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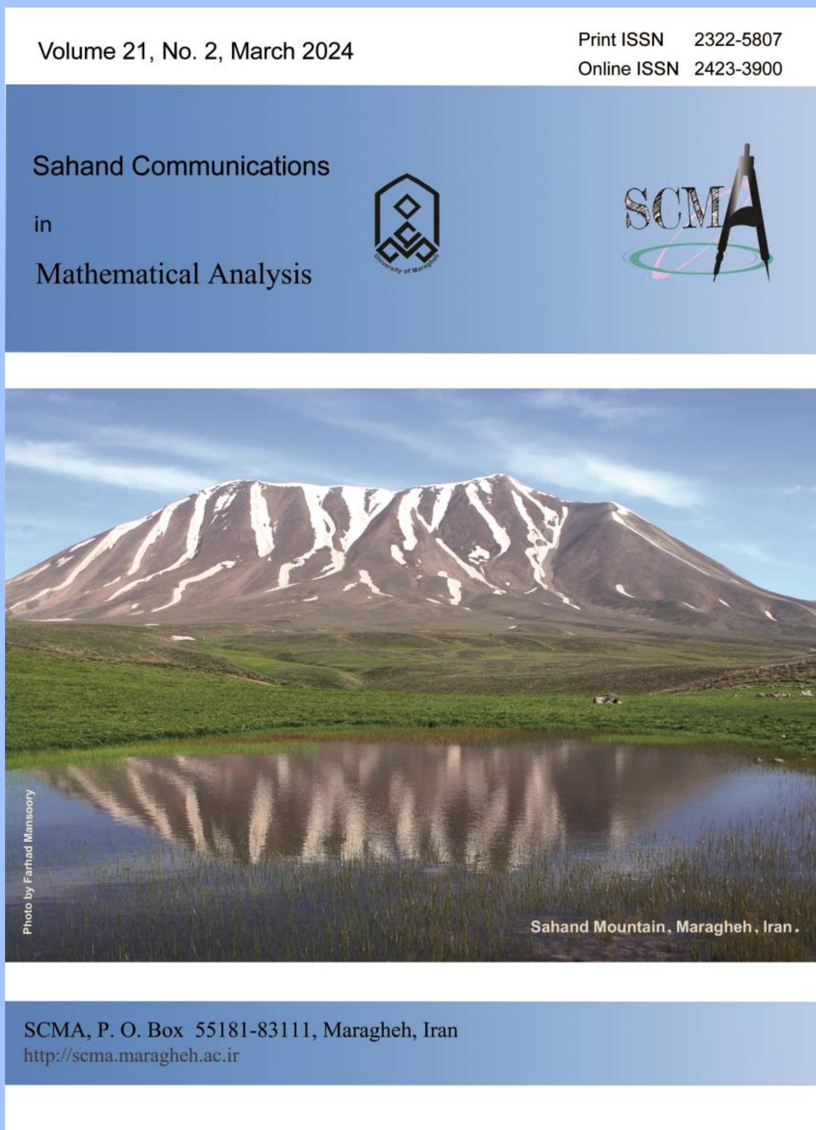

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Some Properties of Close-To-Convex Functions Associated with A Strip Domain

Kadhavoor Ragavan Karthikeyan[1*], Seetharam Varadharajan[2] and Sakkarai Lakshmi[3]

Abstract. Using subordination, we introduce a new class of symmetric functions associated with a vertical strip domain. We have provided some interesting deviations or adaptation which are helpful in unification and extension of various studies of analytic functions. Inclusion relations, geometrical interpretation, coefficient estimates, inverse function coefficient estimates and solution to the Fekete-Szegő problem of the defined class are our main results. Applications of our main results are given as corollaries.

1. Introduction

The function $S_{\theta,\vartheta}(z)$ which plays a central role in the study of functions associated with the vertical domain, is given by

$$S_{\theta,\vartheta}(z) = 1 + \frac{\vartheta - \theta}{\pi} i \log\left(\frac{1 - e^{\frac{2\pi i(1-\theta)}{\vartheta-\theta}} z}{1 - z}\right), \tag{1.1}$$

where $z \in \Delta = \{z \in \mathbb{C} : |z| < 1\}$, θ and ϑ are real numbers with $\theta < 1$ and $\vartheta > 1$. Note that function $S_{\theta,\vartheta}(z)$ is analytic and univalent in Δ with $S_{\theta,\vartheta}(0) = 1$. Precisely, $S_{\theta,\vartheta}(z)$ maps the unit disc on to a vertical strip domain $\theta < \Re\{\omega\} < \vartheta$ (see Figure 1a). Figure 1b is the mapping of annular region under the transformation $S_{\theta,\vartheta}(z)$. Further, we note that the function $S_{\theta,\vartheta}(z)$ defined by (1.1) has a power series of the form

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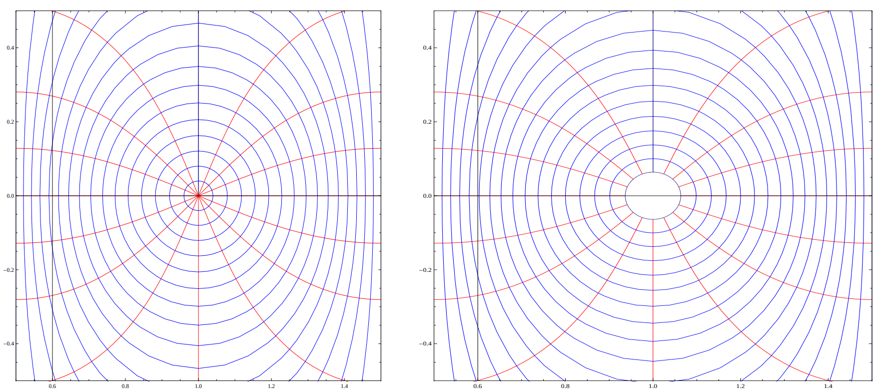
(see [27])

$$S_{\theta,\vartheta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, \tag{1.2}$$

where

$$B_n = \frac{\vartheta - \theta}{n\pi} i \left(1 - e^{\frac{2n\pi i(1-\theta)}{\vartheta - \theta}}\right), \quad n \in \mathbb{N} = \{1, 2, \ldots\}. \tag{1.3}$$

The study by Kuroki and Owa in [27] brought a renewed interest to study the class of functions associated with vertical domain. Recently several authors have defined various subclasses of analytic functions related to a vertical domain, see [12, 18, 22, 28, 30, 32, 40, 41, 43–45].



(A) $|z| < 1$ (B) if $0 <| z |< 1$

FIGURE 1. Images of $S_{\frac{1}{2},\frac{3}{2}}(z)$

Let $\mathcal{P}$ denote the family consisting of functions having a series representation of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

analytic in the unit disc and satisfying the condition $\operatorname{Re}(p(z)) > 0$. Li et al. [30] introduced and studied a class $P(L, M; V, W)$ of functions unifying the well-known Janowski class and the class associated with vertical domain, which is defined as follows.

Definition 1.1 ([30])**.** Let $-1 \leq M < L \leq 1, V \neq W$ and $-1 \leq W \leq 1$. Then the analytic function $p(z) \in P(L, M; V, W)$ if and only if $p(z)$ satisfies each of the following two subordination relationships:

$$p(z) \prec h_1(z) = \frac{1 + Lz}{1 + Mz}, \tag{1.4}$$

and

$$p(z) \prec h_2(z) = \frac{1+Vz}{1+Wz}. \tag{1.5}$$

For $L = 1-2\theta, (0 \leq \theta < 1), M = -1, V = 1-2\vartheta(\vartheta > 1)$ and $W = -1$ in $P(L, M; V, W)$, we obtain the following relationship:

$$p(z) \in P(\theta, \vartheta) = P(1-2\theta, -1; 1-2\vartheta, -1) \;\Leftrightarrow\; \theta < \Re\{p(z)\} < \vartheta. \tag{1.6}$$

From (1.1) and (1.6), we have

$$p(z) \in P(\theta, \vartheta) \quad \Leftrightarrow \quad p(z) \prec S_{\theta,\vartheta}(z). \tag{1.7}$$

Further from Definition 1.1, Li et al. [30] introduced the following subclasses of $P(L, M; V, W)$.

Definition 1.2 ([30])**.** Let $p(z) = 1 + \sum\limits_{n=1}^{\infty} c_n z^n \in \mathcal{P}$ and

$$\begin{aligned} \tilde{P}(\rho_1) &= \{p(z) \in \mathcal{P} : \Re(p(z)) > \rho_1\}, \\ \tilde{P}(\rho_2) &= \{p(z) \in \mathcal{P} : \Re(p(z)) < \rho_2\}, \\ \tilde{P}(\rho_1, \rho_2) &= \{p(z) \in \mathcal{P} : \rho_1 < \Re(p(z)) < \rho_2\}, \end{aligned}$$

and

$$\tilde{P}(\rho_3, \rho_4) = \{p(z) \in \mathcal{P} : \rho_3 < \Re\{p(z)\}, \Re\{2 - p(z)\} < 1 + \rho_4\},$$

where

(1.8)

$$\begin{cases} \rho_1 = \max\left\{\frac{1-L}{1-M}, \frac{1+V}{1+W}\right\}, & -1 < M < L \leq 1, -1 < V < W < 1, \\ \rho_2 = \min\left\{\frac{1+L}{1+M}, \frac{1-V}{1-W}\right\}, & -1 < M < L \leq 1, -1 < V < W < 1, \\ \rho_3 = \left\{\frac{1-L}{2}\right\}, & M = -1, \\ \rho_4 = \left\{\frac{1-V}{2}\right\}, & W = 1. \end{cases}$$

Here, we let Π to denote the class of functions $\chi(z)$ normalized by

$$\chi(z) = z + \sum_{n=2}^{\infty} \ell_n z^n \tag{1.9}$$

which are analytic in the open unit disc Δ. Also, let $\mathcal{S}$ denote the subclass of Π consisting of all functions which are univalent in Δ (see [14]). Also, let $\mathcal{S}^*(\theta)$ denote the class of starlike functions of order θ, $(0 \leq \theta < 1)$.

Koebe 1/4 theorem states that every function $\chi \in \mathcal{S}$ of the form (1.9) has an inverse χ^{-1}, defined by $\chi^{-1}(\chi(z)) = z (z \in \Delta)$ and $\chi^{-1}(\chi(\omega)) = \omega\left(|\omega| < r; r \geq \frac{1}{4}\right)$, where

$$\chi^{-1}(\omega) = \omega - \ell_2\omega^2 + (2\ell_2 - \ell_3)\,\omega^3 - \left(5\ell_2^2 - 5\ell_2\ell_3 + \ell_4\right)\omega^4 + \cdots. \tag{1.10}$$

Here, we study a new family of analytic function associated with vertical domain using q-difference operator. Formally, the q-derivative is defined by

$$D_q\chi(z) = \frac{\chi(qz) - \chi(z)}{(q-1)z}.$$

As $\lim_{q\to 1^-}$, $D_q\chi(z)$ reduces to the classical derivative. We denote

$$[n]_q = \sum_{k=1}^{n} q^{k-1}, \qquad [0]_q = 0, \quad (q \in \mathbb{C}),$$

and the q-*shifted factorial* by

$$(\eta; q)_n = \begin{cases} 1, & n = 0, \\ (1-\eta)(1-\eta q)\dots\left(1-\eta q^{n-1}\right), & n = 1, 2, \dots. \end{cases}$$

Using Hadamard product, Reddy et al. [36] defined the following q-differential operator $\mathcal{J}_\lambda^m(\eta_1, \nu_1;\, q,\, z)\chi : \Delta \to \Delta$ given by

$$\mathcal{J}_\lambda^m(\eta_1, \nu_1;\, q,\, z)\chi = z + \sum_{n=2}^{\infty} [1 - \lambda + \lambda[n]_q]^m \; \Gamma_n \ell_n z^n, \tag{1.11}$$

where $m \in N_0 = N \cup \{0\}$ and $\lambda \geq 0$, and

$$\Gamma_n = \frac{(\eta_1; q)_{n-1}(\eta_2; q)_{n-1}\dots(\eta_r; q)_{n-1}}{(q; q)_{n-1}(\nu_1; q)_{n-1}\dots(\nu_s; q)_{n-1}}, \quad (|q| < 1)\,.$$

Remark 1.3. For details pertaining with the operator $\mathcal{J}_\lambda^m(\eta_1, \nu_1;\, q,\, z)\chi$, refer to [21, 24, 34, 36].

Motivated by Breaz et al. [9, 10], Karthikeyan et al. [19, 20], Prajapat [35] and Wang and Chen [48], we now define the following definition.

Definition 1.4. Let $m \in \mathbb{N}_0, 0 \leq \lambda, -1 \leq M < L \leq 1, -1 < V < W \leq 1$, and $\chi(z) \in \Pi$. Then, the function $\chi(z) \in S_m^{\lambda,t}(\eta_1, \nu_1, q, L, M; V, W)$ if and only if $\chi(z)$ satisfies the following condition:

$$\frac{tz^2\left[\mathcal{J}_\lambda^m(\eta_1, \nu_1; q, z)\chi(z)\right]'}{\left[\mathcal{J}_\lambda^m(\eta_1, \nu_1; q, z)\psi(z)\right]\left[\mathcal{J}_\lambda^m(\eta_1, \nu_1; q, z)\psi(tz)\right]} \in P(L, M; V, W), \tag{1.12}$$

where $\mathcal{J}_\lambda^m(\eta_1, \nu_1; q, z)\psi(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(1/2)$.

If we let $L = 1 - 2\theta(0 \leq \theta < 1), M = -1, V = 1 - 2\vartheta(\vartheta > 1)$, $W = -1$ $m = 0$, $r = 2$, $s = 1$, $\eta_1 = \nu_1$, $\nu_2 = q$, $q \to 1^-$ and $t = -1$ in the Definition 1.4, then the function class $S_m^{\lambda,t}(\eta_1, \nu_1, q, L, M; V, W)$ reduces to

$$\mathcal{K}_s(\theta,\vartheta)=\left\{\chi\in\Pi,\ \psi\in\mathcal{S}^*(1/2);\ \theta<\frac{z^2\chi'(z)}{-\psi(z)\psi(-z)}<\vartheta\right\}.$$

Further letting $\theta=0$ and $\vartheta>1$, the class $\mathcal{K}_s(\theta,\vartheta)$ will reduce to the family that would be closely related to class recently studied by Cho et al. [13].

2. Prelimanries

To prove the main results in the paper, we need the following lemmas.

Lemma 2.1 ([30]). *The function $p(z)\in P(L,M;V,W)$ if and only if $p(z)$ satisfies each of the following two conditions:*

(2.1)
$$\begin{cases} |p(z)-\sigma_i|<r_i, & i=1,2,\ -1<M<L\le 1,\ -1<V<W<1,\\ \rho_3<\Re\{p(z)\}, & M=-1,\ \Re\{2-p(z)\}<1+\rho_4,\ W=1,\end{cases}$$

where

$$\begin{cases} \sigma_1=\frac{1-LM}{1-M^2}, & \text{and} \quad r_1=\frac{L-M}{1-M^2},\\ \sigma_2=\frac{1-VW}{1-W^2}, & \text{and} \quad r_2=\frac{W-V}{1-W^2},\end{cases} \tag{2.2}$$

and ρ_3,ρ_4 are given by (1.8).

Lemma 2.2 ([30]). *Let $j=1,2,3,4;-1<M<L\le 1$ and $-1<V<W<1; S_{\theta,\vartheta}(z)$ is defined by (1.1). If $p(z)\in P(L,M;V,W)$, then $p(z)\prec p_j(z)$ with*

(2.3)

$$p_j(z)$$
$$=\begin{cases} p_1(z)=S_{\frac{1-L}{1-M},\frac{1-V}{1-W}}(z), & MV-LW\ge|L-M+V-W|, \quad j=1,\\ p_2(z)=S_{\frac{1+V}{1+W},\frac{1+L}{1+M}}(z), & LW-MV\ge|L-M+V-W|, \quad j=2,\\ p_3(z)=S_{\frac{1-L}{1-M},\frac{1+L}{1+M}}(z), & |LW-MV|\le M-L+W-V, \quad j=3,\\ p_4(z)=S_{\frac{1+V}{1+W},\frac{1-V}{1-W}}(z), & |LW-MV|\le L-M+V-W, \quad j=4,\end{cases}$$

where $p_j(0) = 1$ *and*

$$p_j(z) = \begin{cases} p_1(z) = 1 + \sum\limits_{n=1}^{\infty} B_{n,1} z^n, & j = 1, \\ p_2(z) = 1 + \sum\limits_{n=1}^{\infty} B_{n,2} z^n, & j = 2, \\ p_3(z) = 1 + \sum\limits_{n=1}^{\infty} B_{n,3} z^n, & j = 3, \\ p_4(z) = 1 + \sum\limits_{n=1}^{\infty} B_{n,4} z^n, & j = 4, \end{cases} \tag{2.4}$$

for

$$B_{n,j} = \begin{cases} B_{n,1} = \frac{\frac{1-V}{1-W} - \frac{1-L}{1-M}}{n\pi} i \left(1 - e^{2n\pi i \left(1 - \frac{1-L}{1-M}\right)/\left(\frac{1-V}{1-W} - \frac{1-L}{1-M}\right)}\right), j = 1, \\ B_{n,2} = \frac{\frac{1+L}{1+M} - \frac{1+V}{1+W}}{n\pi} i \left(1 - e^{2n\pi i \left(1 - \frac{1+V}{1+W}\right)/\left(\frac{1+L}{1+M} - \frac{1+V}{1+W}\right)}\right), j = 2, \\ B_{n,3} = \frac{\frac{1+L}{1+M} - \frac{1-L}{1-M}}{n\pi} i \left(1 - e^{2n\pi i \left(1 - \frac{1-L}{1-M}\right)/\left(\frac{1+L}{1+M} - \frac{1-L}{1-M}\right)}\right), j = 3, \\ B_{n,4} = \frac{\frac{1-V}{1-W} - \frac{1+V}{1+W}}{n\pi} i \left(1 - e^{2n\pi i \left(1 - \frac{1+V}{1+W}\right)/\left(\frac{1-V}{1-W} - \frac{1+V}{1+W}\right)}\right), j = 4. \end{cases} \tag{2.5}$$

Lemma 2.3 ([37])**.** *Let* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *be analytic and univalent in* Δ*, and suppose that* $p(z)$ *maps* Δ *onto a convex domain. If* $q(z) = 1 + q_1 z + q_2 z^2 + \cdots$ *is analytic in* Δ *and satisfies the following subordination:*

$$q(z) \prec p(z), \quad z \in \Delta,$$

then

$$|q_n| \leq |c_1|, \quad n = 1, 2, \ldots.$$

Using Lemma 2.2 and the definition of subordination, we can obtain the following lemma.

Lemma 2.4 ([30])**.** *Let* $-1 \leq M < L \leq 1, -1 < V < W \leq 1, i = 1, 2; j = 1, 2, 3, 4$ *and* $\tilde{P}(\rho_1), \tilde{P}(\rho_2), \tilde{P}(\rho_1, \rho_2)$ *and* $\tilde{P}(\rho_3, \rho_4)$ *are given by Definition 1.2. If* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in P(L, M; V, W)$*, then*

$$|c_n| \leq \phi(\delta_i; \rho_j) = \begin{cases} 2\delta_1, & p \in \tilde{P}(\rho_1), \\ 2\delta_2, & p \in \tilde{P}(\rho_2), \\ 2\min\{\delta_1, \delta_2\}, & p \in \tilde{P}(\rho_1, \rho_2), \\ 2\min\left\{\frac{1+L}{2}, \frac{1-V}{2}\right\}, & p \in \tilde{P}(\rho_3, \rho_4), \end{cases} \tag{2.6}$$

where

$$\begin{cases} \delta_1 = \min\left\{\dfrac{L-M}{1-M}, \dfrac{W-V}{1+W}\right\}, \\ \delta_2 = \min\left\{\dfrac{L-M}{1+M}, \dfrac{W-V}{1-W}\right\}, \end{cases} \tag{2.7}$$

and ρ_j are given by (1.8).

Lemma 2.5 ([20, 42])**.** *Let $\psi(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$ and $0 < |t| \leq 1$, then $\frac{\psi(z)\psi(tz)}{tz} \in \mathcal{S}^*$.*

Lemma 2.6 ([20, 42])**.** *Let $\mathcal{J}_\lambda^m\left(\eta_1, \nu_1; q, z\right)\psi \in \mathcal{S}^*\left(\frac{1}{2}\right)$, then*

$$
\begin{aligned}
G(z) &= \frac{\left[\mathcal{J}_\lambda^m\left(\eta_1, \nu_1; q, z\right)\psi(z)\right]\left[\mathcal{J}_\lambda^m\left(\eta_1, \nu_1; q, z\right)\psi(tz)\right]}{tz} \\
&= z + \sum_{n=2}^{\infty} d_n z^n \in \mathcal{S}^*, \qquad \text{and} \qquad |d_n| \leq n,
\end{aligned} \tag{2.8}
$$

where

$$
\begin{aligned}
d_n &= \left[\Psi_n b_n + \Psi_{n-1}\Psi_2 b_{n-1} b_2 t + \cdots + \Psi_n b_n t^{n-1}\right], \\
\Psi_n &= \left[1 - \lambda + [n]_q \lambda\right]^m \Gamma_n.
\end{aligned}
$$

3. Coefficient Inequality

Theorem 3.1. *Let $m \in \mathbb{N}_0, \lambda \geq 0, |\ell_1| = 1$ and the function $\chi(z)$ be given by (1.9). If $\chi(z) \in S_m^{\lambda,t}(\eta_1, \nu_1, q, L, M; V, W)$, then*

$$
\begin{aligned}
|\ell_n| &\leq M_{n,j}(m, \lambda) \\
&= \begin{cases} \frac{|B_{1,j}|}{2\Gamma_2[1-\lambda+[2]_q\lambda]^m}, & n = 2, \\ \frac{|B_{1,j}|}{n\Gamma_n[1-\lambda+[n]_q\lambda]^m} \prod_{k=2}^n \left(1 + \frac{|B_{1,j}|}{k\Gamma_k[1-\lambda+[k]_q\lambda]^m}\right), & n \geq 3, \end{cases}
\end{aligned} \tag{3.1}
$$

where $|B_{1,j}|\,(j = 1, 2, 3, 4)$ are defined by (2.5).

Proof. From the Definition 1.4, we have

$$
\frac{tz^2\left[\mathcal{J}_\lambda^m(\eta_1, \nu_1; q, z)\chi(z)\right]'}{\left[\mathcal{J}_\lambda^m(\eta_1, \nu_1; q, z)\psi(z)\right]\left[\mathcal{J}_\lambda^m(\eta_1, \nu_1; q, z)\psi(tz)\right]} \in h_1(\Delta), \tag{3.2}
$$

and

$$
\frac{tz^2\left[\mathcal{J}_\lambda^m(\eta_1, \nu_1; q, z)\chi(z)\right]'}{\left[\mathcal{J}_\lambda^m(\eta_1, \nu_1; q, z)\psi(z)\right]\left[\mathcal{J}_\lambda^m(\eta_1, \nu_1; q, z)\psi(tz)\right]} \in h_2(\Delta), \tag{3.3}
$$

where $h_1(z)$ and $h_2(z)$ are given by (1.4) and (1.5), respectively.

Note that (3.2) and (3.3) implies there exist a $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in P(L, M, V, W)$ such that

$$
\frac{tz^2\left[\mathcal{J}_\lambda^m(\eta_1, \nu_1; q, z)\chi(z)\right]'}{\left[\mathcal{J}_\lambda^m(\eta_1, \nu_1; q, z)\psi(z)\right]\left[\mathcal{J}_\lambda^m(\eta_1, \nu_1; q, z)\psi(tz)\right]} = p(z),
$$

or, equivalently,

$$
z\left[\mathcal{J}_\lambda^m(\eta_1, \nu_1; q, z)\chi(z)\right]' = p(z)G(z), \tag{3.4}
$$

where $G(z)$ is defined as in (2.8). Using the respective power series expansion in (3.4) and equating the coefficients of z^n on both sides, we have

$$n\Gamma_n\left[1-\lambda+[n]_q\lambda\right]^m \ell_n = d_n + c_1 d_{n-1} + c_2 d_{n-2} + \cdots + c_{n-1}.$$

Using Lemma 2.2 and Lemma 2.3 in the above equation, we obtain

$$\begin{aligned}|\ell_n| &\le \frac{1}{n\Gamma_n\left[1-\lambda+[n]_q\lambda\right]^m}\left(|c_{n-1}| + |c_{n-2}|\,|d_2| + \cdots + |c_1|\,|d_{n-1}|\right)\\ &\le \frac{|B_{1,j}|}{n\Gamma_n\left[1-\lambda+[n]_q\lambda\right]^m}\sum_{k=1}^{n-1}|d_k|, \quad (d_1 = 1).\end{aligned}$$

Hence, we have $|\ell_2| \le M_{2,j}(m,\lambda)$. To prove the assertion, we need to establish

$$\sum_{k=1}^{n}|d_k| \le \prod_{k=2}^{n}\left(1+\frac{|B_{1,j}|}{k\Gamma_k\left[1-\lambda+[k]_q\lambda\right]^m}\right), \tag{3.5}$$

for $n = 3,4,5,\ldots$. Assume that the inequality (3.5) holds for $n = p$. Consider

$$\begin{aligned}&\sum_{k=1}^{p+1}|d_k|\\ &= \sum_{k=1}^{p}|d_k| + |d_{p+1}| \le \sum_{k=1}^{p}|d_k| + \frac{|B_{1,j}|}{(p+1)\Gamma_{p+1}\left[1-\lambda+[p+1]_q\lambda\right]^m}\sum_{k=1}^{p}|d_k|\\ &= \left(1+\frac{|B_{1,j}|}{(p+1)\Gamma_{p+1}\left[1-\lambda+[p+1]_q\lambda\right]^m}\right)\sum_{k=1}^{p}|d_k|\\ &\le \left(1+\frac{|B_{1,j}|}{(p+1)\Gamma_{p+1}\left[1-\lambda+[p+1]_q\lambda\right]^m}\right)\prod_{k=2}^{p}\left(1+\frac{|B_{1,j}|}{k\Gamma_k\left[1-\lambda+[k]_q\lambda\right]^m}\right)\\ &= \prod_{k=2}^{p+1}\left(1+\frac{|B_{1,j}|}{k\Gamma_k\left[1-\lambda+[k]_q\lambda\right]^m}\right),\end{aligned}$$

which implies that the inequality (3.5) holds for $n = p+1$. Hence by induction, the desired estimate for $|\ell_n|$ ($n \ge 3$) follows, as asserted in (3.1). This completes the proof of Theorem 3.1. □

Letting $L = 1-2\theta(0 \le \theta < 1), M = -1, V = 1-2\vartheta(\vartheta > 1)$, $W = -1$ $m = 0$, $r = 2$, $s = 1$, $\eta_1 = \nu_1$, $\nu_2 = q$, $q \to 1^-$ and $t = -1$ in Theorem 3.1, we have the following result.

Corollary 3.2 ([11])**.** *Let χ belong to $\mathcal{K}_s(\theta,\vartheta)$, then*

$$|\ell_{2n}| \le \frac{(\vartheta-\theta)}{\pi}\sin\frac{\pi(1-\theta)}{\vartheta-\theta}, \quad (n \in \mathbb{N}),$$

and

$$|\ell_{2n+1}| \leq \frac{1 + \frac{2(\vartheta-\theta)n}{\pi} \sin \frac{\pi(1-\theta)}{\vartheta-\theta}}{(2n+1)}, \quad (n \in \mathbb{N}).$$

In Corollary 3.2, letting $\vartheta = \infty$ we get

Corollary 3.3. *If $\chi \in \mathcal{K}_s(\theta)$, then*

$$|\ell_{2n}| \leq 1 - \theta, \quad (n \in \mathbb{N})$$

and

$$|\ell_{2n+1}| \leq \frac{1 + 2(1-\theta)}{2n+1}, \quad (n \in \mathbb{N}).$$

In Corollary 3.3, letting $\theta = 0$ $\vartheta = \infty$ we have.

Corollary 3.4 ([16])**.** *If $\chi \in \mathcal{K}_s$, then*

$$|\ell_n| \leq 1, \quad (n = 2, 3, \ldots).$$

4. Fekete-Szegő Inequality of Functions in $S_m^{\lambda,t}(\eta_1, \nu_1, q, L, M; V, W)$

We need the following well-known results.

Lemma 4.1 ([23])**.** *If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is a function with positive real part, then for each complex number μ*

$$\left|p_2 - \mu p_1^2\right| \leq 2 \max(1, |2\mu - 1|), \tag{4.1}$$

and the result is sharp for the functions given by $p(z) = \frac{1+z^2}{1-z^2}, p(z) = \frac{1+z}{1-z}$.

Lemma 4.2 ([25])**.** *If*

$$G(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{S}^*,$$

then for each complex number λ we have $\left|c_3 - \lambda c_2^2\right| \leq \max(1, |3 - 4\lambda|)$ and the result is sharp for the Koebe functon

$$k(z) = \frac{z}{(1-z)^2}, \quad \text{if} \quad \left|\lambda - \frac{3}{4}\right| \geq \frac{1}{4},$$

and for

$$k^{\frac{1}{2}}\left(z^2\right) = \frac{z}{1-z^2}, \quad \text{if} \quad \left|\lambda - \frac{3}{4}\right| \leq \frac{1}{4}.$$

Lemma 4.3. *A function $\psi \in \Pi$ is said to be convex function in Δ if both $\psi(z)$ and $\psi^{-1}(z)$ are convex in Δ, Nehari [33], subsequently by Koepf [26] and Ian Graham and Gabriela Kohr [17], Corollary 2.2.19 gives*

$$\left|b_3 - b_2^2\right| \leq \frac{1}{3}. \tag{4.2}$$

This estimate is sharp.

Theorem 4.4. *Let* $m \in \mathbb{N} \cup \{0\}, \lambda \geq 0, -1 < M < L \leq 1, -1 < V < W < 1, 0 \leq \mu \leq 1$ *and* $p_j(z) = 1 + \sum\limits_{n=1}^{\infty} B_{n,j} z^n (j = 1,2,3,4)$*. If* $\chi(z) \in S_m^{\lambda,t}(\eta_1, \nu_1, q, L, M; V, W)$*, then for each complex number* μ *we have*

(4.3)

$$\left|\ell_3 - \mu \ell_2^2\right| \leq \frac{|B_{1,j}|}{3\Gamma_3(1-\lambda+[3]_q\lambda)^m} \max\{1, |2\theta - 1|\} + \frac{1}{3}\max\{1, |3 - 4\vartheta|\} + \frac{2\,|B_{1,j}||}{\Gamma_2(1-\lambda+[2]_q\lambda)^m}\left|\frac{1}{3} - \frac{\mu}{2}\right|,$$

where

$$\theta = \frac{1}{2}\left(1 - \frac{B_{2,j}}{B_{1,j}} + \frac{3\mu B_{1,j}\Gamma_3(1-\lambda+[3]_q\lambda)^m}{4\Gamma_2^2(1-\lambda+[2]_q\lambda)^{2m}}\right), \qquad \vartheta = \frac{3\mu}{4},$$

and $|B_{i,j}|\,(i = 1,2; j = 1,2,3,4)$ *are defined by (2.5).*

Proof. If $\chi(z) \in S_m^{\lambda,t}(\eta_1, \nu_1, q, L, M; V, W)$, then there exists a Schwarz function $\omega(z)$ in Δ such that

(4.4)

$$\frac{tz^2\left[\mathcal{J}_\lambda^m(\eta_1,\nu_1;q,z)\chi(z)\right]'}{\left[\mathcal{J}_\lambda^m(\eta_1,\nu_1;q,z)\psi(z)\right]\left[\mathcal{J}_\lambda^m(\eta_1,\nu_1;q,z)\psi(tz)\right]} = p_j(\omega(z)), \quad z \in \Delta,$$

where $p_j(z)(j = 1,2,3,4)$ are defined by (2.4).

Let the function $p(z)$ be given by

$$p(z) = \frac{tz^2\left[\mathcal{J}_\lambda^m(\eta_1,\nu_1;q,z)\chi(z)\right]'}{\left[\mathcal{J}_\lambda^m(\eta_1,\nu_1;q,z)\psi(z)\right]\left[\mathcal{J}_\lambda^m(\eta_1,\nu_1;q,z)\psi(tz)\right]}. \tag{4.5}$$

Then, from (4.4) and (4.5) we have $p(z) \prec p_j(z)$. Let

$$q(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + q_1 z + q_2 z^2 + \cdots. \tag{4.6}$$

Then $q(z)$ is analytic and has positive real part in Δ. From (4.6), we get

$$\omega(z) = \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right)z^2 + \cdots\right]. \tag{4.7}$$

We see from (4.7) that

$$p(z) = p_j\left(\frac{q(z)-1}{q(z)+1}\right) \tag{4.8}$$

$$= 1 + \frac{1}{2}B_{1,j}q_1 z + \left[\frac{1}{2}B_{1,j}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{B_{2,j}q_1^2}{4}\right]z^2 + \cdots .$$

Using (4.5) and (4.8), we obtain

$$\Gamma_2(1-\lambda+[2]_q\lambda)^m(2\ell_2 - c_2) = \frac{B_{1,j}q_1}{2},$$

$$\Gamma_3(1-\lambda+[3]_q\lambda)^m(3\ell_3 - 2\ell_2c_2 - c_3 + c_2^2) = \frac{B_{1,j}q_2}{2} - \frac{q_1^2}{4}\left(B_{1,j} - B_{2,j}\right),$$

which imply that

$$\ell_3 - \mu\ell_2^2 = \frac{B_{1,j}}{6\Gamma_3(1-\lambda+[3]_q\lambda)^m}\left[q_2 - \theta q_1^2\right] + \frac{1}{3}\left[c_3 - \vartheta c_2^2\right] + \frac{B_{1,j}c_2q_1}{2\Gamma_2(1-\lambda+[2]_q\lambda)^m}\left(\frac{1}{3} - \frac{\mu}{2}\right), \tag{4.9}$$

where

$$\theta = \frac{1}{2}\left(1 - \frac{B_{2,j}}{B_{1,j}} + \frac{3\mu B_{1,j}\Gamma_3(1-\lambda+[3]_q\lambda)^m}{4\Gamma_2^2(1-\lambda+[2]_q\lambda)^{2m}}\right), \qquad \vartheta = \frac{3\mu}{4}.$$

Therefore, we have

$$\left|\ell_3 - \mu\ell_2^2\right| = \frac{|B_{1,j}|}{6\Gamma_3(1-\lambda+[3]_q\lambda)^m}\left|q_2 - \theta q_1^2\right| + \frac{1}{3}\left|c_3 - \vartheta c_2^2\right| + \frac{|B_{1,j}|\,|c_2|\,|q_1|}{2\Gamma_2(1-\lambda+[2]_q\lambda)^m}\left|\frac{1}{3} - \frac{\mu}{2}\right|.$$

Using Lemma 4.1 and Lemma 4.2, we complete the proof. □

Corollary 4.5 ([11])**.** *If $\chi \in \mathcal{K}_s(\theta,\vartheta)$, then for any real number μ,*

$$\left|\ell_3 - \mu\ell_2^2\right| \le \frac{1}{3} + \frac{2(\vartheta-\theta)}{3\pi}\sin\frac{\pi(1-\theta)}{\vartheta-\theta} \times \max\left\{1, \left|\cos\frac{\pi(1-\theta)}{\vartheta-\theta} - \mu\frac{3(\vartheta-\theta)}{2\pi}\sin\frac{\pi(1-\theta)}{\vartheta-\theta}\right|\right\}.$$

Letting $\vartheta \to \infty$ in corollary 4.5, we can get the following.

Corollary 4.6. *If $\chi \in \mathcal{K}_s(\theta)$, then for any real number μ,*

$$\left|\ell_3 - \mu\ell_2^2\right| \le \frac{1}{3} + \frac{2(1-\theta)}{3}\max\left\{1, \left|1 - \frac{3(1-\theta)}{2}\mu\right|\right\}.$$

Letting $\theta = 0$ and $\vartheta \to \infty$ in Corollary 4.5, we have

Corollary 4.7. *If $\chi \in \mathcal{K}_s$, then for any real number μ,*

$$\left|\ell_3 - \mu\ell_2^2\right| \le \frac{1}{3} + \frac{2}{3}\max\left\{1, \left|1 - \frac{3}{2}\mu\right|\right\}.$$

5. Coefficient Inequality for a Subclass of Bi-Univalent Functions

Lewin in [29] introduced the so-called class of *bi-univalent functions*, which consists of functions f analytic in unit disc Δ such that both f and f^{-1} are univalent in Δ. Here, we let $\mathcal{BS}$ to denote the class of bi-univalent functions. Examples of functions belonging to the class $\mathcal{BS}$ include

$$f_1(z) = \frac{z}{1-z}, \qquad f_2(z) = -\log(1-z), \qquad f_3(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), \ldots .$$

Figure 2 is the mapping of f_2 and f_2^{-1} respectively, if the domain is unit disc.

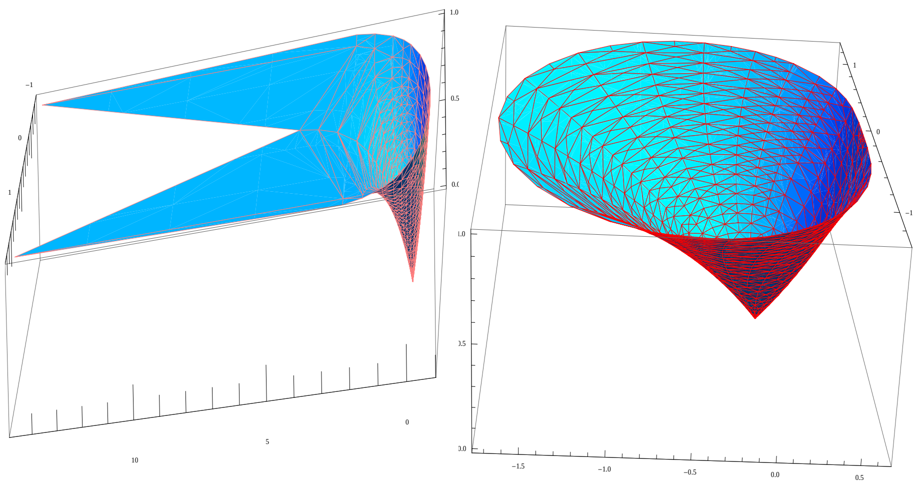


Figure 2. Mapping of $w = -\log(1-z)$ and its inverse $z = 1 - e^{-w}$

On the other hand, the function $\frac{z}{1-z^2}$ belongs to class $\mathcal{S}$ but does not belong to $\mathcal{BS}$. Recently, several researchers introduced and studied various subclasses of bi-univalent functions, see [1–8, 15, 31, 46, 47, 49, 50].

Now, we let $S\Sigma_m^{\lambda,t}(\eta_1, \nu_1, q, L, M; V, W)$ to denote the class of functions Π which satisfies

$$\chi \in S_m^{\lambda,t}(\eta_1, \nu_1, q, L, M; V, W) \quad \text{and} \quad \chi^{-1} \in S_m^{\lambda,t}(\eta_1, \nu_1, q, L, M; V, W),$$

where χ^{-1} is the inverse function of χ given by (1.10).

Remark 5.1. Now, we present only few special cases of the class $S\Sigma_m^{\lambda,t}(\eta_1, \nu_1, q, L, M; V, W)$.

(i) Letting $L = 1 - 2\theta\,(0 \le \theta < 1), M = -1, V = 1 - 2\vartheta$, $(\vartheta > 1)$, $W = -1$ $m = 0$, $r = 2$, $s = 1$, $\eta_1 = \nu_1$, $\nu_2 = q$, $q \to 1^-$ and $t = -1$ in $S\Sigma_m^{\lambda,t}(\eta_1, \nu_1, q, L, M; V, W)$, we get the class $\mathcal{K}_{\Sigma_s}(\theta, \vartheta)$ introduced and studied by Bulut [11, Definition 4.].

(ii) Letting $\vartheta \to \infty$ in the class $\mathcal{K}_{\Sigma_s}(\theta, \vartheta)$, we have the class $\mathcal{K}_{\Sigma_s}(\theta)$ of bi-close-to-convex functions of order θ. The class $\mathcal{K}_{\Sigma_s}(\theta)$ was recently introduced and studied by Şeker and Eker in[38](also see [39]).

We now obtain the estimates of the initial coefficients of functions belonging to $S\Sigma_m^{\lambda,t}(\eta_1, \nu_1, q, L, M; V, W)$.

Theorem 5.2. *Let* $m \in \mathbb{N}_0, \lambda \ge 0, -1 < M < L \le 1, -1 < V < W < 1$. *If* $\chi \in S\Sigma_m^{\lambda,t}(\eta_1, \nu_1, q, L, M; V, W)$, *then*

$$|\ell_2| \le \frac{|B_{1,j}|}{\Gamma_2[1-\lambda+[2]_q\lambda]^m} + \sqrt{\frac{1}{\Gamma_3[1-\lambda+[3]_q\lambda]^m}\left(|B_{1,j}-B_{2,j}| - |B_{1,j}| + \frac{|B_{1,j}|^2}{\Gamma_2^2[1-\lambda+[2]_q\lambda]^{2m}}\right)},$$

and

$$|\ell_3| \le \frac{|B_{1,j}|^2}{\Gamma_2^2[1-\lambda+[2]_q\lambda]^{2m}}\left(\frac{1}{\Gamma_3[1-\lambda+[3]_q\lambda]^m} + 1\right) + |B_{1,j}|\left(\frac{1}{3} + \frac{1}{\Gamma_3[1-\lambda+[3]_q\lambda]^m}\right) + \frac{1}{9}, \tag{5.1}$$

where $|B_{i,j}|\,(i = 1, 2; j = 1, 2, 3, 4)$ *are defined by* (2.5).

Proof. If $\chi(z) \in S\Sigma_m^{\lambda,t}(\eta_1, \nu_1, q, L, M; V, W)$, then

$$\chi(z) \in S_m^{\lambda,t}(\eta_1, \nu_1, q, L, M; V, W)$$

and

$$\psi = \chi^{-1} \in S_m^{\lambda,t}(\eta, \nu, q, L, M; V, W).$$

Hence

$$I(z) = \frac{z\left[\mathcal{J}_\lambda^m(\eta_1, \nu_1; q, z)\chi(z)\right]'}{G(z)} \prec p_j(z), \quad z \in \Delta;\ j = 1, 2, 3, 4,$$

$$H(w) = \frac{w\left[\mathcal{J}_\lambda^m(\eta_1, \nu_1; q, z)k(w)\right]'}{L(w)} \prec p_j(w), \quad z \in \Delta;\ j = 1, 2, 3, 4,$$

where the function $p_j(z)$ is given by (2.3). Let

$$\varsigma(z) = \frac{1 + p_j^{-1}(I(z))}{1 - p_j^{-1}(I(z))} = 1 + \varsigma_1 z + \varsigma_2 z^2 + \cdots, \quad z \in \Delta;\ j = 1, 2, 3, 4,$$

and

$$\tau(z)=\frac{1+p_j^{-1}(H(z))}{1-p_j^{-1}(H(z))}=1+\tau_1 z+\tau_2 z^2+\cdots, \quad z\in\Delta;\ j=1,2,3,4.$$

Then ς and τ are analytic and have positive real part in Δ, and satisfy the estimates

$$|\varsigma_n|\le 2, \qquad |\tau_n|\le 2, \quad n\in\mathbb{N}. \tag{5.2}$$

Therefore, we have

$$I(z)=p_j\left(\frac{\varsigma(z)-1}{\varsigma(z)+1}\right), \qquad H(z)=p_j\left(\frac{\tau(z)-1}{\tau(z)+1}\right), \quad z\in\Delta;\ j=1,2,3,4.$$

By comparing the coefficients, we get

$$\Gamma_2(1-\lambda+[2]_q\lambda)^m(2\ell_2-c_2)=\frac{B_{1,j}\varsigma_1}{2}, \tag{5.3}$$

$$\Gamma_3(1-\lambda+[3]_q\lambda)^m(3\ell_3-2\ell_2c_2-c_3+c_2^2)=\frac{B_{1,j}\varsigma_2}{2}-\frac{\varsigma_1^2}{4}\left(B_{1,j}-B_{2,j}\right), \tag{5.4}$$

$$\Gamma_2(1-\lambda+[2]_q\lambda)^m(c_2-2\ell_2)=\frac{B_{1,j}\tau_1}{2} \tag{5.5}$$

and

$$\begin{aligned}\Gamma_3(1-\lambda+[3]_q\lambda)^m+(6\ell_2^2-3\ell_3-2\ell_2c_2-c_2^2+c_3)\\ =\frac{B_{1,j}\tau_2}{2}-\frac{\tau_1^2}{4}\left(B_{1,j}-B_{2,j}\right),\end{aligned} \tag{5.6}$$

where $B_{i,j}(i=1,2;j=1,2,3,4)$ are given by (2.5). From (5.3) and (5.5), we obtain

$$\varsigma_1=-\tau_1. \tag{5.7}$$

From (5.4),(5.6) and using (5.3),(5.7), we see that

$$\begin{aligned}\ell_2=\frac{B_{1,j}\varsigma_1}{2\Gamma_2[1-\lambda+[2]_q\lambda]^m}+\sqrt{\frac{1}{\Gamma_3[1-\lambda+[3]_q\lambda]^m}}\\ \times\sqrt{\left(\frac{(B_{1,j}-B_{2,j})\varsigma_1^2}{4}-\frac{B_{1,j}(\varsigma_2+\varsigma_1)}{4}+\frac{B_{1,j}^2\varsigma_1^2}{4\Gamma_2^2[1-\lambda+[2]_q\lambda]^{2m}}\right)},\end{aligned}$$

and

$$\ell_3=\ell_2^2+\frac{1}{3}\left(\varsigma_2^2-\varsigma_3\right)+\frac{B_{1,j}}{12}(\varsigma_2-\tau_2).$$

These equations, together with (5.2) and Lemma 4.3, give the bounds on $|\ell_2|$ and $|\ell_3|$ as asserted in (5.1). This completes the proof of Theorem 5.2. $\square$

Corollary 5.3 ([11])**.** *Let* $\chi \in \mathcal{K}_{\Sigma_s}(\theta, \vartheta)\,(0 \leq \theta < 1 < \vartheta)$*, then*

$$|\ell_2| \leq \frac{(\vartheta - \theta)}{\pi} \sin \frac{\pi(1-\theta)}{\vartheta - \theta},$$

and

$$|\ell_3| \leq \frac{1 + \frac{2(\vartheta-\theta)}{\pi} \sin \frac{\pi(1-\theta)}{\vartheta-\theta}}{3}.$$

Letting $\vartheta \to \infty$ in Corollary (5.3), we get the result obtained by Bulut [11, Corollary 4.2.]

Corollary 5.4. *If* $\chi \in \mathcal{K}_{\Sigma_s}(\theta)$, $(0 \leq \theta < 1)$*, then*

$$|\ell_2| \leq 1 - \theta$$

and

$$|\ell_3| \leq \frac{3 - 2\theta}{3}.$$

Letting $\theta = 0$ and $\vartheta \to \infty$ in Corollary (5.3), we get

Corollary 5.5 ([11, Corollary 4.4.])**.** *If* χ *belongs to* $\mathcal{K}_{\Sigma_s}$*, then*

$$|\ell_2| \leq 1, \qquad |\ell_3| \leq 1.$$

6. Conclusion

We defined a new family of starlike functions which connects Janowski starlike functions and the class of starlike functions associated with vertical domain. To make this study more comprehensive, we have defined the class of functions using a differential operator which helps in amalgamating the study of several classes of well-known analytic functions. Inclusion relations, geometrical interpretation, coefficient estimates, inverse function coefficient estimates and solution to the Fekete-Szegő problem are the foremost results of this paper. Also, we have pointed out appropriate connections and applications of our main results, which are mostly presented in the form of corollaries and remarks.

This study can be further extended by replacing the respective superordinate function in (1.4) and (1.2) with a function which is not Carathéodory (see [19]). Further, this study can be extended by taking an trigonometric hyperbolic function, Gegenbauer polynomial, Laguerre polynomial, Chebyshev polynomial, Fibonacci sequence, or q-Hermite polynomial instead of considering an arbitrary $h_1(z)$ and $h_1(z)$ in Definition 1.1.

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