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The Stability of Bi-Drygas Functional Equation

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Abstract. In this paper, we introduce and solve a system of bi-Drygas functional equations

$$\begin{cases} f(x+y,z)+f(x-y,z)=2f(x,z)+f(y,z)+f(-y,-z) \\ f(x,y+z)+f(x,y-z)=2f(x,y)+f(x,z)+f(-x,-z) \end{cases}$$

for all $x,y,z \in X$. We will also investigate the Hyers-Ulam stability of the system of bi-Drygas functional equations.

1. Introduction

Let $\mathcal{H}$ be an operator and a solution collection $\{\eta\}$ with the property that $\mathcal{H}(\eta)=0$. When does $\mathcal{H}(\theta) \leq \epsilon$ for an $\epsilon \geq 0$ imply that $|\eta-\theta| \leq k\epsilon$ for some η and for some $k>0$? This problem is referred to as the stability of the functional [48]. An answer to the problem was provided by Hyers [20] in the setting of Banach spaces. Since then the stability problems have been extensively investigated for various functional equations and spaces.

Hyers was the first mathematician to establish the concept stability of functional equations. He provided an answer to Ulam's question for the case of approximate additive mappings, assuming that G_1 and G_2 are Banach spaces (see [20]).

The method introduced by Hyers [20] which produces the additive function will be referred a direct method. This method is the most significant and powerful tool concerning the stability of different functional equations. That is, the exact solution of the functional equation

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is explicitly constructed as a limit of a sequence, starting from the given approximate solution [5, 7, 15, 46].

The stability problems of several functional equations have been extensively studied by a number of authors leading to many intriguing results containing homomorphisms, derivations, Cauchy-Jensen mappings, additive mappings and quadratic-additive mapping. For more details on this topic, refer to [8, 10–12, 19, 23–28, 30, 39, 41].

In recent years, a number of papers have been investigated on various generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings, for example, functional equations, Laguerre differential equations, differential equations, hypergeometric differential equations and convex functins (see [1–4, 9, 21, 22, 40, 45, 47]). Furthermore, we refer to [29, 31, 36] for stability results of radical cubic functional equations in various spaces and to [13, 14, 32, 44, 49] for recent monograph on stability.

In [6] J. Brzdek et al, presented a method that allows to study the stability of some systems of functional equations connected with the Cauchy, Jensen and quadratic equations.

Recently, some authors extended the stability problem from functional equations to system of functional equations (see [37, 43]).

I. EL. Fassi [18] conducted a study on the hyperstability of the bi-additive functional equation and obtained inequalities characterizing bi-additive mappings and inner product spaces.

Let X and Y be vector spaces. A mapping $f : X \times X \to Y$ is said to be bi-additive if it satisfies

$$f(x+y, z+w) = f(x,z) + f(x,w) + f(y,z) + f(y,w) \tag{1.1}$$

for all $x, y, z, w \in X$.

A mapping $f : X \times X \to Y$ is called bi-quadratic if f satisfies the system of equations

$$\begin{cases} f(x+y,z) + f(x-y,z) = 2f(x,z) + 2f(y,z) \\ f(x,y+z) + f(x,y-z) = 2f(x,y) + 2f(x,z) \end{cases} \tag{1.2}$$

for all $x, y, z \in X$.

A mapping $f : X \times X \to Y$ is called bi-quadratic if f satisfies the functional equation

$$\begin{aligned} &f(x+y, z+w) + f(x+y, z-w) + f(x-y, z+w) + f(x-y, z-w) \\ &\quad = 4\left[f(x,z) + f(x,w) + f(y,z) + f(y,w)\right] \end{aligned}$$

for all $x, y, z, w \in X$.

A mapping $g : X \to Y$ is called Drygas if g satisfies the functional equation

$$g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y) \tag{1.3}$$

for all $x, y \in X$. This functional equation, introduced by Drygas [16] serves as a tool for distinguishing quasi-inner product spaces. The functional equation (1.3) has been considered (see [17, 27]).

In this paper, we introduce the bi-Drygas functional equation in a vector space.

Definition 1.1. Let X and Y be vector spaces. A mapping $f : X \times X \to Y$ is said to be bi-Drygas if f satisfies the system of equations

$$\begin{cases} f(x+y,z) + f(x-y,z) = 2f(x,z) + f(y,z) + f(-y,-z) \\ f(x,y+z) + f(x,y-z) = 2f(x,y) + f(x,z) + f(-x,-z) \end{cases} \tag{1.4}$$

for all $x, y, z \in X$.

Example 1.2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,z) = k_1 x + k_2 z + k_3 x^2 z^2$ where k_1, k_2 and k_3 are real constants. Then f is a solution of (1.4).

For a mapping $f : X \times X \to Y$, consider the functional equation

$$\begin{aligned} & f(x+y, z+w) + f(x+y, z-w) \\ & \quad + f(x-y, z+w) + f(x-y, z-w) \\ & = 4f(x,z) + 2[f(x,w) + f(y,z) + f(y,w)] \\ & \quad + 2[f(-x,-w) + f(-y,-z) + f(-y,-w)] \end{aligned} \tag{1.5}$$

for all $x, y, z, w \in X$.

Throughout this paper, f^e and f^o denote the even and odd parts of f, respectively, i.e.,

$$f^e(x,z) = \frac{f(x,z) + f(-x,-z)}{2}, \qquad f^o(x,z) = \frac{f(x,z) - f(-x,-z)}{2}$$

for all $x, z \in X$.

In this note, we establish and solve bi-Drygas functional equation. Additionally, we will investigate the stability of this functional equation.

2. Some Auxiliary Results

The Hyers-Ulam stability of the bi-quadratic functional equation was initially demonstrated by Park et al. [38] for functions from a vector space into a Banach space.

The next lemma follows from Theorem 6 in [38] by $\tilde{\varphi}(x, y, z,) = \epsilon_1$ and $\tilde{\psi}(x, y, z,) = \epsilon_2$ for all $x, y, z, w \in X$.

Lemma 2.1. *Let X be a vector space and let Y be a Banach space. Let $f : X \times X \to Y$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\|f(x+y,z)+f(x-y,z)-2f(x,z)-2f(y,z)\| \le \epsilon_1$$
$$\|f(x,y+z)+f(x,y-z)-2f(x,y)-2f(x,z)\| \le \epsilon_2$$

for some $\epsilon_1, \epsilon_2 \ge 0$ and for all $x, y, z \in X$. Then there exist two bi-quadratic mappings $Q_1, Q_2 : X \times X \to Y$ such that

$$\|f(x,z)-Q_1(x,z)\| \le \frac{5}{3}\epsilon_1$$
$$\|f(x,z)-Q_2(x,z)\| \le \frac{5}{3}\epsilon_2$$

for all $x, z \in X$.

Also, Lemma 2.2 follows from Theorem 7 in [38] by $\tilde{\varphi}(x, y, z, w) = \epsilon$ for all $x, y, z, w \in X$.

Lemma 2.2. *Let X be a vector space and let Y be a Banach space. Let $f : X \times X \to Y$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\begin{aligned}\Big\|&f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)\\ &-4[f(x,z)+f(x,w)+f(y,z)+f(y,w)]\Big\|\\ &\le \epsilon\end{aligned}$$

for some $\epsilon \ge 0$ and for all $x, y, z, w \in X$. Then there exists a unique bi-quadratic mapping $Q : X \times X \to Y$ such that

$$\|f(x,z)-Q(x,z)\| \le \frac{\epsilon}{15}$$

for all $x, z \in X$.

Lemma 2.3. *Let X be a vector space and let Y be a Banach space. Let $f : X \times X \to Y$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\begin{aligned}\Big\|&f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)\\ &+f(x-y,z-w)-4f(x,z)\Big\|\\ &\le \epsilon\end{aligned} \tag{2.1}$$

for some $\epsilon \ge 0$ and for all $x, y, z \in X$. Then there exist a unique bi-additive mapping $A : X \times X \to Y$ such that

$$\|f(x,z)-A(x,z)\| \le \frac{\epsilon}{3} \tag{2.2}$$

for all $x, z \in X$.

Proof. Putting $z = x$ and $w = z$ in (2.1), we get

$$\|f(2x,2z)-4f(x,z)\| \le \epsilon$$

for all $x, z \in X$. By induction it follows that

$$\left\| \frac{1}{4^n} f(2^n x, 2^n z) - f(x,z) \right\| \leq \sum_{j=1}^{n} \frac{\epsilon}{4^j} \tag{2.3}$$

for all $x, z \in X$ and $n = 1, 2, 3, \ldots$.

By (2.3), the sequence $\left\{\frac{1}{4^n} f(2^n x, 2^n z)\right\}$ is a Cauchy sequence in Banach space Y, and as a result, the sequence $\left\{\frac{1}{4^n} f(2^n x, 2^n z)\right\}$ converges. Define the mapping $A : X \times X \to Y$ by:

$$A(x,z) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n z)$$

for all $x, z \in X$.

Allowing n tending to infinity, we get

$$\begin{aligned} \|A(x,z) - f(x,z)\| &\leq \epsilon \sum_{j=0}^{\infty} \frac{1}{4^j} \\ &= \frac{\epsilon}{3} \end{aligned}$$

for all $x, z \in X$.

Next, we show that A is bi-additive mapping. Define

$$\begin{aligned} D(x,y,z,w) := &\ f(x+y, z+w) + f(x+y, z-w) + f(x-y, z+w) \\ &+ f(x-y, z-w) - 4f(x,z) \end{aligned}$$

for all $x, y, z, w \in X$. So

$$\begin{aligned} &A(x+y, z+w) + A(x+y, z-w) + A(x-y, z+w) + A(x-y, z-w) \\ &= \lim_{n\to\infty} \frac{1}{4^n} [f(2^n(x+y), 2^n(z+w)) + f(2^n(x+y), 2^n(z-w)) \\ &\quad + f(2^n(x-y), 2^n(z+w)) + f(2^n(x-y), 2^n(z-w))] \\ &= \lim_{n\to\infty} \frac{1}{4^n} 4f(2^n x, 2^n z) + \lim_{n\to\infty} \frac{1}{4^{2n}} D(2^n x, 2^n y, 2^n z, 2^n w) \\ &= 4A(x,z) \end{aligned}$$

for all $x, y, z, w \in X$.

To prove the uniqueness, let $A' : X \times X \to Y$ be another bi-additive mapping such that

$$\left\|A'(x,z) - f(x,z)\right\| \leq \frac{\epsilon}{3}$$

for all $x, z \in X$ and

$$\left\|A(x_0, z_0) - A'(x_0, z_0)\right\| = \delta > 0$$

for some $x_0, z_0 \in X$. Moreover

$$\left\|A(x,z) - A'(x,z)\right\| \leq \|A(x,z) - f(x,z)\| + \left\|f(x,z) - A'(x,z)\right\|$$

$$\leq \frac{2}{3}\epsilon$$

for all $x, z \in X$. It follows that

$$\begin{aligned}\frac{2}{3}\epsilon &\geq \left\|A(2^n x_0, 2^n z_0) - A'(2^n x_0, 2^n z_0)\right\| \\ &= \left\|4^n A(x_0, z_0) - 4^n A'(x_0, z_0)\right\| \\ &= 4^n \delta\end{aligned}$$

for all $n = 1, 2, 3, \ldots$ and for some $x_0, z_0 \in X$, a contradiction. □

Lemma 2.4. *Let X be a vector space and let Y be a Banach space. Let $f : X \times X \to Y$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\|f(x+y, z) + f(x-y, z) - 2f(x, z)\| \leq \epsilon_1, \tag{2.4}$$

$$\|f(x, y+z) + f(x, y-z) - 2f(x, y)\| \leq \epsilon_2 \tag{2.5}$$

for some $\epsilon_1, \epsilon_2 \geq 0$ and for all $x, y, z \in X$. Then there exist two bi-additive mappings $A_1, A_2 : X \times X \to Y$ such that

$$\|f(x, z) - A_1(x, z)\| \leq \epsilon_1, \tag{2.6}$$

$$\|f(x, z) - A_2(x, z)\| \leq \epsilon_2 \tag{2.7}$$

for all $x, z \in X$.

Proof. Putting $y = x$ in (2.4) and $y = z$ in (2.5), we get

$$\|f(2x, z) - 2f(x, z)\| \leq \epsilon_1,$$

$$\|f(x, 2z) - 2f(x, z)\| \leq \epsilon_2$$

for all $x, z \in X$. By induction it follows that

$$\left\|\frac{1}{2^n} f(2^n x, z) - f(x, z)\right\| \leq \sum_{j=1}^{n} \frac{\epsilon_1}{2^j}, \tag{2.8}$$

$$\left\|\frac{1}{2^n} f(x, 2^n z) - f(x, z)\right\| \leq \sum_{j=1}^{n} \frac{\epsilon_2}{2^j} \tag{2.9}$$

for all $x, z \in X$ and $n = 1, 2, 3, \ldots$.

By (2.8) and (2.9), the sequences $\left\{\frac{1}{2^n} f(2^n x, z)\right\}$ and $\left\{\frac{1}{2^n} f(x, 2^n z)\right\}$ are Cauchy sequence in Banach space Y and hence sequences $\left\{\frac{1}{2^n} f(2^n x, z)\right\}$ and $\left\{\frac{1}{2^n} f(x, 2^n z)\right\}$ converges. Define the mapping $A_1, A_2 : X \times X \to Y$ by:

$$A_1(x, z) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x, z), \qquad A_2(x, z) := \lim_{n\to\infty} \frac{1}{2^n} f(x, 2^n z)$$

for all $x, z \in X$.

Similarly, to the proof of Lemma 2.3, the mappings A_1 and A_2 are unique bi-additive and satisfying (2.6) and (2.7). □

Lemma 2.5. *Let X be a vector space and let E be an inner product space over $\mathbb{K}$. Let $f : X \times X \to E$ be a mapping satisfying*

$$\|f(x,z) + f(y,z)\| \le \|f(x+y,z)\|, \tag{2.10}$$

$$\|f(x,y) + f(x,z)\| \le \|f(x,y+z)\| \tag{2.11}$$

for all $x, y, z \in X$. Then f is bi-additive.

Proof. Taking $x = y = 0$ in (2.10), we have $f(0,z) = 0$ for all $z \in X$. Setting $y = z = 0$ in (2.11), we have $f(x,0) = 0$ for all $x \in X$.

Letting $y = -x$ in (2.10) and $y = -z$ in (2.11), we get

$$f(-x,z) = f(x,-z) = -f(x,z) \tag{2.12}$$

for all $x, z \in X$. Inequalities (2.10) and (2.11) lead to

$$\|f(x,z)\|^2 + 2\mathrm{Re}\,\langle f(x,z), f(y,z)\rangle + \|f(y,z)\|^2 \le \|f(x+y,z)\|^2, \tag{2.13}$$

$$\|f(x,y)\|^2 + 2\mathrm{Re}\,\langle f(x,y), f(x,z)\rangle + \|f(x,z)\|^2 \le \|f(x,y+z)\|^2 \tag{2.14}$$

for all $x, y, z \in X$.

Substituting x by $x + y$ and y by $-y$ in (2.13) and y by $y + z$ and z by $-z$ in (2.14), we gain

$$\|f(x+y,z)\|^2 + 2\mathrm{Re}\,\langle f(x+y,z), f(-y,z)\rangle + \|f(-y,z)\|^2 \le \|f(x,z)\|^2,$$

$$\|f(x,y+z)\|^2 + 2\mathrm{Re}\,\langle f(x,y+z), f(x,-z)\rangle + \|f(x,-z)\|^2 \le \|f(x,y)\|^2$$

for all $x, y, z \in X$. By (2.12),

$$\|f(x+y,z)\|^2 - 2\mathrm{Re}\,\langle f(x+y,z), f(y,z)\rangle + \|f(y,z)\|^2 \le \|f(x,z)\|^2 \tag{2.15}$$

and

$$\|f(x,y+z)\|^2 - 2\mathrm{Re}\,\langle f(x,y+z), f(x,z)\rangle + \|f(x,z)\|^2 \le \|f(x,y)\|^2 \tag{2.16}$$

for all $x, y, z \in X$.

Adding (2.13) to (2.15) and (2.14) to (2.16), we obtain

$$\mathrm{Re}\,\langle f(x,z) + f(y,z) - f(x+y,z), f(y,z)\rangle \le 0, \tag{2.17}$$

$$\mathrm{Re}\,\langle f(x,y) + f(x,z) - f(x,y+z), f(x,z)\rangle \le 0 \tag{2.18}$$

for all $x, y, z \in X$.

Interchanging the roles of x and y in (2.17)and the roles of y and z in (2.18), we obtain

$$\mathrm{Re}\,\langle f(x,z) + f(y,z) - f(x+y,z), f(x,z)\rangle \le 0, \tag{2.19}$$

$$\mathrm{Re}\,\langle f(x,y) + f(x,z) - f(x,y+z), f(x,y)\rangle \le 0 \tag{2.20}$$

for all $x, y, z \in X$.

Now, replacing x by $x+y$ and y by $-y$ in (2.19) and y by $y+z$ and z by $-z$ in (2.20), we arrive at

$$\operatorname{Re}\langle f(x,z)+f(y,z)-f(x+y,z),-f(x+y,z)\rangle \leq 0, \tag{2.21}$$

$$\operatorname{Re}\langle f(x,y)+f(x,z)-f(x,y+z),-f(x,y+z)\rangle \leq 0 \tag{2.22}$$

for all $x, y, z \in X$.

It follows from (2.17), (2.19) and (2.21) that

$$\|f(x,z)+f(y,z)-f(x+y,z)\|^2 \leq 0$$

and also, from (2.18), (2.20) and (2.22), we gain

$$\|f(x,y)+f(x,z)-f(x,y+z)\|^2 \leq 0$$

for all $x, y, z \in X$. Thus f is bi-additive. □

Lemma 2.6. *Let X be a vector space and let E be an inner product space over $\mathbb{K}$. Let $f : X \times X \to E$ be a mapping satisfying*

$$\|2f(x,z)+2f(y,z)-f(x-y,z)\| \leq \|f(x+y,z)\|, \tag{2.23}$$

$$\|2f(x,y)+2f(x,z)-f(x,y-z)\| \leq \|f(x,y+z)\| \tag{2.24}$$

for all $x, y, z \in X$. Then f is a solution of the system of bi-quadratic functional equation (1.2).

Proof. Letting $x = y = 0$ in (2.23), we gain $f(0,z) = 0$ for all $z \in X$ and putting $y = z = 0$ in (2.24), we obtain $f(x,0) = 0$ for all $x \in X$.

Taking $y = -x$ in (2.23) and $y = -z$ in (2.24), we have

$$f(2x,z) = 2f(x,z)+2f(-x,z), \tag{2.25}$$

$$f(x,-2z) = 2f(x,-z)+2f(x,z) \tag{2.26}$$

for all $x, z \in X$. Replacing x by $-x$ in (2.25) and z by $-z$ in (2.26), to get

$$f(-2x,z) = 2f(-x,z)+2f(x,z), \tag{2.27}$$

$$f(x,2z) = 2f(x,z)+2f(x,-z) \tag{2.28}$$

for all $x, z \in X$. By (2.25) and (2.27), we arrive at $f(2x,z) = f(-2x,z)$ implying

$$f(x,z) = f(-x,z) \tag{2.29}$$

for all $x, z \in X$. Also, from (2.26) and (2.28), we obtain $f(x,2z) = f(x,-2z)$ which concludes

$$f(x,z) = f(x,-z) \tag{2.30}$$

for all $x, z \in X$.

Using (2.25), (2.26), (2.29) and (2.30), we see that

$$f(2x,z)=4f(x,z),\qquad f(x,2z)=4f(x,z) \tag{2.31}$$

for all $x,z\in X$.

Squaring the two sides of (2.23) and (2.24) gives

$$\|2f(x,z)+2f(y,z)-f(x-y,z)\|^2\le\|f(x+y,z)\|^2,$$
$$\|2f(x,y)+2f(x,z)-f(x,y-z)\|^2\le\|f(x,y+z)\|^2$$

for all $x,z\in X$. Hence

$$\begin{aligned}\|2f(x,z)+2f(y,z)\|^2&+\|f(x-y,z)\|^2\\ &-2\mathrm{Re}\,\langle 2f(x,z)+2f(y,z),f(x-y,z)\rangle\\ \le\|f(x+y,z)\|^2&\end{aligned} \tag{2.32}$$

and

$$\begin{aligned}\|2f(x,y)+2f(x,z)\|^2&+\|f(x,y-z)\|^2\\ &-2\mathrm{Re}\,\langle 2f(x,y)+2f(x,z),f(x,y-z)\rangle\\ \le\|f(x,y+z)\|^2&\end{aligned} \tag{2.33}$$

for all $x,y,z\in X$. Replace y by $-y$ in (2.32) and z by $-z$ in (2.33), to obtain

$$\begin{aligned}\|2f(x,z)+2f(-y,z)\|^2&+\|f(x+y,z)\|^2\\ &-2\mathrm{Re}\,\langle 2f(x,z)+2f(-y,z),f(x+y,z)\rangle\\ \le\|f(x-y,z)\|^2&\end{aligned} \tag{2.34}$$

and

$$\begin{aligned}\|2f(x,y)+2f(x,-z)\|^2&+\|f(x,y+z)\|^2\\ &-2\mathrm{Re}\,\langle 2f(x,y)+2f(x,-z),f(x,y+z)\rangle\\ \le\|f(x,y-z)\|^2&\end{aligned} \tag{2.35}$$

for all $x,y,z\in X$.

Applying (2.29) and adding (2.32) and (2.34), we get

$$\begin{aligned}2\,\|2f(x,z)&+2f(y,z)\|^2\\ &-2\mathrm{Re}\,\langle 2f(x,z)+2f(y,z),f(x+y,z)+f(x-y,z)\rangle\\ \le 0&\end{aligned}$$

and hence

$$\begin{aligned}2\mathrm{Re}\,&\langle -2f(x,z)-2f(y,z),-2f(x,z)-2f(y,z)\rangle\\ &+2\mathrm{Re}\,\langle -2f(x,z)-2f(y,z),f(x+y,z)+f(x-y,z)\rangle\\ \le 0&\end{aligned}$$

for all $x, y, z \in X$. So

$$\begin{aligned}\mathrm{Re}\,\langle -2f(x,z) - 2f(y,z), f(x+y,z) + f(x-y,z) - 2f(x,z) - 2f(y,z)\rangle \\ \leq 0\end{aligned} \tag{2.36}$$

for all $x, y, z \in X$. Substituting x by $-x-y$ and y by $y-x$ in (2.32) yields

$$\begin{aligned}&\|2f(-x-y,z) + 2f(y-x,z)\|^2 + \|f(-2y,z)\|^2 \\ &\quad - 2\mathrm{Re}\,\langle 2f(-x-y,z) + 2f(y-x,z), f(-2y,z)\rangle \\ &\leq \|f(-2x,z)\|^2\end{aligned}$$

for all $x, y, z \in X$. By (2.29) and (2.31),

$$\begin{aligned}&\|2f(x+y,z) + 2f(x-y,z)\|^2 + \|f(-2y,z)\|^2 \\ &\quad - 2\mathrm{Re}\,\langle 2f(x+y,z) + 2f(x-y,z), 4f(y,z)\rangle \\ &\leq \|f(-2x,z)\|^2\end{aligned} \tag{2.37}$$

for all $x, y, z \in X$. Interchanging x and y in (2.37) and using (2.29), we gain

$$\begin{aligned}&\|2f(x+y,z) + 2f(x-y,z)\|^2 + \|f(-2x,z)\|^2 \\ &\quad - 2\mathrm{Re}\,\langle 2f(x+y,z) + 2f(x-y,z), 4f(x,z)\rangle \\ &\leq \|f(-2y,z)\|^2\end{aligned} \tag{2.38}$$

for all $x, y, z \in X$. Adding (2.37) and (2.38), we have

$$\begin{aligned}&8\,\|f(x+y,z) + f(x-y,z)\|^2 \\ &\quad - 8\mathrm{Re}\,\langle f(x+y,z) + f(x-y,z), 2f(x,z) + 2f(y,z)\rangle \\ &\leq 0\end{aligned}$$

for all $x, y, z \in X$. Consequently,

$$\begin{aligned}&\mathrm{Re}\,\langle f(x+y,z) + f(x-y,z), f(x+y,z) + f(x-y,z)\rangle \\ &\quad + \mathrm{Re}\,\langle f(x+y,z) + f(x-y,z), -2f(x,z) - 2f(y,z)\rangle \\ &\leq 0\end{aligned}$$

implying

$$\begin{aligned}&\mathrm{Re}\langle f(x+y,z) + f(x-y,z) \\ &\quad , f(x+y,z) + f(x-y,z) - 2f(x,z) - 2f(y,z)\rangle \\ &\leq 0\end{aligned} \tag{2.39}$$

for all $x, y, z \in X$. It follows from (2.36) and (2.39) that

$$\mathrm{Re}\langle f(x+y,z) + f(x-y,z) - 2f(x,z) - 2f(y,z)$$

$$, f(x+y,z)+f(x-y,z)-2f(x,z)-2f(y,z)\rangle$$
$$\leq 0$$

for all $x, y, z \in X$. Therefore

$$f(x+y,z)+f(x-y,z)=2f(x,z)+2f(y,z)$$

for all $x, y, z \in X$.

Similarly, from (2.30), (2.31), (2.33) and (2.35), one can show that

$$f(x,y+z)+f(x,y-z)=2f(x,y)+2f(x,z)$$

for all $x, y, z \in X$. Hence, f is bi-quadratic. □

3. Solution and Stability of Bi-Drygas Functional Equation

In this section, we solve both the functional equation (1.4) and we will investigate the Hyers-Ulam stability of this functional equation.

Theorem 3.1. *The mapping $f : X \times X \to Y$ satisfies the system of functional equations* (1.4) *if and only if f satisfies* (1.5)*.*

Proof. If f satisfies (1.4), then we see that

$$\begin{aligned}
f(x+y,z+w)&+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)\\
&=2f(x+y,z)+f(x+y,w)+f(-x+(-y),-w)\\
&\quad+2f(x-y,z)+f(x-y,w)+f(-x-(-y),-w)\\
&=4f(x,z)+2[f(x,w)+f(y,z)+f(y,w)]\\
&\quad+2\left[f(-x,-w)+f(-y,-z)+f(-y,-w)\right]
\end{aligned}$$

for all $x, y, z, w \in X$.

Conversely, assume that f satisfies (1.5). Taking $x=y=z=w=0$ in (1.5), we gain $f(0,0)=0$. Setting $y=z=w=0$, we get

$$f(x,0)+f(-x,0)=0$$

for all $x \in X$. Letting $w=0$ in (1.5), we arrive at

$$f(x+y,z)+f(x-y,z)=2f(x,z)+f(y,z)+f(-y,-z)$$

for all $x, y, z \in X$.

Putting $x=y=w=0$ in (1.5), we get

$$f(0,z)+f(0,-z)=0$$

for all $z \in X$. Letting $y=0$ in (1.5) and replacing z by y and w by z, we obtain

$$f(x,y+z)+f(x,y-z)=2f(x,y)+f(x,z)+f(-x,-z)$$

for all $x, y, z \in X$. □

Theorem 3.2. *Let X be a vector space and let Y be a Banach space. Let $f : X \times X \to Y$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

(3.1)
$$\begin{cases} \|f(x+y,z)+f(x-y,z)-2f(x,z)-f(y,z)-f(-y,-z)\| \leq \epsilon_1 \\ \|f(x,y+z)+f(x,y-z)-2f(x,y)-f(x,z)-f(-x,-z)\| \leq \epsilon_2 \end{cases}$$

for some $\epsilon_1 \geq 0, \epsilon_2 \geq 0$ and for all $x, y, z \in X$. Then there exist two bi-additive mappings $A_1, A_2 : X \times X \to Y$ and two bi-quadratic mappings $Q_1, Q_2 : X \times X \to Y$ such that

$$\begin{cases} \|f(x,z)-A_1(x,z)-Q_1(x,z)\| \leq \frac{8}{3}\epsilon_1 \\ \|f(x,z)-A_2(x,z)-Q_2(x,z)\| \leq \frac{8}{3}\epsilon_2 \end{cases} \tag{3.2}$$

for all $x, z \in X$.

Proof. In (3.1), replace x, y and z by $-x, -y$ and $-z$, respectively, to get

(3.3)
$$\begin{cases} \|f(-x-y,-z)+f(y-x,-z)-2f(-x,-z)-f(-y,-z)-f(y,z)\| \\ \quad \leq \epsilon_1 \\ \|f(-x,-y-z)+f(-x,z-y)-2f(-x,-y)-f(-x,-z)-f(x,z)\| \\ \quad \leq \epsilon_2 \end{cases}$$

for all $x, y, z \in X$. By (3.1) and (3.3) we obtain

$$\begin{cases} \|f^e(x+y,z)+f^e(x-y,z)-2f^e(x,z)-2f^e(y,z)\| \leq \epsilon_1 \\ \|f^e(x,y+z)+f^e(x,y-z)-2f^e(x,y)-2f^e(x,z)\| \leq \epsilon_2 \end{cases}$$

for all $x, y, z \in X$. Applying Lemma 2.1 to f^e, we get two bi-quadratic mapping $Q_1, Q_2 : X \times X \to Y$ such that

$$\begin{cases} \|f^e(x,z)-Q_1(x,z)\| \leq \frac{5}{3}\epsilon_1 \\ \|f^e(x,z)-Q_2(x,z)\| \leq \frac{5}{3}\epsilon_2 \end{cases} \tag{3.4}$$

for all $x, z \in X$.

Similarly, using (3.1) and (3.3) yields

$$\begin{cases} \|f^o(x+y,z)+f^o(x-y,z)-2f^o(x,z)\| \leq \epsilon_1 \\ \|f^o(x,y+z)+f^o(x,y-z)-2f^o(x,y)\| \leq \epsilon_2 \end{cases}$$

for all $x, y, z \in X$. Applying Lemma 2.4 to f^o, we get two bi-additve mapping $A_1, A_2 : X \times X \to Y$ such that

$$\begin{cases} \|f^o(x,z) - A_1(x,z)\| \le \epsilon_1 \\ \|f^o(x,z) - A_2(x,z)\| \le \epsilon_2 \end{cases} \tag{3.5}$$

for all $x, z \in X$. Combining (3.4) and (3.5) gives (3.2). □

Theorem 3.3. *Let X be a vector space and let Y be a Banach space. Let $f : X \times X \to Y$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\begin{aligned} &\big\| f(x+y, z+w) + f(x+y, z-w) + f(x-y, z+w) \\ &\qquad + f(x-y, z-w) - 4f(x,z) \\ &\qquad - 2\left[f(x,w) + f(y,z) + f(y,w)\right] \\ &\qquad - 2\left[f(-x,-w) + f(-y,-z) + f(-y,-w)\right] \big\| \\ &\le \epsilon \end{aligned} \tag{3.6}$$

for some $\epsilon \ge 0$ and for all $x, y, z, w \in X$. Then there exist a unique bi-additive mapping $A : X \times X \to Y$ and a unique bi-quadratic mapping $Q : X \times X \to Y$ such that

$$\|f(x,z) - A(x,z) - Q(x,z)\| \le \frac{2}{5}\epsilon \tag{3.7}$$

for all $x, z \in X$.

Proof. In (3.6), replace x, y, z and w by $-x, -y, -z$ and $-w$, respectively, to obtain

$$\begin{aligned} &\big\| f(-x-y, -z-w) + f(-x-y, -z+w) \\ &\qquad + f(-x+y, -z-w) + f(-x+y, -z+w) - 4f(-x,-z) \\ &\qquad - 2\left[f(-x,-w) + f(-y,-z) + f(-y,-w)\right] \\ &\qquad - 2\left[f(x,w) + f(y,z) + f(y,w)\right] \big\| \\ &\le \epsilon \end{aligned} \tag{3.8}$$

for all $x, y, z, w \in X$. It follows from (3.6) and (3.8) that

$$\begin{aligned} &\big\| f^e(x+y, z+w) + f^e(x+y, z-w) + f^e(x-y, z+w) \\ &\qquad + f^e(x-y, z-w) - 4[f^e(x,z) + f^e(x,w) + f^e(y,z) + f^e(y,w)] \big\| \\ &\le \epsilon \end{aligned}$$

for all $x, y, z, w \in X$. By Lemma 2.2, there exists a unique bi-quadratic mapping $Q : X \times X \to Y$ such that

$$\|f^e(x,z) - Q(x,z)\| \le \frac{\epsilon}{15} \tag{3.9}$$

for all $x, z \in X$.

Also, from (3.6) and (3.8), to get

$$\begin{aligned}&\big\|f^o(x+y,z+w)+f^o(x+y,z-w)+f^o(x-y,z+w)\\&\quad+f^o(x-y,z-w)-4f^o(x,z)\big\|\\&\le\epsilon\end{aligned}$$

for all $x,z\in X$.

Applying Lemma 2.3 to f^o we get a unique bi-additive mapping $A: X\times X\to Y$ satisfying

$$\|f^e(x,z)-A(x,z)\|\le\frac{\epsilon}{3}\tag{3.10}$$

for all $x,z\in X$.

Using (3.9) and (3.10) we get

$$\|f(x,z)-A(x,z)-Q(x,z)\|\le\frac{2}{5}\epsilon$$

for all $x,z\in X$. Which proves the validity of (3.7). □

Theorem 3.4. *Let X be a vector space and let Y be a Banach space. Let $f: X\times X\to Y$ be a mapping satisfying $f(x,0)=f(0,z)=0$ and*

$$\|2f(x,z)+f(y,z)+f(-y,-z)-f(x-y,z)\|\le\|f(x+y,z)\|,\tag{3.11}$$

$$\|2f(x,y)+f(x,z)+f(-x,-z)-f(x,y-z)\|\le\|f(x,y+z)\|\tag{3.12}$$

for all $x,y,z\in X$. Then f is a solution of the system of bi-Drygas functional equations (1.4).

Proof. Replacing y by $-x$ in (3.11), we have

$$2f(x,z)+f(-x,z)+f(x,-z)=f(2x,z)\tag{3.13}$$

for all $x,z\in X$. Setting $y=-z$ in (3.12), we obtain

$$2f(x,-z)+f(x,z)+f(-x,-z)=f(x,-2z)\tag{3.14}$$

for all $x,z\in X$. Replacing z by $-z$ in (3.14)

$$2f(x,z)+f(x,-z)+f(-x,z)=f(x,2z)\tag{3.15}$$

for all $x,z\in X$. It follows from (3.13) and (3.15) that

$$f(2x,z)=f(x,2z)\tag{3.16}$$

for all $x,z\in X$.

Replacing x by $-x$ in (3.14), we gain

$$2f(-x,-z)+f(-x,z)+f(x,-z)=f(-x,-2z)\tag{3.17}$$

for all $x,z\in X$. From (3.13) and (3.17), to obtain

$$2f(x,z)-2f(-x,-z)=f(2x,z)-f(-x,-2z)\tag{3.18}$$

for all $x, z \in X$. Using (3.16) and (3.18), we have

$$2f(x,z) - 2f(-x,-z) = f(2x,z) - f(-2x,-z)$$

By using $f = f^e + f^o$, we get

$$f^o(2x,z) = 2f^o(x,z) \tag{3.19}$$

for all $x, z \in X$. (3.13) and (3.19) yield

$$f^e(2x,z) = 2f^e(x,z) + 2f^e(-x,z) \tag{3.20}$$

for all $x, z \in X$. In (3.20), replace z by $-z$, to get

$$f^e(2x,-z) = 2f^e(x,-z) + 2f^e(x,z) \tag{3.21}$$

for all $x, z \in X$. It follows from (3.20) and (3.21) that

$$f^e(x,z) = f^e(x,-z) \tag{3.22}$$

for all $x, z \in X$. From (3.20) and (3.22), we arrive at

$$f^e(2x,z) = 4f^e(x,z) \tag{3.23}$$

for all $x, z \in X$. From (3.19), (3.23) and using induction, to obtain

$$f^e(x,z) = \frac{1}{4^n} f^e(2^n x,z), \qquad f^o(x,z) = \frac{1}{2^n} f^o(2^n x,z) \tag{3.24}$$

for all $n \in \mathbb{N}$ and all $x, z \in X$.

In (3.11) and (3.12), substituting x, y and z by $-x, -y$ and $-z$, respectively, to get

$$\begin{aligned} &\|2f(-x,-z) + f(-y,-z) + f(y,z) - f(y-x,-z)\| \\ &\quad \le \|f(-x-y,-z)\| \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} &\|2f(-x,-y) + f(-x,-z) + f(x,z) - f(-x,z-y)\| \\ &\quad \le \|f(-x,-y-z)\| \end{aligned} \tag{3.26}$$

for all $x, y, z \in X$. (3.11) and (3.25) yield

$$\begin{aligned} &\|4f^e(x,z) + 4f^e(y,z) - 2f^e(x-y,z)\| \\ &\quad \le \|f(x+y,z)\| + \|f(-x-y,-z)\| \end{aligned} \tag{3.27}$$

and (3.12) and (3.26) implies that

$$\begin{aligned} &\|4f^e(x,y) + 4f^e(x,z) - 2f^e(x,y-z)\| \\ &\quad \le \|f(x,y+z)\| + \|f(-x,-y-z)\| \end{aligned} \tag{3.28}$$

for all $x, y, z \in X$.

Replacing $2^n x$ instead of x and $2^n y$ instead of y in (3.27), we get

$$\begin{aligned} &\|4f^e(2^n x,z) + 4f^e(2^n y,z) - 2f^e(2^n(x-y),z)\| \\ &\quad \le \|f^e(2^n(x+y),z) + f^o(2^n(x+y),z)\| \end{aligned} \tag{3.29}$$

$$+ \|f^e(2^n(-x-y),-z) + f^o(2^n(-x-y),-z)\|$$

for all $n \in \mathbb{N}$ and all $x, y, z \in X$. By using (3.24) and dividing (3.29) by 4^n, we gain

$$\begin{aligned} &\|4f^e(x,z) + 4f^e(y,z) - 2f^e(x-y,z)\| \\ &\quad \leq \left\|f^e(x+y,z) + 2^{-n}f^o((x+y),z)\right\| \\ &\quad\quad + \left\|f^e(x+y,-z) + 2^{-n}f^o(-x-y,-z)\right\| \end{aligned}$$

for all $n \in \mathbb{N}$ and all $x, y, z \in X$. By setting $n \to \infty$, we have

$$\|2f^e(x,z) + 2f^e(y,z) - f^e(x-y,z)\| \leq \|f^e(x+y,z)\| \tag{3.30}$$

for all $x, y, z \in X$.

Similarly, using (3.28) yields

$$\|2f^e(x,y) + 2f^e(x,z) - f^e(x,y-z)\| \leq \|f^e(x,y+z)\| \tag{3.31}$$

for all $x, y, z \in X$. Therefore by (3.30), (3.31) and Lemma 2.6, f^e is a solution of the system of bi-quadratic functional equations

$$\begin{cases} f^e(x+y,z) + f^e(x-y,z) = 2f^e(x,z) + 2f^e(y,z) \\ f^e(x,y+z) + f^e(x,y-z) = 2f^e(x,y) + 2f^e(x,z) \end{cases} \tag{3.32}$$

for all $x, y, z \in X$.

Inequalities (3.11) and (3.12) can be rewritten as

$$\begin{aligned} &\|2f^e(x,z) + 2f^o(x,z) + 2f^e(y,z) - f^e(x-y,z) - f^o(x-y,z)\| \\ &\quad \leq \|f(x+y,z)\| \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} &\|2f^e(x,y) + 2f^o(x,y) + 2f^e(x,z) - f^e(x,y-z) - f^o(x,y-z)\| \\ &\quad \leq \|f(x,y+z)\| \end{aligned} \tag{3.34}$$

for all $x, y, z \in X$. Since f^e satisfies the bi-quadratic functional equations (3.32), from (3.33) and (3.34), we obtain

$$\|f^e(x+y,z) + 2f^o(x,z) - f^o(x-y,z)\| \leq \|f(x+y,z)\|, \tag{3.35}$$

$$\|f^e(x,y+z) + 2f^o(x,y) - f^o(x,y-z)\| \leq \|f(x,y+z)\| \tag{3.36}$$

for all $x, y, z \in X$.

Interchanging the roles of x and y in (3.35) and the roles of y and z in (3.36), to get

$$\|f^e(x+y,z) + 2f^o(y,z) - f^o(y-x,z)\| \leq \|f(x+y,z)\|, \tag{3.37}$$

$$\|f^e(x,y+z) + 2f^o(x,z) - f^o(x,z-y)\| \leq \|f(x,y+z)\| \tag{3.38}$$

for all $x, y, z \in X$.

Adding (3.35) to (3.37) and (3.36) with (3.38) and using triangle inequality yield

$$\begin{aligned}&\|f^e(x+y,z)+f^o(x,z)+f^o(y,z)\|\\&\quad\le\|f^e(x+y,z)+f^o(x+y,z)\|\end{aligned}$$

and

$$\begin{aligned}&\|f^e(x,y+z)+f^o(x,y)+f^o(x,z)\|\\&\quad\le\|f^e(x,y+z)+f^o(x,y+z)\|\end{aligned}$$

which implies that

$$\begin{aligned}&\|f^o(x,z)+f^o(y,z)\|^2+\|f^e(x+y,z)\|^2\\&\qquad+2\mathrm{Re}\,\langle f^o(x,z)+f^o(y,z),f^e(x+y,z)\rangle\\&\quad\le\|f^e(x+y,z)\|^2+\|f^o(x+y,z)\|^2+2\mathrm{Re}\,\langle f^o(x+y,z),f^e(x+y,z)\rangle\end{aligned}$$

and

$$\begin{aligned}&\|f^o(x,y)+f^o(x,z)\|^2+\|f^e(x,y+z)\|^2\\&\qquad+2\mathrm{Re}\,\langle f^o(x,y)+f^o(x,z),f^e(x,y+z)\rangle\\&\quad\le\|f^e(x,y+z)\|^2+\|f^o(x,y+z)\|^2+2\mathrm{Re}\,\langle f^o(x,y+z),f^e(x,y+z)\rangle\end{aligned}$$

for all $x,y,z\in X$. Therefore,

$$\begin{aligned}&\|f^o(x,z)+f^o(y,z)\|^2\\&\qquad+2\mathrm{Re}\,\langle f^o(x,z)+f^o(y,z)-f^o(x+y,z),f^e(x+y,z)\rangle\\&\quad\le\|f^o(x+y,z)\|^2\end{aligned}\tag{3.39}$$

and

$$\begin{aligned}&\|f^o(x,y)+f^o(x,z)\|^2\\&\qquad+2\mathrm{Re}\,\langle f^o(x,y)+f^o(x,z)-f^o(x,y+z),f^e(x,y+z)\rangle\\&\quad\le\|f^o(x,y+z)\|^2\end{aligned}\tag{3.40}$$

for all $x,y,z\in X$.

In (3.39), substituting $-x$ and $-y$ instead of x and y, respectievly and add to (3.39), to gain

$$\|f^o(x,z)+f^o(y,z)\|\le\|f^o(x+y,z)\|\tag{3.41}$$

and similarly, replacing y by $-y$ and z by $-z$ in (3.40) and add with (3.40), lead to

$$\|f^o(x,y)+f^o(x,z)\|\le\|f^o(x,y+z)\|\tag{3.42}$$

for all $x,y,z\in X$. From (3.41), (3.42) and by Lemma 2.5, f^o is bi-additive.

Finally, $f(x,z) = f^e(x,z) + f^o(x,z)$ is a solution of the system of bi-Drygas functional equations (1.4). □

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