. We have:*

(i) $j(AB) \leq \|A\| j(B)$.
(ii) *If A is invertible, then* $\gamma(A) j(A^{-1}B) \leq j(B)$.
(iii) $j(A + B) \leq \max\{j(A), j(B)\}$.
(iv) *For all* $n \in \mathbb{N}$, $j(nA) \leq j(A)$.
(v) *For all* $n \geq 1$, $j(A^n) \leq \|A^{n-1}\| j(A)$.

Proof. Since $A, B \in B(X)$. Then

(i)

$$\begin{aligned} j(AB) &= \inf_{x \in X \setminus \{0\}} \frac{\|ABx\|}{\|x\|} \\ &\leq \|A\| \inf_{x \in X \setminus \{0\}} \frac{\|Bx\|}{\|x\|} \\ &= \|A\| j(B). \end{aligned}$$

(ii) Since A is invertible. Then by Proposition 3.1, $\gamma(A) = \|A^{-1}\|^{-1}$. Thus

$$\begin{aligned} j(A^{-1}B) &= \inf_{x \in X \setminus \{0\}} \frac{\|A^{-1}Bx\|}{\|x\|} \\ &\leq \|A^{-1}\| \inf_{x \in X \setminus \{0\}} \frac{\|Bx\|}{\|x\|} \\ &= \|A^{-1}\| j(B). \end{aligned}$$

Hence

$$\begin{aligned} \gamma(A) j(A^{-1}B) &= \|A^{-1}\|^{-1} j(A^{-1}B) \\ &\leq j(B). \end{aligned}$$

(iii) We have:

$$\begin{aligned} j(A+B) &= \inf_{x\in X\setminus\{0\}} \frac{\|(A+B)x\|}{\|x\|} \\ &\leq \inf_{x\in X\setminus\{0\}} \max\left\{\frac{\|Ax\|}{\|x\|}, \frac{\|Bx\|}{\|x\|}\right\} \\ &\leq \max\left\{\inf_{x\in X\setminus\{0\}} \frac{\|Ax\|}{\|x\|}, \inf_{x\in X\setminus\{0\}} \frac{\|Bx\|}{\|x\|}\right\} \\ &= \max\left\{j(A), j(B)\right\}. \end{aligned}$$

(iv) Since for all $n \in \mathbb{N},\ |n| \leq 1$, then $j(nA) = |n|j(A) \leq j(A)$.

(v) For all $n \geq 1$, we have

$$\begin{aligned} j(A^n) &= \inf_{x\in X\setminus\{0\}} \frac{\|A^n x\|}{\|x\|} \\ &= \inf_{x\in X\setminus\{0\}} \frac{\|A^{n-1}Ax\|}{\|x\|} \\ &\leq \|A^{n-1}\| \inf_{x\in X\setminus\{0\}} \frac{\|Ax\|}{\|x\|} \\ &= \|A^{n-1}\| j(A). \end{aligned}$$

□

Definition 3.17. Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A \in B(X)$, A is said to be bounded below if for all $x \in X,\ M\|x\| \leq \|Ax\|$ for some $M > 0$.

We have the following results.

Lemma 3.18. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A \in B(X)$. Then A is bounded below if and only if A is injective and $R(A)$ is closed.*

Proof. Suppose that A is bounded below, then for all $x \in X,\ M\|x\| \leq \|Ax\|$ for some $M > 0$. If $\|Ax\| = 0$, then $x = 0$, hence A is injective. Let $(x_n)_{n\in\mathbb{N}} \subset X$ such that $\lim\limits_{n\to\infty} Ax_n = y$. Then for each $m, n \in \mathbb{N},\ M\|x_n - x_m\| \leq \|Ax_n - Ax_m\|$. Hence $(x_n)_{n\in\mathbb{N}}$ is Cauchy sequence in X. Since X is complete, we conclude that there is $x \in X$ such that $\lim\limits_{n\to\infty} x_n = x$. So $\lim\limits_{n\to\infty} Ax_n = Ax = y$ i.e., $y \in R(A)$. Thus $R(A)$ is closed. The converse is clear. □

Proposition 3.19. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A \in B(X)$ be bijective, then $j(A) = \|A^{-1}\|^{-1}$.*

Proof. Since A is bijective and set $y = Ax$, we have

$$j(A) = \inf_{x\in X\setminus\{0\}} \frac{\|Ax\|}{\|x\|}$$

$$
\begin{aligned}
&= \inf_{x\in X\setminus\{0\}} \left(\frac{1}{\frac{\|x\|}{\|Ax\|}} \right) \\
&= \left(\sup_{x\in X\setminus\{0\}} \frac{\|x\|}{\|Ax\|} \right)^{-1} \\
&= \left(\sup_{y\in X\setminus\{0\}} \frac{\|A^{-1}y\|}{\|y\|} \right)^{-1} \\
&= \|A^{-1}\|^{-1}. \qquad \square
\end{aligned}
$$

Proposition 3.20. *Let X be a non-Archimedean Banach space over $\mathbb{K}$ such that $\|X\| \subseteq |\mathbb{K}|$. If $A, B \in B(X)$, then $|j(A) - j(B)| \le \|A - B\|$.*

Proof. Let $x \in X$ such that $\|x\| = 1$, then

$$
\begin{aligned}
|\|Ax\| - \|Bx\|| &\le \|Ax - Bx\| \\
&\le \|A - B\|.
\end{aligned}
$$

Thus

$$j(A) - j(B) \le \|A - B\|.$$

Moreover

$$j(B) - j(A) \le \|A - B\|.$$

Consequently $|j(A) - j(B)| \le \|A - B\|$. $\square$

We finish with the following theorem.

Theorem 3.21. *Suppose that $\mathbb{K}$ is spherically complete. Let X be a non-Archimedean Banach space over $\mathbb{K}$ such that $\|X\| \subseteq |\mathbb{K}|$ and let $A \in B(X)$. Then*

$$j(A) = \sup\{r > 0 : A - B \ \ \textit{bounded below for all}\ B \in B(X),\ \|B\| < r\}.$$

Proof. Let $A, B \in B(X)$ such that $\|B\| < j(A)$. It is easy to see that

$$
\begin{aligned}
j(A - B) &\ge j(A) - j(B) \\
&\ge j(A) - \|B\| \\
&> 0.
\end{aligned}
$$

So, we conclude that $A - B$ is bounded below. Conversely, let $\varepsilon > 0$ there is a $x_0 \in X\setminus\{0\}$ such that $\|x_0\| = 1$ and $\|Ax_0\| < \varepsilon + j(A)$. By Theorem 2.9, there is $f \in X'$ such that $f(x_0) = 1$ and $\|f\| = 1$. Consider the operator B defined on X by for all $x \in X$, $Bx = f(x)Ax_0$. Then $Bx_0 = f(x_0)Ax_0 = Ax_0$ i.e., $(B - A)x_0 = 0$. Hence $B - A$ is not one to one. Thus $A - B$ is not bounded below. $\square$

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Department of Mathematics and Computer Science, Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar El Mahraz, B.P. 1796 Atlas, Fès, Morocco.

Email address: jawad.ettayb@usmba.ac.ma