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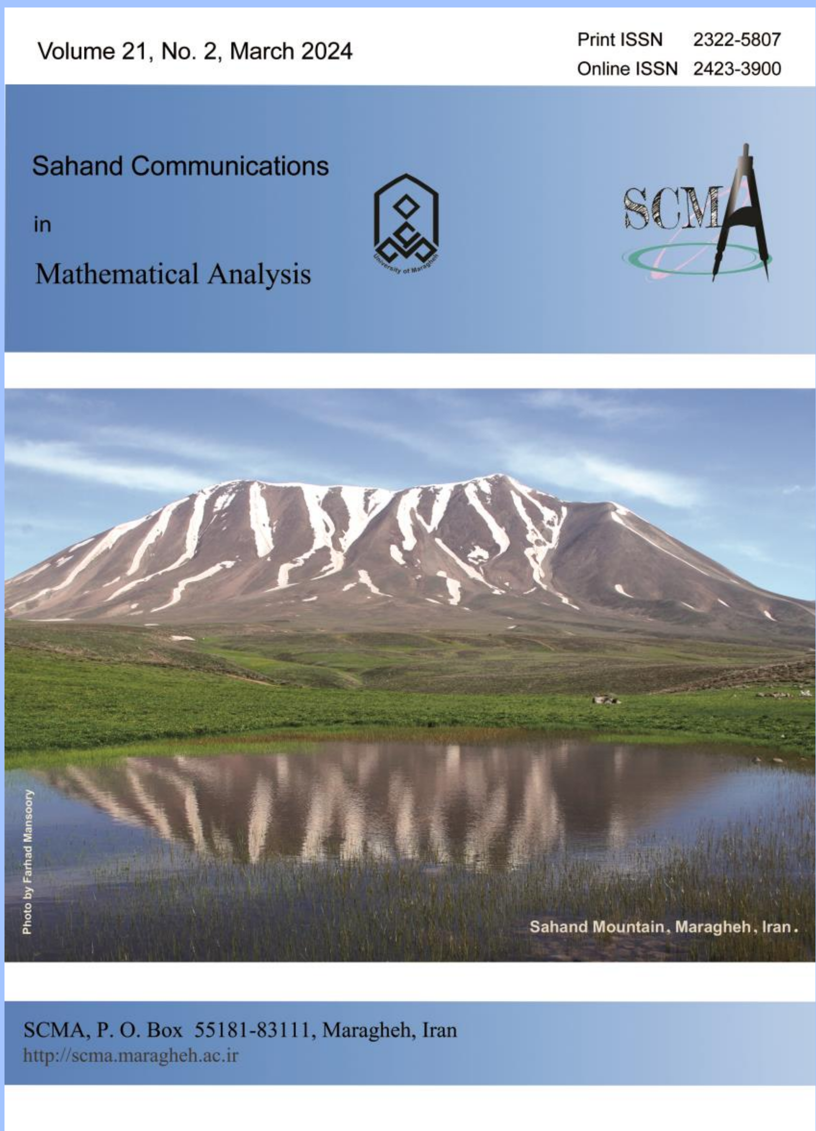

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On Saigo Fractional q-Calculus of a General Class of q-Polynomials

Biniyam Shimelis[1] and Dayalal Suthar[2*]

Abstract. In this paper, we derive Saigo fractional q-integrals of the general class of q-polynomials and demonstrate their application by investigating q-Konhouser biorthogonal polynomial, q-Jacobi polynomials and basic analogue of the Kampé de Fériet function. We have also derived polynomials as a specific example of our significant findings.

1. Introduction and Preliminaries

The computation of fractional q-derivatives (and fractional q-integrals) of special functions of one or more variables is significant. These results can be used to evaluate q-integrals and solve q-integral and q-differential equations. The theory of q-hypergeometric functions of one and more variables has several applications in Applied mathematics, Engineering and Physical Sciences, such as Lie theory, Number theory, Computational complexity, Partition theory, Quantum field theory, and so on. Due to their use in domains including combinatorics, orthogonal polynomials, calculus of variations, basic hypergeometric functions, the theory of relativity and mechanics, quantum difference operators are significant in mathematics [3]. Numerous basic quantum calculus ideas are covered in the book by Kac and Cheung [8]. Researchers have recently paid a lot of attention to q-calculus, and [1, 12, 14, 19, 20, 26], along with other references cited therein, present numerous new findings.

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The fractional calculus has been acknowledged as a tool for the explanation of several phenomena in kinematics, biology, chemistry, finance, etc. over the past two centuries (see [11]). Moreover, q-calculus has been as a method for handling discrete variations of continuous scientific problems (see [8] for more information). It was only a matter of time before those ideas came together.

Inspired by these possibilities, a number of researchers have employed fractional q-calculus operators in the theory of special functions of one or more variables. Recently, Kumar et al. [9, 10] derived several fractional q-integral formulas for the q-analogues of I-function and Aleph function of one and two variables. Vyas et al. [22–24] analyzed the q-analogues of the numerous special functions and subsequent implementation. It has been derived for basic q-generating series, q-trigonometric functions and q-exponential function by Al-Omari et al. [13] utilized for different forms of q-Bessel functions and some power series of special type. Purohit et al. [15–17] derived a unified class of spiral-like functions, including analytic functions with different types of fractional operators in quantum calculus.

To our opinion, there is far too little literature on fractional q-integrability and q-differentiability. As a result of the foregoing research, the goal of this study is to obtain the Saigo fractional q-calculus formula, using series expansion method, for the general class of basic (or q-) polynomials. The term 'general class of basic polynomials' itself indicates the importance of the results, as it enables the derivation of numerous fractional q-calculus formulae for various classical orthogonal q-polynomials.

In q-series theory (see [2]), for a real or complex a and $|q| < 1$, the q-shifted factorial is defined as:

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}\left(1 - aq^k\right), \quad (n \in \mathbb{N}).$$

Equivalently

$$(a;q)_n = \frac{\Gamma_q(a+n)}{\Gamma_q(a)}(1-q)^n.$$

According to Gasper and Rahman [2], the q-gamma function is defined as

$$\begin{aligned}\Gamma_q(\varepsilon) &= \frac{(q;q)_\infty}{(q^\varepsilon;q)_\infty}(1-q)^{1-\varepsilon}\\ &= \frac{(q;q)_{\varepsilon-1}}{(1-q)^{\varepsilon-1}}, \quad (\varepsilon \neq 0,-1,-2,\ldots).\end{aligned}$$

Also, the power series in terms of q-analogue $(a\pm b)^n$ is defined (see [20])

$$(a\pm b)^{(n)} = (a\pm b)_n = a^n\left(\mp b/a;q\right)_n = a^n\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q q^{k(k-1)/2}\left(\pm b/a\right)^k, \quad (n\in\mathbb{N}), \tag{1.1}$$

where the q-binomial coefficient is defined as:

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{(q^{-n};q)_k}{(q;q)_k}\left(-q^n\right)^k q^{-k(k-1)/2}.$$

For a bounded sequence of real or complex numbers, let $f(z) = \sum\limits_{n=-\infty}^{+\infty} A_n z^n$ be a power series in z, then (see, for instance, [4], p. 502, eqn. (3.18) and [6])

$$f[(z\pm x)] = \sum_{n=-\infty}^{+\infty} A_n z^n\left(\mp x/z;q\right)_n. \tag{1.2}$$

The generalized basic hypergeometric series (cf. Gasper and Rahman [2]) is given by

$${}_r\varphi_s\left[\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array};q,x\right] = \sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}x^n\left\{(-1)^n q^{(n(n-1)/2)}\right\}^{(1+s-r)},$$

where for convergence, we have $|q|<1$ and $|x|<1$ if $r=s+1$, and for any x if $r\le s$.

The abnormal type of generalized basic hypergeometric series ${}_2\phi_1(.)$ is defined as

$${}_r\phi_s\left[\begin{array}{c} a_1,\ldots,a_r;\ q,x \\ b_1,\ldots,b_s;\ q^{\lambda} \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}x^n\left\{q^{(\lambda n(n+1)/2)}\right\}$$

where $\lambda>0$ and $|q|<1$.

In terms of a bounded complex sequence $\{S_{n,q}\}_{n=0}^{\infty}$, the family of basic (or q-) polynomials $f_{n,N}(x;q)$ (cf. Srivastava and Agarwal [21]) is defined as:

$$f_{n,N}(x;q) = \sum_{j=0}^{[n/N]}\left[\begin{array}{c} n \\ Nj \end{array}\right]_q S_{n,q}\,x^j, \quad (n=0,1,2,\ldots), \tag{1.3}$$

where N is a positive integer.

On suitably specializing the coefficient $S_{n,q}$ the q-polynomials family $f_{n,N}(x;q)$ yields a number of known q-polynomials as its special cases. These include, among others,the q-Laguerre polynomials, the q-Hermite polynomials, the Wall polynomials, the q-Jacobi polynomials, the q-Konhauser polynomials and several others.

2. q-Analogues of Saigo's Fractional Integrals and Derivatives

Recently, Purohit and Yadav [18] recently defined q-analogues of Saigo's fractional integral operators with the constraint that one of the parameters η is a non-negative integer. With that constraint, it was not possible to provide a definition of fractional derivatives. Garg and Chanchalani [5] provide the following formulations of q-analogues of Saigo's fractional integral operators to address these issues.

For $\Re(\varepsilon) > 0$, ϑ and η being real or complex, the generalized fractional q-integral operators $I_q^{\varepsilon,\vartheta,\eta}(.)$ and $K_q^{\varepsilon,\vartheta,\eta}(.)$ defined in the following manner:

$$I_q^{\varepsilon,\vartheta,\eta} f(x) = \frac{x^{-\vartheta-1}}{\Gamma_q(\varepsilon)} \int_0^x (tq/x;q)_{\varepsilon-1} \sum_{m=0}^{\infty} \frac{\left(q^{\varepsilon+\vartheta};q\right)_m \left(q^{-\eta};q\right)_m}{\left(q^{\varepsilon};q\right)_m (q;q)_m} \times (-1)^m q^{(\eta-\vartheta)m} q^{-\binom{m}{2}} \left(\frac{t}{x}-1\right)_q^m f(t)\, d_q t, \tag{2.1}$$

and

$$K_q^{\varepsilon,\vartheta,\eta} f(x) = \frac{q^{-\varepsilon(\varepsilon+1)/2-\vartheta}}{\Gamma_q(\varepsilon)} \int_x^{\infty} (x/t;q)_{\varepsilon-1} t^{-\vartheta-1} \times \sum_{m=0}^{\infty} \frac{\left(q^{\varepsilon+\vartheta};q\right)_m \left(q^{-\eta};q\right)_m}{\left(q^{\varepsilon};q\right)_m (q;q)_m} (-1)^m q^{(\eta-\vartheta)m} \times q^{-\binom{m}{2}} \left(\frac{x}{qt}-1\right)_q^m f\left(tq^{1-\varepsilon}\right) d_q t, \tag{2.2}$$

where $0 < |q| < 1$, for a real valued function $f(x)$ on $(0,\infty)$.

The q-integrals of a function $f(t)$ are defined as [2] follows

$$\int_0^a f(t)\, d_q t = a(1-q) \sum_{k=0}^{\infty} q^k f\left(aq^k\right), \tag{2.3}$$

and

$$\int_a^{\infty} f(t)\, d_q t = a(1-q) \sum_{k=1}^{\infty} q^{-k} f\left(aq^{-k}\right). \tag{2.4}$$

In view of (2.3) and (2.4), the definitions given by (2.1) and (2.2), can be expressed as

$$I_q^{\varepsilon,\vartheta,\eta} f(x) = x^{-\vartheta} (1-q)^{\varepsilon} \sum_{m=0}^{\infty} \frac{\left(q^{\varepsilon+\vartheta};q\right)_m (q^{-\eta};q)_m}{(q;q)_m} \times q^{(\eta-\vartheta+1)m} \sum_{k=0}^{\infty} q^k \frac{(q^{\varepsilon+m};q)_k}{(q;q)_k} f\left(xq^{k+m}\right),$$

and

$$K_q^{\varepsilon,\vartheta,\eta} f(x) = x^{-\vartheta} q^{-\varepsilon(\varepsilon+1)/2} (1-q)^{\varepsilon} \sum_{m=0}^{\infty} \frac{\left(q^{\varepsilon+\vartheta};q\right)_m (q^{-\eta};q)_m}{(q;q)_m} \times q^{\eta m} \sum_{k=0}^{\infty} q^{\vartheta k} \frac{(q^{\varepsilon+m};q)_k}{(q;q)_k} f\left(xq^{-\varepsilon-k-m}\right).$$

The q-analogues of Saigo fractional integrals of order $\varepsilon \in \mathbb{C}$, $(m-1 < \Re(\varepsilon) \leq m)$, $m \in \mathbb{N}$, for a real valued function $f(x)$ on $(0,\infty)$ are defined as follows

$$D_q^{\varepsilon,\vartheta,\eta} f(x) = D_q^m I_q^{-\varepsilon+m,-\vartheta-m,\varepsilon+\eta-m} f(x),$$

and

$$p_q^{\varepsilon,\vartheta,\eta} f(x) = q^{\varepsilon(\varepsilon+\vartheta)} \left(-q^{-(\varepsilon+\vartheta)} D_q\right)^m K_q^{-\varepsilon+m,-\vartheta-m,\varepsilon+\eta} f(x),$$

where $0<|q|<1$; ϑ and η being real or complex, $I_q^{\varepsilon,\vartheta,\eta}$ and $K_q^{\varepsilon,\vartheta,\eta}$ are given by (2.1) and (2.2) respectively.

For $0<|q|<1$; $\Re(\varepsilon)>0$, ϑ and η being real or complex. Images of power function under fractional q-integrals $I_q^{\varepsilon,\vartheta,\eta}$ and $K_q^{\varepsilon,\vartheta,\eta}$ are given by:

$$I_q^{\varepsilon,\vartheta,\eta}(x^\mu) = \frac{\Gamma_q(\mu+1)\Gamma_q(\mu-\vartheta+\eta+1)}{\Gamma_q(\mu-\vartheta+1)\Gamma_q(\mu+\varepsilon+\eta+1)} x^{\mu-\vartheta}, \tag{2.5}$$

provided $\Re(\mu+1)>0$ and $\Re(\mu-\vartheta+\eta+1)>0$. Also

$$K_q^{\varepsilon,\vartheta,\eta}(x^\mu) = \frac{\Gamma_q(\vartheta-\mu)\Gamma_q(\eta-\mu)}{\Gamma_q(-\mu)\Gamma_q(\vartheta+\varepsilon-\mu+\eta)} x^{\mu-\vartheta} q^{-\varepsilon\mu-\varepsilon(\varepsilon+1)/2}, \tag{2.6}$$

provided $\Re(\vartheta-\mu)>0$ and $\Re(\eta-\mu)>0$.

For $0<|q|<1$; $n-1<\Re(\varepsilon)\leq n$, $\quad n\in\mathbb{N}, \vartheta, \eta$ and μ being real or complex. Images of power function under fractional q-derivatives $D_q^{\varepsilon,\vartheta,\eta}$ and $P_q^{\varepsilon,\vartheta,\eta}$ are given by

$$D_q^{\varepsilon,\vartheta,\eta}(x^\mu) = \frac{\Gamma_q(\mu+1)\Gamma_q(\mu+\varepsilon+\vartheta+\eta+1)}{\Gamma_q(\mu+\vartheta+1)\Gamma_q(\mu+\eta+1)} x^{\mu+\vartheta},$$

provided $\Re(\mu+1)>0$ and $\Re(\mu+\varepsilon+\vartheta+\eta+1)>0$.

$$P_q^{\varepsilon,\vartheta,\eta}(x^{\mu})=\frac{\Gamma_q(-\vartheta-\mu)\,\Gamma_q(\varepsilon+\eta-\mu)}{\Gamma_q(-\mu)\,\Gamma_q(-\vartheta+\eta-\mu)}x^{\mu-\vartheta}q^{\varepsilon(\vartheta+\mu)+\varepsilon(\varepsilon-1)/2},$$

provided $\Re(-\vartheta-\mu)>0$ and $\Re(\varepsilon+\eta-\mu)>0$.

Furthermore, the operators defined in (2.1) and (2.2) can be consider extensions of the Riemann-Liouville, Weyl and Kober fractional q-integral operators with the following functional relations:

$$\begin{aligned}
&I_q^{\varepsilon,0,\eta}f(x)=I_q^{\eta,\varepsilon}f(x),\\
&I_q^{\varepsilon,-\varepsilon,\eta}f(x)=I_q^{\varepsilon}f(x),\\
&K_q^{\varepsilon,0,\eta}f(x)=q^{-\varepsilon(\varepsilon+1)/2}K_q^{\eta,\varepsilon}f(x),\\
&K_q^{\varepsilon,-\varepsilon,\eta}f(x)=K_q^{\varepsilon}f(x).
\end{aligned}$$

3. Main Results

In this section, we shall evaluate the following Saigo fractional q-integrals involving a general class of basic polynomials. The main theorems are as follows:

Theorem 3.1. *Consider a x^k-weighted basic binomial function and the family of q-polynomials, and then generalized fractional q-integral of Saigo type $I_q^{\varepsilon,\vartheta,\eta}(.)$ is given by*

$$\begin{aligned}
(3.1)\quad & I_q^{\varepsilon,\vartheta,\eta}\left\{x^k\,(x+\xi)^{(-\lambda)}\,f_{n,N}\left(x^{\rho}\left(xq^{-\lambda}+\xi\right)^{(-\sigma)};q\right)\right\}\\
&=x^{k-\vartheta}\,\xi^{-\lambda}\sum_{j=0}^{[n/N]}\begin{bmatrix} n\\ Nj\end{bmatrix}_q S_{n,q}\left(\frac{x^{\rho}}{\xi^{\sigma}}\right)^{j}\\
&\times\frac{\Gamma_q(k+\rho j+1)\;\Gamma_q(k+\rho j+1-\vartheta+\eta)}{\Gamma_q(k+\rho j+1-\vartheta)\;\Gamma_q(k+\rho j+1+\eta+\varepsilon)}\\
&\times{}_3\varphi_2\left[\begin{matrix} q^{\lambda+\sigma j},q^{1+k+\rho j},\,q^{1+k-\vartheta+\eta+\rho j};\\ q^{1+k-\vartheta+\rho j},\,q^{1+k+\eta+\varepsilon+\rho j};\end{matrix}\;q,-\left(xq^{-\lambda-\sigma j}/\xi\right)\right],
\end{aligned}$$

where $\Re(k+\rho j+1)>0$, $\Re(k+\rho j+1-\vartheta+\eta)>0$, $\left|xq^{-\lambda-\sigma j}/\xi\right|<1$, $\min(k,\lambda,\sigma,\rho)>0$, $n=0,1,2,\ldots$ *and N is a positive integer.*

Proof. For a simple and direct proof of the generalized fractional q-integral formula (3.1), we express the general class of q-polynomials $f_{n,N}$ occurring on its left hand side (say L) in the series form given by equation (1.3), then with help of relation (1.2) it yields to

$$L=I_q^{\varepsilon,\vartheta,\eta}\left\{x^k\,(x+\xi)^{(-\lambda)}\sum_{j=0}^{[n/N]}\begin{bmatrix} n\\ Nj\end{bmatrix}_q S_{n,q}x^{\rho j}\xi^{-\sigma j}\left(\frac{-xq^{-\lambda}}{\xi};q\right)_{(-\sigma j)}\right\},$$

On using the q-binomial theorem given by (1.1), we get

$$L=\xi^{-\lambda}\sum_{j=0}^{[n/N]}\begin{bmatrix} n \\ Nj \end{bmatrix}_q S_{n,q}\,\xi^{-\sigma j}\sum_{m=0}^{\infty}\frac{\left(q^{\lambda+\sigma j};q\right)_m}{(q;q)_m}\left(\frac{-q^{-\lambda-\sigma j}}{\xi}\right)^m \times I_q^{\varepsilon,\vartheta,\eta}\left\{x^{k+m+\rho j}\right\},$$

On using (2.5), we arrive at

$$L=\xi^{-\lambda}\sum_{j=0}^{[n/N]}\begin{bmatrix} n \\ Nj \end{bmatrix}_q S_{n,q}\,\xi^{-\sigma j}\sum_{m=0}^{\infty}\frac{\left(q^{\lambda+\sigma j};q\right)_m}{(q;q)_m}\left(\frac{-q^{-\lambda-\sigma j}}{\xi}\right)^m \times\frac{\Gamma_q\left(k+m+\rho j+1\right)\Gamma_q\left(k+m+\rho j-\vartheta+\eta+1\right)}{\Gamma_q\left(k+m+\rho j-\vartheta+1\right)\Gamma_q\left(k+m+\rho j+\varepsilon+\eta+1\right)} \times x^{k+m+\rho j-\vartheta}.$$

Rearrangement of parameters and simple calculation, we obtain

$$L=\xi^{-\lambda}x^{k-\vartheta}\sum_{j=0}^{[n/N]}\begin{bmatrix} n \\ Nj \end{bmatrix}_q S_{n,q}\left(\frac{x^{\rho}}{\xi^{\sigma}}\right)^j \times\sum_{m=0}^{\infty}\frac{\left(q^{\lambda+\sigma j};q\right)_m}{(q;q)_m}\left(\frac{-q^{-\lambda-\sigma j}}{\xi}\right)^m\frac{\Gamma_q\left(k+\rho j+1\right)}{\Gamma_q\left(k+\rho j-\vartheta+1\right)} \times\frac{\Gamma_q\left(k+\rho j-\vartheta+\eta+1\right)\left(q^{k+\rho j+1},q\right)_m\left(q^{k+\rho j-\vartheta+\eta+1},q\right)_m}{\Gamma_q\left(k+\rho j+\varepsilon+\eta+1\right)\left(q^{k+\rho j-\vartheta+1},q\right)_m\left(q^{k+\rho j+\varepsilon+\eta+1},q\right)_m}.$$

Furthermore, by utilizing the series expansion of ${}_3\phi_2(.)$ during the course of analysis, we arrive at the required result (3.1). □

If we set $\vartheta=0$ and $\vartheta=-\varepsilon$, in the Theorem 3.1, then we obtain the following corollaries:

Corollary 3.2. *Consider x^k-weighted basic binomial function and the family of q-polynomials, and then fractional q-integral of Kober type $I_q^{\eta,\varepsilon}(.)$ is given by*

$$I_q^{\eta,\varepsilon}\left\{x^k\left(x+\xi\right)^{(-\lambda)}f_{n,N}\left(x^{\rho}\left(xq^{-\lambda}+\xi\right)^{(-\sigma)};q\right)\right\} =x^k\,\xi^{-\lambda}\sum_{j=0}^{[n/N]}\begin{bmatrix} n \\ Nj \end{bmatrix}_q S_{n,q}\left(\frac{x^{\rho}}{\xi^{\sigma}}\right)^j\frac{\Gamma_q\left(k+\rho j+1+\eta\right)}{\Gamma_q\left(k+\rho j+1+\eta+\varepsilon\right)} \times{}_2\varphi_1\left[\begin{matrix} q^{\lambda+\sigma j},q^{1+k-\vartheta+\eta+\rho j}; \\ q^{1+k+\eta+\varepsilon+\rho j}; \end{matrix}\; q,-\left(xq^{-\lambda-\sigma j}/\xi\right)\right],$$

where $\Re(k+\rho j+1+\eta)>0$, $\left|xq^{-\lambda-\sigma j}/\xi\right|<1$, $\min(k,\lambda,\sigma,\rho)>0$, $n=0,1,2,\ldots$ *and* N *is a positive integer.*

Corollary 3.3. *Consider a* x^k*-weighted basic binomial function and the family of* q*-polynomials, and then fractional* q*-integral of Riemann-Liouville type* $I_q^{\varepsilon}(.)$ *is given by*

$$\begin{aligned} & I_q^{\varepsilon}\left\{x^k\,(x+\xi)^{(-\lambda)}\,f_{n,N}\left(x^{\rho}\left(xq^{-\lambda}+\xi\right)^{(-\sigma)};q\right)\right\} \\ & = x^{k+\varepsilon}\,\xi^{-\lambda}\sum_{j=0}^{[n/N]}\left[\begin{array}{c} n \\ Nj \end{array}\right]_q S_{n,q}\left(\frac{x^{\rho}}{\xi^{\sigma}}\right)^j\frac{\Gamma_q\left(k+\rho j+1\right)}{\Gamma_q\left(k+\rho j+1+\varepsilon\right)} \\ & \times {}_2\varphi_1\left[\begin{array}{c} q^{\lambda+\sigma j},q^{1+k+\rho j}\,; \\ q^{1+k-\vartheta+\rho j}\,; \end{array}\ q,-\left(xq^{-\lambda-\sigma j}/\xi\right)\right], \end{aligned}$$

where $\Re(k+\rho j+1)>0$, $\left|xq^{-\lambda-\sigma j}/\xi\right|<1$, $\min(k,\lambda,\sigma,\rho)>0$, $n=0,1,2,\ldots$ *and* N *is a positive integer.*

Theorem 3.4. *Consider a* x^k*-weighted basic binomial function and the family of* q*-polynomials, and then generalized fractional* q*-integral of Saigo type* $K_q^{\varepsilon,\vartheta,\eta}(.)$ *is given by*

(3.2)

$$\begin{aligned} & K_q^{\varepsilon,\vartheta,\eta}\left\{x^k\,(x+\xi)^{(-\lambda)}\,f_{n,N}\left(x^{\rho}\left(xq^{-\lambda}+\xi\right)^{(-\sigma)};q\right)\right\} \\ & = x^{k-\vartheta}q^{-\varepsilon k-\varepsilon(\varepsilon+1)/2}\,\xi^{-\lambda}\sum_{j=0}^{[n/N]}\left[\begin{array}{c} n \\ Nj \end{array}\right]_q S_{n,q}\,\xi^{-\sigma j} \\ & \times\frac{\Gamma_q\left(\vartheta-k-\rho j\right)\,\Gamma_q\left(\eta-k-\rho j\right)}{\Gamma_q\left(-k-\rho j\right)\,\Gamma_q\left(\vartheta+\varepsilon-k+\eta-\rho j\right)}\left(\frac{x}{q^{\varepsilon}}\right)^{\rho j} \\ & \times\ {}_3\varphi_2\left[\begin{array}{c} q^{\lambda+\sigma j},\ q^{1+k+\rho j},\ q^{1-\vartheta-\varepsilon+k-\eta+\rho j}; \\ q^{1-\vartheta+k+\rho j},\ q^{1-\eta+k+\rho j}\ q; \end{array}\ q,-\left(xq^{-\varepsilon-\lambda-\sigma j}/\xi\right)\right], \end{aligned}$$

where $\Re\left(\vartheta-k-\rho j\right)>0$, $\Re\left(\eta-k-\rho j\right)>0$, $\left|xq^{-\varepsilon-\lambda-\sigma j}/\xi\right|<1$, $\min(k,\lambda,\sigma,\rho)>0$, $n=0,1,2,\ldots$ *and* N *is a positive integer.*

Proof. For a simple and direct proof of the generalized fractional q-integral formula (3.2), we express the general class of q-polynomials $f_{n,N}$ occurring its left hand side (say ℓ) in the series form given by equation (1.3), then with help of relation (1.2) it yields to

$$\ell=K_q^{\varepsilon,\vartheta,\eta}\left\{x^k\,(x+\xi)^{(-\lambda)}\sum_{j=0}^{[n/N]}\left[\begin{array}{c} n \\ Nj \end{array}\right]_q S_{n,q}x^{\rho j}\xi^{-\sigma j}\left(\frac{-xq^{-\lambda}}{\xi};q\right)_{(-\sigma j)}\right\}$$

On using the q-binomial theorem given by (1.1), we get

$$\ell = \xi^{-\lambda} \sum_{j=0}^{[n/N]} \left[\begin{array}{c} n \\ Nj \end{array} \right]_q S_{n,q}\, \xi^{-\sigma j} \sum_{m=0}^{\infty} \frac{\left(q^{\lambda+\sigma j}; q\right)_m}{(q;q)_m} \times \left(\frac{-q^{-\lambda-\sigma j}}{\xi} \right)^m K_q^{\varepsilon,\vartheta,\eta} \left\{ x^{k+m+\rho j} \right\},$$

On using (2.6), we obtain

$$\ell = \xi^{-\lambda} \sum_{j=0}^{[n/N]} \left[\begin{array}{c} n \\ Nj \end{array} \right]_q S_{n,q}\, \xi^{-\sigma j} \sum_{m=0}^{\infty} \frac{\left(q^{\lambda+\sigma j}; q\right)_m}{(q;q)_m} \left(\frac{-q^{-\lambda-\sigma j}}{\xi} \right)^m \times \frac{\Gamma_q(\vartheta - k - m - \rho j)\, \Gamma_q(\eta - k - m - \rho j)}{\Gamma_q(-k-m-\rho j)\, \Gamma_q(\vartheta + \varepsilon - k - m - \rho j + \eta)} \times x^{k+m+\rho j - \vartheta} q^{-\varepsilon(k+m+\rho j) - \varepsilon(\varepsilon+1)/2},$$

Using the formula $\left(q^{\eta-k-\rho j}, q\right)_{-m} = \frac{\Gamma_q(\eta-k-m-\rho j)}{\Gamma_q(\eta-k-\rho j)} (1-q)^{-m}$, we get

$$\ell = \xi^{-\lambda} \sum_{j=0}^{[n/N]} \left[\begin{array}{c} n \\ Nj \end{array} \right]_q S_{n,q} \xi^{-\sigma j} \sum_{m=0}^{\infty} \frac{\left(q^{\lambda+\sigma j}; q\right)_m}{(q;q)_m} \left(\frac{-q^{-\lambda-\sigma j}}{\xi} \right)^m \times \frac{\Gamma_q(\vartheta - k - \rho j)\ \Gamma_q(\eta - k - \rho j)\ \left(q^{\vartheta-k-\rho j}, q\right)_{-m}}{\Gamma_q(-k-\rho j)\ \Gamma_q(\vartheta + \varepsilon - k + \eta - \rho j)\ \left(q^{-k-\rho j}, q\right)_{-m}} \times \frac{\left(q^{\eta-k-\rho j}, q\right)_{-m}}{\left(q^{\vartheta+\varepsilon-k+\eta-\rho j}, q\right)_{-m}} x^{k+m+\rho j-\vartheta} q^{\varepsilon(-k-m-\rho j) - \varepsilon(\varepsilon+1)/2}.$$

Further using the series expansion of ${}_3\phi_2(.)$ during the course of analysis, we arrive at the required result (3.2). □

If we set $\vartheta = 0$ and $\vartheta = -\varepsilon$, in the Theorem 3.4, then we obtain the following corollaries:

Corollary 3.5. *Consider a x^k-weighted basic binomial function and the family of q-polynomials, and then fractional q-integral of generalized Weyl type $K_q^{\eta,\varepsilon}(.)$ is given by*

$$K_q^{\eta,\varepsilon} \left\{ x^k (x+\xi)^{(-\lambda)} f_{n,N} \left(x^\rho \left(xq^{-\lambda} + \xi \right)^{(-\sigma)}; q \right) \right\} = q^{-\varepsilon(\varepsilon+1)/2}\, \xi^{-\lambda} \left(\frac{x}{q^\varepsilon} \right)^k \sum_{j=0}^{[n/N]} \left[\begin{array}{c} n \\ Nj \end{array} \right]_q \times S_{n,q}\, \xi^{-\sigma j} \left(\frac{x}{q^\varepsilon} \right)^{\rho j} \frac{\Gamma_q(\eta - k - \rho j)}{\Gamma_q(\varepsilon + \eta - k - \rho j)}$$

$$\times {}_2\varphi_1 \left[\begin{array}{c} q^{\lambda+\sigma j}, q^{1-\eta-\varepsilon+k+\rho j}; \\ q^{1-\eta+k+\rho j}; \end{array} q, -\left(xq^{-\varepsilon-\lambda-\sigma j}/\xi\right)\right],$$

where $\Re(\eta - k - \rho j) > 0$, $\left|xq^{-\varepsilon-\lambda-\sigma j}/\xi\right| < 1$, $\min(k, \lambda, \sigma, \rho) > 0$, $n = 0, 1, 2, \ldots$ *and* N *is a positive integer.*

Corollary 3.6. *Consider a* x^k*-weighted basic binomial function and the family of q-polynomials, and then fractional q-integral of Weyl type* $K_q^{\varepsilon}(.)$ *is given by*

$$K_q^{\varepsilon}\left\{x^k (x+\xi)^{(-\lambda)} f_{n,N}\left(x^{\rho}\left(xq^{-\lambda}+\xi\right)^{(-\sigma)}; q\right)\right\}$$
$$= x^{k+\varepsilon} q^{\varepsilon(-k)-\varepsilon(\varepsilon+1)/2}\ \xi^{-\lambda} \sum_{j=0}^{[n/N]} \left[\begin{array}{c} n \\ Nj \end{array} \right]_q$$
$$\times S_{n,q}\, \xi^{-\sigma j} \left(\frac{x}{q^{\varepsilon}}\right)^{\rho j} \frac{\Gamma_q(-\varepsilon-k-\rho j)}{\Gamma_q(-k-\rho j)}$$
$$\times {}_2\varphi_1 \left[\begin{array}{c} q^{\lambda+\sigma j},\ q^{1+k+\rho j}; \\ q^{1+\varepsilon+k+\rho j}\ ; \end{array} q, -\left(xq^{-\varepsilon-\lambda-\sigma j}/\xi\right)\right],$$

where $\Re(-\varepsilon - k - \rho j) > 0$, $\left|xq^{-\varepsilon-\lambda-\sigma j}/\xi\right| < 1$, $\min(k, \lambda, \sigma, \rho) > 0$, $n = 0, 1, 2, \ldots$ *and* N *is a positive integer.*

Theorem 3.7. *If* $\Re(k + \rho j + 1) > 0$, $\Re(\eta + k + \rho j + \delta i + 1 - \vartheta) > 0$, $\left|\frac{xq^{-\lambda-\sigma j}}{\xi}\right| < 1$, $\min(k, \lambda, \rho, \delta, \sigma) > 0$, *and* $I_q^{\varepsilon,\vartheta,\eta}(.)$ *be generalized fractional q-integral operator* (2.1) *of Saigo type, then the following result holds:*

(3.3)

$$I_q^{\varepsilon,\vartheta,\eta}\left\{x^k (x+\xi)^{(-\lambda)} f_{n,N}\left(x^{\rho}\left(xq^{-\lambda}+\xi\right)^{(-\sigma)}; q\right) f_{m,M}\left(x^{\delta}; q\right)\right\}$$
$$= x^{k-\vartheta}\, \xi^{-\lambda} \sum_{j=0}^{[n/N]} \sum_{i=0}^{[m/M]} \left[\begin{array}{c} n \\ Nj \end{array} \right]_q \left[\begin{array}{c} m \\ Mi \end{array} \right]_q S_{n,q}\, S_{m,q}\, \frac{x^{\rho j+\delta i}}{\xi^{\sigma j}}$$
$$\times \frac{\Gamma_q(k+\rho j+\delta i+1)}{\Gamma_q(k+\rho j+\delta i+1-\vartheta)} \frac{\Gamma_q(k+\rho j+\delta i+1-\vartheta+\eta)}{\Gamma_q(k+\rho j+\delta i+1+\eta+\varepsilon)}$$
$$\times\ {}_3\phi_2 \left[\begin{array}{c} q^{\lambda+\sigma j}, q^{1+k+\rho j+\delta i}, q^{1+k-\vartheta+\eta+\rho j+\delta i}; \\ q^{1+k-\vartheta+\rho j+\delta i}, q^{1+k+\eta+\varepsilon+\rho j+\delta i}; \end{array} q, -\left(xq^{-\lambda-\sigma j}/\xi\right)\right],$$

where $n, m = 0, 1, 2, \ldots$ *and N, M are arbitrary positive integer.*

Proof. Proceeding as in Theorem 3.1, one can easily prove the Theorem 3.7. □

Again, if we set $\vartheta = 0$, and $\vartheta = -\varepsilon$ then by using the functional relations (2.9) and (2.10), the Theorem 3.7 yields to the following corollaries:

Corollary 3.8. *If* $\Re(\eta + k + \rho j + \delta i + 1) > 0, \min\ (k, \lambda, \rho, \delta, \sigma) > 0$, $\left|\frac{xq^{-\lambda-\sigma j}}{\xi}\right| < 1$, *and* $I_q^{\eta,\varepsilon}(.)$ *be fractional q-integral operator of Kober type, then the following result holds:*

$$I_q^{\eta,\varepsilon}\left\{x^k\,(x+\xi)^{(-\lambda)}\,f_{n,N}\left(x^{\rho}\left(xq^{-\lambda}+\xi\right)^{(-\sigma)};q\right)\,f_{m,M}\left(x^{\delta};q\right)\right\}$$

$$= x^k\,\xi^{-\lambda}\sum_{j=0}^{[n/N]}\sum_{i=0}^{[m/M]}\left[\begin{array}{c} n \\ Nj \end{array}\right]_q\left[\begin{array}{c} m \\ Mi \end{array}\right]_q S_{n,q}S_{m,q}\frac{x^{\rho j+\delta i}}{\xi^{\sigma j}}$$

$$\times\frac{\Gamma_q\,(k+\rho j+\delta i+1+\eta)}{\Gamma_q\,(k+\rho j+\delta i+1+\eta+\varepsilon)}$$

$$\times\ {}_2\varphi_1\left[\begin{array}{c} q^{\lambda+\sigma j},\, q^{1+k-\vartheta+\eta+\rho j+\delta i}\,; \\ q^{1+k+\eta+\varepsilon+\rho\, j+\delta i}\,; \end{array}\ q, -\left(xq^{-\lambda-\sigma\, j}/\xi\right)\right],$$

where $n, m\ = 0, 1, 2, \ldots,$ *and N, M are arbitrary positive integer.*

Corollary 3.9. *If* $\Re(k + \rho j + \delta i + 1) > 0, \left|\frac{xq^{-\lambda-\sigma j}}{\xi}\right| < 1, \min(k, \lambda, \rho, \delta, \sigma) > 0$, *and* $I_q^{\varepsilon}(.)$ *be fractional q-integral operator of Riemann-Liouville type the following result holds:*

$$I_q^{\varepsilon}\left\{x^k\,(x+\xi)^{(-\lambda)}\,f_{n,N}\left(x^{\rho}\left(xq^{-\lambda}+\xi\right)^{(-\sigma)};q\right)\,f_{m,M}\left(x^{\delta};q\right)\right\}$$

$$= x^{k+\varepsilon}\,\xi^{-\lambda}\sum_{j=0}^{[n/N]}\sum_{i=0}^{[m/M]}\left[\begin{array}{c} n \\ Nj \end{array}\right]_q\left[\begin{array}{c} m \\ Mi \end{array}\right]_q\frac{\Gamma_q\,(k+\rho j+\delta i+1)}{\Gamma_q\,(k+\rho j+\delta i+1+\varepsilon)}$$

$$\times\, S_{n,q}\,S_{m,q}\frac{x^{\rho j+\delta i}}{\xi^{\sigma j}}\ {}_2\,\varphi_1\left[\begin{array}{c} q^{\lambda+\sigma j},\, q^{1+k+\rho j+\delta i}\,; \\ q^{1+k+\varepsilon+\rho\, j+\delta i}\,; \end{array}\ q, -\left(xq^{-\lambda-\sigma\, j}/\xi\right)\right],$$

where $n, m\ = 0, 1, 2, \ldots,$ *and N, M are arbitrary positive integer.*

Theorem 3.10. *If* $\Re(\vartheta - k - \rho j - \delta i) > 0, \quad \Re(\eta - k - \rho j - \delta i) > 0$, $\left|\frac{xq^{-\varepsilon-\lambda-\sigma j}}{\xi}\right| < 1$, *min* $(k, \lambda, \rho, \delta, \sigma) > 0$, *and* $K_q^{\varepsilon,\vartheta,\eta}(.)$ *be generalized fractional q-integral operator* (2.2) *of Saigo type, then the following result holds:*

(3.4)

$$K_q^{\varepsilon,\vartheta,\eta}\left\{x^k\,(x+\xi)^{(-\lambda)}\,f_{n,N}\left(x^{\rho}\left(xq^{-\lambda}+\xi\right)^{(-\sigma)};q\right)\,f_{m,M}\left(x^{\delta};q\right)\right\}$$

$$= x^{k-\vartheta} q^{-\varepsilon k-\varepsilon(\varepsilon+1)/2}\ \xi^{-\lambda} \sum_{j=0}^{[n/N]} \sum_{i=0}^{[m/M]} \begin{bmatrix} n \\ Nj \end{bmatrix}_q \begin{bmatrix} m \\ Mi \end{bmatrix}_q S_{n,q} S_{m,q}$$
$$\times\, \xi^{-\sigma j} \left(\frac{x}{q^{\varepsilon}}\right)^{\rho j+\delta i} \frac{\Gamma_q(\vartheta-k-\rho j-\delta i)\ \Gamma_q(\eta-k-\rho j-\delta i)}{\Gamma_q(-k-\rho j-\delta i)\ \Gamma_q(\vartheta+\varepsilon-k+\eta-\rho j-\delta i)}$$
$$\times\, {}_3\varphi_2 \left[\begin{matrix} q^{\lambda+\sigma j}, q^{1+k+\rho j+\delta i}, q^{1-\vartheta-\varepsilon+k-\eta+\rho j+\delta i}; \\ q^{1-\vartheta+k+\rho j+\delta i}, q^{1-\eta+k+\rho j+\delta i}; \end{matrix}\ q, -\left(xq^{-\varepsilon-\lambda-\sigma j}/\xi\right) \right],$$

where $n, m = 0, 1, 2, \ldots,$ *and* N, M *are arbitrary positive integer.*

Proof. Proceeding as in Theorem 3.4, one can easily prove the Theorem 3.10. □

Again, if we set $\vartheta = 0$, and $\vartheta = -\varepsilon$; then on using the functional relation (2.11) and (2.12), the Theorem 3.10 yields to the following corollaries:

Corollary 3.11. *If* $\Re(\eta-k-\rho j-\delta i) > 0, \left|\frac{xq^{\varepsilon-\lambda-\sigma j}}{\xi}\right| < 1, \min(k, \lambda, \rho, \delta, \sigma) > 0,$ *and* $K_q^{\eta,\varepsilon}(.)$ *be generalized fractional q-integral operator of generalized Weyl type, then the following result holds:*

$$K_q^{\eta,\varepsilon}\left\{x^k (x+\xi)^{(-\lambda)} f_{n,N}\left(x^{\rho}\left(xq^{-\lambda}+\xi\right)^{(-\sigma)}; q\right) f_{m,M}\left(x^{\delta}; q\right)\right\}$$
$$= x^{k} q^{-\varepsilon k-\varepsilon(\varepsilon+1)/2}\ \xi^{-\lambda} \sum_{j=0}^{[n/N]} \sum_{i=0}^{[m/M]} \begin{bmatrix} n \\ Nj \end{bmatrix}_q \begin{bmatrix} m \\ Mi \end{bmatrix}_q$$
$$\times\ S_{n,q}\, S_{m,q}\ \xi^{-\sigma j} \left(\frac{x}{q^{\varepsilon}}\right)^{\rho j+\delta i} \frac{\Gamma_q(\eta-k-\rho j-\delta i)}{\Gamma_q(\varepsilon-k+\eta-\rho j-\delta i)}$$
$$\times\ {}_2\varphi_1 \left[\begin{matrix} q^{\lambda+\sigma j}, q^{1-\vartheta-\varepsilon+k-\eta+\rho j+\delta i}; \\ q^{1-\eta+k+\rho j+\delta i}; \end{matrix}\ q, -\left(xq^{-\varepsilon-\lambda-\sigma j}/\xi\right) \right],$$

where $n, m = 0, 1, 2, \ldots,$ *and* N, M *are arbitrary positive integer.*

Corollary 3.12. *If* $\Re(-\varepsilon-k-\rho j-\delta i) > 0,\ \left|\frac{xq^{\varepsilon-\lambda-\sigma j}}{\xi}\right| < 1,\ \min(k, \lambda, \rho, \delta, \sigma) > 0,$ *and* $K_q^{\varepsilon}(.)$ *be fractional q-integral operator of Weyl type, then the following result holds:*

$$K_q^{\varepsilon}\left\{x^k (x+\xi)^{(-\lambda)} f_{n,N}\left(x^{\rho}\left(xq^{-\lambda}+\xi\right)^{(-\sigma)}; q\right) f_{m,M}\left(x^{\delta}; q\right)\right\}$$
$$= x^{k+\varepsilon} q^{-\varepsilon k-\varepsilon(\varepsilon+1)/2}\ \xi^{-\lambda} \sum_{j=0}^{[n/N]} \sum_{i=0}^{[m/M]} \begin{bmatrix} n \\ Nj \end{bmatrix}_q \begin{bmatrix} m \\ Mi \end{bmatrix}_q S_{n,q} S_{m,q}$$
$$\times\, \xi^{-\sigma j} \left(\frac{x}{q^{\varepsilon}}\right)^{\rho j+\delta i} \frac{\Gamma_q(-\varepsilon-k-\rho j-\delta i)}{\Gamma_q(-k-\rho j-\delta i)}$$

$$\times\ {}_2\varphi_1\left[\begin{array}{c} q^{\lambda+\sigma j}, q^{1+k+\rho j+\delta i}; \\ q^{1+\varepsilon+k+\rho j+\delta i}; \end{array} q, -\left(xq^{-\varepsilon-\lambda-\sigma j}/\xi\right)\right],$$

where $n, m = 0, 1, 2, \ldots,$ *and* N, M *are arbitrary positive integer.*

4. Special Cases

As the special cases of our main results, if we take $\sigma = 0$ in the results (3.1), (3.2) (3.3) and (3.4), we deduce the following results

$$I_q^{\varepsilon,\vartheta,\eta}\left\{x^k (x+\xi)^{(-\lambda)} f_{n,N}(x^\rho; q)\right\} \tag{4.1}$$
$$= x^{k-\vartheta}\,\xi^{-\lambda}\sum_{j=0}^{[n/N]}\left[\begin{array}{c} n \\ Nj \end{array}\right]_q S_{n,q} x^{\rho j}$$
$$\times\ \frac{\Gamma_q(k+\rho j+1)\,\Gamma_q(k-\vartheta+\eta+\rho j+1)}{\Gamma_q(k-\vartheta+\rho j+1)\,\Gamma_q(k+\eta+\varepsilon+\rho j+1)}$$
$$\times\ {}_3\varphi_2\left[\begin{array}{c} q^{\lambda}, q^{1+k+\rho j}, q^{1+k-\vartheta+\eta+\rho j}; \\ q^{1+k-\vartheta+\rho j}, q^{1+k+\eta+\varepsilon+\rho j}; \end{array} q, -\left(xq^{-\lambda}/\xi\right)\right],$$

$$K_q^{\varepsilon,\vartheta,\eta}\left\{x^k (x+\xi)^{(-\lambda)} f_{n,N}(x^\rho; q)\right\} \tag{4.2}$$
$$= x^{k-\vartheta} q^{-\varepsilon k-\varepsilon(\varepsilon+1)/2}\xi^{-\lambda}\sum_{j=0}^{[n/N]}\left[\begin{array}{c} n \\ Nj \end{array}\right]_q S_{n,q}\left(\frac{x}{q^{\varepsilon}}\right)^{\rho j}$$
$$\times\ \frac{\Gamma_q(\vartheta-k-\rho j)\ \Gamma_q(\eta-k-\rho j)}{\Gamma_q(-k-\rho j)\ \Gamma_q(\vartheta+\varepsilon-k+\eta-\rho j)}$$
$$\times\ {}_3\varphi_2\left[\begin{array}{c} q^{\lambda}, q^{1+k+\rho j}, q^{1-\vartheta-\varepsilon+k-\eta+\rho j}; \\ q^{1-\vartheta+k+\rho j}, q^{1-\eta+k+\rho j}; \end{array} q, -\left(xq^{-\varepsilon-\lambda}/\xi\right)\right],$$

(4.3)

$$I_q^{\varepsilon,\vartheta,\eta}\left\{x^k (x+\xi)^{(-\lambda)} f_{n,N}(x^\rho; q)\ f_{m,M}\left(x^{\delta}; q\right)\right\}$$
$$= x^{k-\vartheta}\,\xi^{-\lambda}\sum_{j=0}^{[n/N]}\sum_{i=0}^{[m/M]}\left[\begin{array}{c} n \\ Nj \end{array}\right]_q\left[\begin{array}{c} m \\ Mi \end{array}\right]_q S_{n,q}\,S_{m,q}$$
$$\times\ x^{\rho j+\delta i}\frac{\Gamma_q(k+\rho j+\delta i+1)}{\Gamma_q(k+\rho j+\delta i+1-\vartheta)}\frac{\Gamma_q(k+\rho j+\delta i+1-\vartheta+\eta)}{\Gamma_q(k+\rho j+\delta i+1+\eta+\varepsilon)}$$
$$\times\ {}_3\varphi_2\left[\begin{array}{c} q^{\lambda}, q^{1+k+\rho j+\delta i}, q^{1+k-\vartheta+\eta+\rho j+\delta i}; \\ q^{1+k-\vartheta+\rho j+\delta i}, q^{1+k+\eta+\varepsilon+\rho j+\delta i}; \end{array} q, -\left(xq^{-\lambda}/\xi\right)\right],$$

and

(4.4)

$$K_q^{\varepsilon,\vartheta,\eta}\left\{x^k\left(x+\xi\right)^{(-\lambda)} f_{n,N}\left(x^\rho;q\right)\, f_{m,M}\left(x^\delta;q\right)\right\}$$
$$= x^{k-\vartheta}q^{-\varepsilon k-\varepsilon(\varepsilon+1)/2}\xi^{-\lambda}\sum_{j=0}^{[n/N]}\sum_{i=0}^{[m/M]}\left[\begin{array}{c} n \\ Nj \end{array}\right]_q\left[\begin{array}{c} m \\ Mi \end{array}\right]_q S_{n,q}S_{m,q}$$
$$\times\left(\frac{x}{q^\varepsilon}\right)^{\rho j+\delta i}\frac{\Gamma_q\left(\vartheta-k-\rho j-\delta i\right)\,\Gamma_q\left(\eta-k-\rho j-\delta i\right)}{\Gamma_q\left(-k-\rho j-\delta i\right)\,\Gamma_q\left(\vartheta+\varepsilon-k+\eta-\rho j-\delta i\right)}$$
$$\times\,{}_3\phi_2\left[\begin{array}{c} q^\lambda, q^{1+k+\rho j+\delta i}, q^{1-\vartheta-\varepsilon+k-\eta+\rho j+\delta i}; \\ q^{1-\vartheta+k+\rho j+\delta i}, q^{1-\eta+k+\rho j+\delta i}\,; \end{array} q,-\left(xq^{-\varepsilon-\lambda}/\xi\right)\right].$$

Further, if we take $\lambda=0$ in the above results (4.1) , (4.2), (4.3) and (4.4), these formulae reduces to

(4.5)
$$I_q^{\varepsilon,\vartheta,\eta}\left\{x^k\, f_{n,N}\left(x^\rho;q\right)\right\}$$
$$= x^{k-\vartheta}\sum_{j=0}^{[n/N]}\left[\begin{array}{c} n \\ Nj \end{array}\right]_q S_{n,q}$$
$$\times\frac{\Gamma_q\left(k+\rho j+1\right)\Gamma_q\left(k+\rho j+1-\vartheta+\eta\right)}{\Gamma_q\left(k+\rho j+1-\vartheta\right)\Gamma_q\left(k+\rho j+1+\eta+\varepsilon\right)}\,x^{\rho j},$$

(4.6)
$$K_q^{\varepsilon,\vartheta,\eta}\left\{x^k f_{n,N}\left(x^\rho;q\right)\right\}$$
$$= x^{k-\vartheta}q^{-\varepsilon k-\varepsilon(\varepsilon+1)/2}\sum_{j=0}^{[n/N]}\left[\begin{array}{c} n \\ Nj \end{array}\right]_q S_{n,q}$$
$$\times\frac{\Gamma_q\left(\vartheta-k-\rho j\right)\,\Gamma_q\left(\eta-k-\rho j\right)}{\Gamma_q\left(-k-\rho j\right)\,\Gamma_q\left(\vartheta+\varepsilon-k+\eta-\rho j\right)}\left(\frac{x}{q^\varepsilon}\right)^{\rho j},$$

(4.7)

$$I_q^{\varepsilon,\vartheta,\eta}\left\{x^k f_{n,N}\left(x^\rho;q\right)\, f_{m,M}\left(x^\delta;q\right)\right\}$$
$$= x^{k-\vartheta}\sum_{j=0}^{[n/N]}\sum_{i=0}^{[m/M]}\left[\begin{array}{c} n \\ Nj \end{array}\right]_q\left[\begin{array}{c} m \\ Mi \end{array}\right]_q S_{n,q}S_{m,q}$$
$$\times\frac{\Gamma_q\left(k+\rho j+\delta i+1\right)\Gamma_q\left(k+\rho j+\delta i+1-\vartheta+\eta\right)}{\Gamma_q\left(k-\vartheta+\rho j+\delta i+1\right)\Gamma_q\left(k+\rho j+\delta i+1+\eta+\varepsilon\right)}x^{\rho j+\delta i},$$

(4.8)

$$K_q^{\varepsilon,\vartheta,\eta}\left\{x^k f_{n,N}\left(x^{\rho};q\right)\,f_{m,M}\left(x^{\delta};q\right)\right\}$$
$$= x^{k-\vartheta}q^{-\varepsilon k-\varepsilon(\varepsilon+1)/2-\vartheta}\sum_{j=0}^{[n/N]}\sum_{i=0}^{[m/M]}\left[\begin{array}{c} n \\ Nj \end{array}\right]_q\left[\begin{array}{c} m \\ Mi \end{array}\right]_q S_{n,q}S_{m,q}$$
$$\times\frac{\Gamma_q\left(\vartheta-k-\rho j-\delta i\right)\,\Gamma_q\left(\eta-k-\rho j-\delta i\right)}{\Gamma_q\left(-k-\rho j-\delta i\right)\,\Gamma_q\left(\vartheta+\varepsilon-k+\eta-\rho j-\delta i\right)}\left(\frac{x}{q^{\varepsilon}}\right)^{\rho j+\delta i}.$$

5. Applications of the Main Results

The Saigo fractional q-integrals formulas deduced in the previous sections, can find many applications by providing the Saigo fractional q-integral operators of all such classical orthogonal q-polynomials which are special cases of the q-polynomial system (1.2). As a simple illustration, we first apply formula (4.5) to obtain the Saigo fractional q-integral operator of one of the q-Konhouser biorthogonal polynomial $Z_n^{\mu}\left(x;q,\rho\right)$ define by ([25] p.185 , equ. (2.1)).

$$Z_n^{\mu}\left(x;q,\rho\right)=\frac{\Gamma_q\left(\rho n+\mu+1\right)}{(q;q)_n}\sum_{j=0}^{n}\left[\begin{array}{c} n \\ j \end{array}\right]_q\frac{(-1)^j q^{j(j-1)}}{\Gamma_q\left(\rho j+\mu+1\right)}x^{\rho j}, \tag{5.1}$$

where $\Re\left(\mu\right)>-1$ and ρ is a positive integer.

By setting

$$N=1,\quad \rho\geq 1,\quad S_{n,q}=\frac{\Gamma_q\left(\rho n+\mu+1\right)(-1)^j\,q^{j(j-1)}}{(q;q)_n\,\Gamma_q\left(\rho j+\mu+1\right)}, \tag{5.2}$$

and replace x by x^{ρ} in equation (1.3), (4.5) and (4.6) in view of (5.1) yields then the results

$$I_q^{\varepsilon,\vartheta,\eta}\left\{x^k Z_n^{\mu}\left(x;q,\rho\right)\right\}$$
$$=\frac{\Gamma_q\left(\rho n+\mu+1\right)}{(q;q)_n}\sum_{j=0}^{\infty}\left[\begin{array}{c} n \\ j \end{array}\right]_q(-1)^j q^{j(j-1)}\frac{\Gamma_q\left(k+\rho j+1\right)}{\Gamma_q\left(\rho j+\mu+1\right)}$$
$$\times\frac{\Gamma_q\left(k+\rho j+1-\vartheta+\eta\right)}{\Gamma_q\left(k+\rho j+1-\vartheta\right)\Gamma_q\left(k+\rho j+1+\eta+\varepsilon\right)}x^{k-\vartheta+\rho j},$$

and

$$K_q^{\varepsilon,\vartheta,\eta}\left\{x^k Z_n^{\mu}\left(x;q,\rho\right)\right\}$$
$$=\frac{\Gamma_q\left(\rho n+\mu+1\right)}{(q;q)_n}x^{k-\vartheta}q^{-\varepsilon k-\varepsilon(\varepsilon+1)/2}\sum_{j=0}^{\infty}\left[\begin{array}{c} n \\ j \end{array}\right]_q(-1)^j q^{j(j-1)}$$

$$\times \frac{\Gamma_q\left(\vartheta-k-\rho j\right)\,\Gamma_q\left(\eta-k-\rho j\right)}{\Gamma_q\left(-k-\rho j\right)\,\Gamma_q\left(\vartheta+\varepsilon-k+\eta-\rho j\right)}\left(\frac{x}{q^{\varepsilon}}\right)^{\rho j}.$$

Also by putting

$$N=\rho=1, \tag{5.3}$$

$$S_{n,q}=\frac{\Gamma_q\left(n+\tau+1\right)(1-q)^n\,\Gamma_q\left(\tau+\varsigma+n+1+j\right)(-1)^j\,q^{(j(j+1)/2)-nj}}{(q;q)_n\,\Gamma_q\left(\tau+\varsigma+n+1\right)\Gamma_q\left(\tau+1+j\right)}$$

the results (4.5) and (4.6) comes involving q-Jacobi polynomials, namely

$$\begin{aligned}
&I_q^{\varepsilon,\vartheta,\eta}\left\{x^k\,p_n^{(\tau,\varsigma)}\left(x;q\right)\right\}\\
&\quad=\frac{\Gamma_q\left(\tau+1+n\right)\,\Gamma_q\left(k+1\right)\Gamma_q\left(k+1+\eta-\vartheta\right)}{\Gamma_q\left(\tau+1\right)\Gamma_q\left(n+1\right)\,\Gamma_q\left(k+1-\vartheta\right)\Gamma_q\left(k+1+\eta+\varepsilon\right)}x^{k-\vartheta}\\
&\quad\times{}_4\varphi_3\left[\begin{array}{c}q^{-n},q^{\tau+n+\varsigma+1},q^{k+1},q^{k+1+\eta-\vartheta}\\ q^{\tau+1},q^{k+1-\vartheta},q^{k+1+\eta+\varepsilon}\end{array};q,xq\right],
\end{aligned}$$

and

$$\begin{aligned}
&K_q^{\varepsilon,\vartheta,\eta}\left\{x^k\,p_n^{(\tau,\varsigma)}\left(x;q\right)\right\}\\
&\quad=q^{-\varepsilon k-\varepsilon(\varepsilon+1)/2}x^{k-\vartheta}\\
&\quad\times\frac{\Gamma_q\left(\tau+1+n\right)\,\Gamma_q\left(\vartheta-k\right)\,\Gamma_q\left(\eta-k\right)}{\Gamma_q\left(\tau+1\right)\Gamma_q\left(n+1\right)\,\Gamma_q\left(-k\right)\,\Gamma_q\left(\vartheta+\varepsilon-k+\eta\right)}\\
&\quad\times{}_4\varphi_3\left[\begin{array}{c}q^{-n},q^{\tau+n+\varsigma+1},q^{1+k},q^{1+k-\eta-\varepsilon}\\ q^{\tau+1},q^{1+k-\vartheta},q^{1+k-\eta-\varepsilon-\vartheta}\end{array};q,xq^{1-\varepsilon}\right],
\end{aligned}$$

where the q-Jacobi polynomials is defined as

$$p_n^{(\tau,\varsigma)}\left(x;q\right)=\frac{\left(q^{\tau+1};q\right)_n}{(q;q)_n}{}_2\phi_1\left[\begin{array}{c}q^{-n},q^{\tau+n+\varsigma+1};\\ q^{\tau+1};\end{array}q,xq\right]. \tag{5.4}$$

Similarly, if we express the polynomials $f_{n,N}\left(x^{\rho};q\right)$,$f_{m,M}\left(x^{\delta};q\right)$ in the form (5.1) and (5.4) in accordance with the choice of parameters given in (5.2) and (5.3) respectively, we get the following result from (4.7) and (4.8):

(5.5)

$$\begin{aligned}
&I_q^{\varepsilon,\vartheta,\eta}\left\{x^k Z_n^{\mu}\left(x;q,\rho\right)p_n^{(\tau,\varsigma)}\left(x;q\right)\right\}\\
&\quad=\frac{\left(q^{\mu+1};q\right)_{\rho n}\left(q^{\tau+1};q\right)_m}{(q;q)_n\,(q;q)_m\,(1-q)^{\rho n}}\frac{\Gamma_q\left(k+1\right)\Gamma_q\left(k+1-\vartheta+\eta\right)}{\Gamma_q\left(k-\vartheta+1\right)\Gamma_q\left(k+1+\eta+\varepsilon\right)}x^{k-\vartheta}\\
&\quad\times\sum_{j=0}^{n}\sum_{i=0}^{[m]}\left[\begin{array}{c}n\\ j\end{array}\right]_q\left[\begin{array}{c}m\\ i\end{array}\right]_q\frac{\left(q^{\tau+m+\varsigma+1};q\right)_i}{\left(q^{\mu+1};q\right)_{\rho j}\left(q^{\tau+1};q\right)_i}\frac{\left(q^{k+1};q\right)_{\rho j+i}}{\left(q^{k-\vartheta+1};q\right)_{\rho j+i}}
\end{aligned}$$

$$\times \frac{\left(q^{k-\vartheta+\eta+1};q\right)_{\rho j+i}}{\left(q^{k+\eta+\varepsilon+1};q\right)_{\rho j+i}}(-1)^{j+i}q^{j(j-1)+i(i+1)/2-mi}\left(1-q\right)^{\rho j}\left(xq^{-\varepsilon}\right)^{\rho j+i},$$

and

(5.6)

$$\begin{aligned}
&K_q^{\varepsilon,\vartheta,\eta}\left\{x^k Z_n^{\mu}\left(x;q,\rho\right)p_n^{(\tau,\varsigma)}\left(x;q\right)\right\}\\
&\quad= x^{k-\vartheta}q^{-\varepsilon k-\varepsilon(\varepsilon+1)/2}\\
&\quad\times \frac{\left(q^{\mu+1};q\right)_{\rho n}\left(q^{\tau+1};q\right)_m \Gamma_q\left(-k\right)\Gamma_q\left(\vartheta+\varepsilon+\eta-k\right)}{\left(q;q\right)_n\left(q;q\right)_m\left(1-q\right)^{\rho n}\Gamma_q\left(\eta-k\right)\Gamma_q\left(\vartheta-k\right)}\\
&\quad\times \sum_{j=0}^{n}\sum_{i=0}^{[m]}\left[\begin{array}{c} n\\ j\end{array}\right]_q\left[\begin{array}{c} m\\ i\end{array}\right]_q\frac{\left(q^{\tau+m+\varsigma+1};q\right)_i\left(q^{\vartheta-k};q\right)_{\rho j+i}}{\left(q^{\mu+1};q\right)_{\rho j}\left(q^{\tau+1};q\right)_i\left(q^{\vartheta+\varepsilon+\eta-k};q\right)_{\rho j+i}}\\
&\quad\times \frac{\left(q^{\eta-k};q\right)_{\rho j+i}}{\left(q^{-k};q\right)_{\rho j+i}}\left(-1\right)^{j+i}q^{j(j-1)+i(i+1)/2-mi}\left(1-q\right)^{\rho j}\left(xq^{-\varepsilon}\right)^{\rho j+i}.
\end{aligned}$$

Furthermore, it is interesting to observe that for $\rho=1$, the above result (5.5) and (5.6) reduces to a new result involving q-Konhouser polynomials, q-Jacobi polynomials and basic analogue of the Kampe de Feriet function as follows:

$$\begin{aligned}
&I_q^{\varepsilon,\vartheta,\eta}\left\{x^k Z_n^{\mu}\left(x;q,1\right)p_n^{(\tau,\varsigma)}\left(x;q\right)\right\}\\
&\quad= \frac{\left(q^{\mu+1};q\right)_n\left(q^{\tau+1};q\right)_m}{\left(q;q\right)_n\left(q;q\right)_m\left(1-q\right)^n}\frac{\Gamma_q\left(k+1\right)\Gamma_q\left(k+1-\vartheta+\eta\right)}{\Gamma_q\left(k-\vartheta+1\right)\Gamma_q\left(k+1+\eta+\varepsilon\right)}x^{k-\vartheta}\\
&\quad\times \varphi_{2:1:1}^{2:1:2}\left[\begin{array}{c} q^{k+1},q^{k+1-\vartheta+\eta}:q^{-n},q^{-m},q^{\tau+m+\varsigma+1}:q,\left(1-q\right)xq^n,xq\\ q^{k-\vartheta+1},q^{k+1+\eta+\varepsilon}:q^{\mu+1};q^{\tau+1}:1,0,0\end{array}\right],
\end{aligned}$$

and

$$\begin{aligned}
&K_q^{\varepsilon,\vartheta,\eta}\left\{x^k Z_n^{\mu}\left(x;q,1\right)p_n^{(\tau,\varsigma)}\left(x;q\right)\right\}\\
&\quad= \frac{\left(q^{\mu+1};q\right)_n\left(q^{\tau+1};q\right)_m \Gamma_q\left(-k\right)\Gamma_q\left(\vartheta+\varepsilon+\eta-k\right)}{\left(q;q\right)_n\left(q;q\right)_m\left(1-q\right)^n\Gamma_q\left(\eta-k\right)\Gamma_q\left(\vartheta-k\right)}x^{k-\vartheta}q^{-\varepsilon k-\varepsilon(\varepsilon+1)/2}\\
&\quad\times \varphi_{2:1:1}^{2:1:2}\left[\begin{array}{c} q^{k-\eta},q^{k-\vartheta}:q^{-n},q^{-m},q^{\tau+m+\varsigma+1}:q,\left(1-q\right)xq^{n-\varepsilon},xq^{1-\varepsilon}\\ q^{k},\ q^{k-\vartheta-\eta-\varepsilon}:q^{\mu+1};q^{\tau+1}:1,0,0\end{array}\right],
\end{aligned}$$

where the q-Kampe de Feriet function is defined by (cf. Jain [7])

$$\phi_{C;D';D"}^{A:B';B"}\left[\begin{array}{cccc} (a) & :\left(b'\right) & :\left(b"\right) & ;q,x,y\\ (c) & :\left(a"\right) & :\left(a"'\right) & ;i,j,k\end{array}\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{\prod_{t=1}^{A}(a_t;q)_{m+n} \prod_{t=1}^{B"}\left(b'_t;q\right)_m \prod_{t=1}^{B"}\left(b"_t;q\right)_n}{\prod_{t=1}^{C}(c_t;q)_{m+n} \prod_{t=1}^{D"}(d'_t;q)_m \prod_{t=1}^{D"}(d"_t;q)_n} \times \frac{x^m y^n q^{im(m-1)/2+jn(n-1)/2+kmn}}{(q;q)_m (q;q)_n},$$

where $|x| < 1$, $|y| < 1$ and i, j, k are integers.

Though several similar results can be obtained from our theorems, we omit further details.

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