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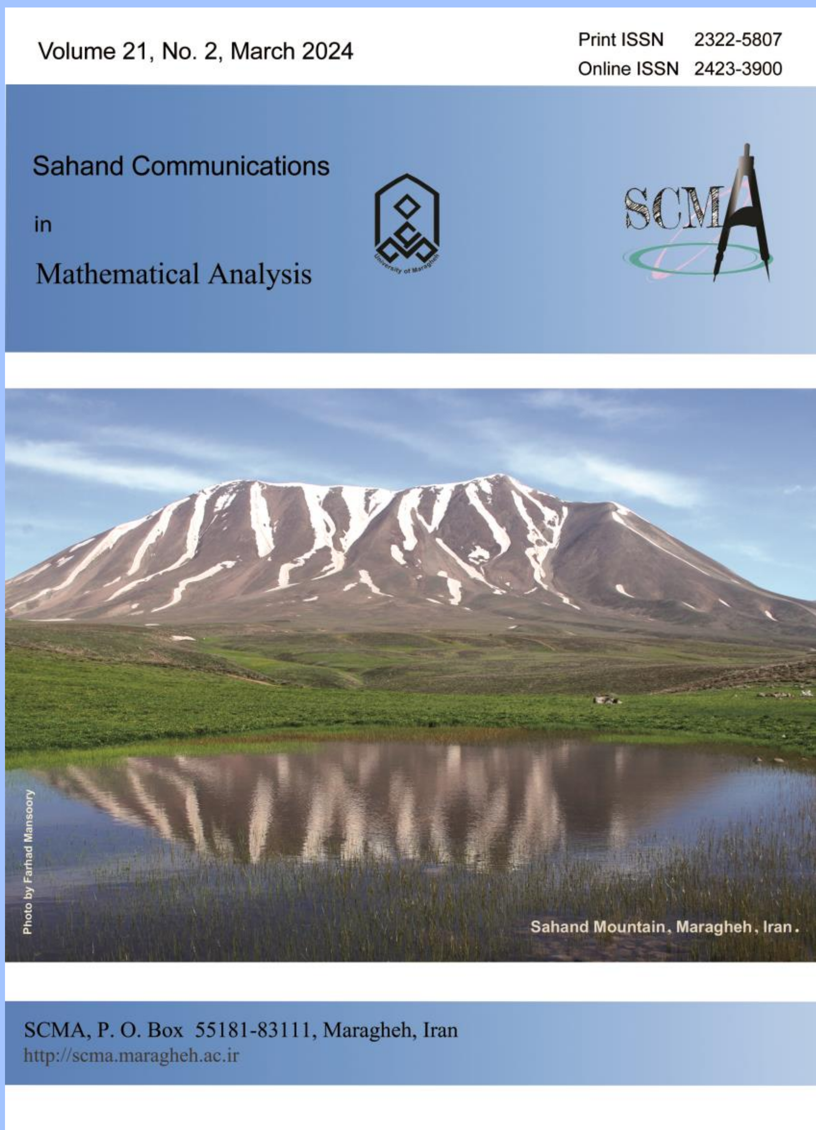
Mohamed Rossafi, Roumaissae Eljazzar and Ram Mohapatra

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 21
Number: 2
Pages: 25-49

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2023.2001846.1318



Sahand Communications in Mathematical Analysis (SCMA) Vol. 21 No. 2 (2024), 25-49
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2023.2001846.1318

Douglas' Factorization Theorem and Atomic System in Hilbert Pro-C^*-Modules

Mohamed Rossafi[1*], Roumaissae Eljazzar[2] and Ram Mohapatra[3]

ABSTRACT. In the present paper, we introduce the generalized inverse operators, which have an exciting role in operator theory. We establish Douglas' factorization theorem type for the Hilbert pro-C^*-module.We introduce the notion of atomic system and K-frame in the Hilbert pro-C^*-module and study their relationship. We also demonstrate some properties of the K-frame by using Douglas' factorization theorem.Finally we demonstrate that the sum of two K-frames in a Hilbert pro-C^*-module with certain conditions is once again a K-frame.

1. Introduction

Douglas [2] has studied the equation $AX = B$ intending to find solution for bounded linear operators on Hilbert spaces. A generalization of the Douglas theorem for the Hilbert C^*-module was given in [4] and [5]. Those authors have extended the Douglas factor decomposition for closed densely defined operators on Hilbert spaces in the context of regular operators in Hilbert C^*-modules.

Frames were introduced by Duffin and Schaefer [3] in 1952 to analyse some deep problems in nonharmonic Fourier series by abstracting the fundamental notion introduced by Gabor [8] for signal processing. Today, frame theory is an exciting, dynamic and fast-paced subject with applications to a wide variety of areas in mathematics and engineering, including sampling theory, operator theory, harmonic analysis, nonlinear sparse approximation, pseudodifferential operators, wavelet theory,

2010 *Mathematics Subject Classification.* 42C15, 46L05, 47A05.
Key words and phrases. Douglas majorization, Atomic system, Hilbert pro-C^*-modules
Received: 07 May 2023, Accepted: 03 June 2023.
* Corresponding author.

wireless communication, data transmission with erasures, filter banks, signal processing, image processing, geophysics, quantum computing, sensor networks and more.

In 2008, Joita [12] extended the theory of frames in Hilbert modules over pro-C^*-algebras. The concept of K-frame was first introduced by Laura Găvruţa [9] to study atomic systems for a given bounded linear operator K in a separable Hilbert space. It is well known that K-frames present a generalization of ordinary frames, which permits the reconstruction of the elements in the domain of a linear and bounded operator in a Hilbert space.

This paper establishes Douglas' Factorization theorem type results for pro-C^*-modules. Moreover, we define the atomic system in the framework of pro-C^*-modules and mention some properties. Also, we present K-frame in Hilbert pro-C^*-modules and establish some results.

This article will be organized as follows: Section 3 briefly recalls some definitions and basic properties of pro-C^*-algebra. Section 4 introduces the concept of generalized inverse, which will be used to prove a Douglas-type theorem on a Hilbert pro-C^*-module.

Section 5, defines the atomic system and shows that every Bessel sequence is an atomic system for the frame operator. Finaly, we show that the sum of two K-frames under certain conditions is again a K-frame.

2. Preliminaries

The basic information about pro-C^*-algebras can be found in the works [6, 7, 10, 14–16].

A C^*-algebra whose topology is induced by a family of continuous C^*-seminorms instead of a C^*-norm is called pro-C^*-algebra. Hilbert pro-C^*-modules are generalizations of Hilbert spaces. The inner product takes values in a pro-C^*-algebra rather than in the field of complex numbers.

Pro-C^*-algebra is defined as a complete Hausdorff complex topological $*$-algebra $\mathcal{A}$ whose topology is determined by its continuous C^*-seminorms in the sens that a net $\{a_\alpha\}$ converges to 0 if and only if $p(a_\alpha)$ converges to 0 for all continuous C^*-seminorms p on $\mathcal{A}$ (see [10, 13, 16]) and we have:

1) $p(ab) \leq p(a)p(b)$
2) $p(a^*a) = p(a)^2$

for all $a, b \in \mathcal{A}$. If the topology of pro-C^*-algebra is determined by only countably many C^*-seminorms, then it is called a σ-C^*-algebra. We denote by $sp(a)$ the spectrum of a such that: $sp(a) = \{\lambda \in \mathbb{C} : \lambda 1_{\mathcal{A}} - a$

is not invertible } for all $a \in \mathcal{A}$, where $\mathcal{A}$ is unital pro-C^*-algebra with unity $1_{\mathcal{A}}$.

Let the set of all continuous C^*-seminorms on $\mathcal{A}$ be denoted by $S(\mathcal{A})$. If $\mathcal{A}^+$ denotes the set of all positive elements of $\mathcal{A}$, then $\mathcal{A}^+$ is a closed convex C^*-seminorms on $\mathcal{A}$.

We also denote by $\mathcal{H}_{\mathcal{A}}$ the set of all sequences $(a_n)_n$ with $a_n \in \mathcal{A}$ such that $\sum_n a_n^* a_n$ converges in $\mathcal{A}$.

Example 2.1. Every C^*-algebra is a pro-C^*-algebra.

Proposition 2.2 ([10])**.** *Let $\mathcal{A}$ be a unital pro-C^*-algebra with an identity $1_{\mathcal{A}}$. Then for any $p \in S(\mathcal{A})$, we have:*

(1) *$p(a) = p(a^*)$ for all $a \in \mathcal{A}$*
(2) *$p(1_{\mathcal{A}}) = 1$*
(3) *If $1_{\mathcal{A}} \leq b$, then b is invertible and $b^{-1} \leq 1_{\mathcal{A}}$*
(4) *If $a, b \in \mathcal{A}^+$ are invertible and $0 \leq a \leq b$, then $0 \leq b^{-1} \leq a^{-1}$*
(5) *If $a, b \in \mathcal{A}^+$ and $a^2 \leq b^2$, then $0 \leq a \leq b$*

Definition 2.3 ([16])**.** A pre-Hilbert module over pro-C^*-algebra $\mathcal{A}$ is a complex vector space E, which is also a left $\mathcal{A}$-module compatible with the complex algebra structure, equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle$ $E \times E \to \mathcal{A}$, which is $\mathbb{C}$-and $\mathcal{A}$-linear in its first variable and satisfies the following conditions:

1) $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for every $\xi, \eta \in E$
2) $\langle \xi, \xi \rangle \geq 0$ for every $\xi \in E$
3) $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$

for every $\xi, \eta \in E$. We say E is a Hilbert $\mathcal{A}$-module (or Hilbert pro-C^*-module over $\mathcal{A}$). If E is complete with respect to the topology determined by the family of seminorms

$$\bar{p}_E(\xi) = \sqrt{p(\langle \xi, \xi \rangle)}, \quad \xi \in E, p \in S(\mathcal{A})$$

Let $\mathcal{A}$ be a pro-C^*-algebra, $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and assume that I and J are countable index sets. A bounded $\mathcal{A}$-module map from $\mathcal{X}$ to $\mathcal{Y}$ is called operators from $\mathcal{X}$ to $\mathcal{Y}$. We denote the set of all operators from $\mathcal{X}$ to $\mathcal{Y}$ by $Hom_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$.

Definition 2.4. An $\mathcal{A}$-module map $T : \mathcal{X} \to \mathcal{Y}$ is adjointable if there is a map $T^* : \mathcal{Y} \to \mathcal{X}$ such that $\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle$ for all $\xi \in \mathcal{X}, \eta \in \mathcal{Y}$ and is called bounded if for all $p \in S(\mathcal{A})$, there is $M_p > 0$ such that $\bar{p}_{\mathcal{Y}}(T\xi) \leq M_p \bar{p}_{\mathcal{X}}(\xi)$ for all $\xi \in \mathcal{X}$.

We denote by $Hom^*_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$, the set of all adjointable operator from $\mathcal{X}$ to $\mathcal{Y}$ and $Hom^*_{\mathcal{A}}(\mathcal{X}) = Hom^*_{\mathcal{A}}(\mathcal{X}, \mathcal{X})$

Definition 2.5. Let $\mathcal{A}$ be a pro-C^*-algebra and $\mathcal{X}, \mathcal{Y}$ be two Hilbert $\mathcal{A}$-modules. The operator $T : \mathcal{X} \to \mathcal{Y}$ is called uniformly bounded below, if there exists $C > 0$ such that for each $p \in S(\mathcal{A})$,

$$\bar{p}_{\mathcal{Y}}(T\xi) \leq C\bar{p}_{\mathcal{X}}(\xi), \quad \text{for all } \xi \in \mathcal{X}$$

and is called uniformly bounded above if there exists $C' > 0$ such that for each $p \in S(\mathcal{A})$,

$$\bar{p}_{\mathcal{Y}}(T\xi) \geq C'\bar{p}_{\mathcal{X}}(\xi), \quad \text{for all } \xi \in \mathcal{X}$$
$$\|T\|_\infty = \inf\{M : M \text{ is an upper bound for } T\}$$
$$\hat{p}_{\mathcal{Y}}(T) = \sup\{\bar{p}_{\mathcal{Y}}(T(x)) : \xi \in \mathcal{X}, \quad \bar{p}_{\mathcal{X}}(\xi) \leq 1\}$$

It's clear to see that, $\hat{p}(T) \leq \|T\|_\infty$ for all $p \in S(\mathcal{A})$.

Definition 2.6 ([12])**.** A sequence $\{\xi_i\}_i$ in $M(\mathcal{X})$ is a standard frame of multipliers in $\mathcal{X}$ if for each $\xi \in \mathcal{X}$, $\sum_i \langle \xi, \xi_i\rangle_{M(\mathcal{X})} \langle \xi_i, \xi\rangle_{M(\mathcal{X})}$ converges in $\mathcal{A}$ and there are two positive constants C and D such that

$$C\langle \xi, \xi\rangle_{\mathcal{X}} \leq \sum_i \langle \xi, \xi_i\rangle_{M(\mathcal{X})} \langle \xi_i, \xi\rangle_{M(\mathcal{X})} \leq D\langle \xi, \xi\rangle_{\mathcal{X}}$$

for all $\xi \in \mathcal{X}$. If $D = C = 1$ we say that $\{\xi_i\}_i$ is a standard normalized frame of multipliers.

Particularly, if the right inequality

$$\sum_i \langle \xi, \xi_i\rangle_{M(\mathcal{X})} \langle \xi_i, \xi\rangle_{M(\mathcal{X})} \leq D\langle \xi, \xi\rangle_{\mathcal{X}}, \quad \forall \xi \in \mathcal{X}$$

holds true, we call $\{\xi_i\}_{i\in I}$ a Bessel sequence.

Definition 2.7. Let $\{\xi_i\}_i$ be a standard frame of multipliers in $\mathcal{X}$. The module morphism $U : \mathcal{X} \to \mathcal{H}_{\mathcal{A}}$ defined by $U(x) = \left(\langle \xi_i, x\rangle_{M(\mathcal{X})}\right)_i$ is called the frame transform for $\{\xi_i\}_i$.

Definition 2.8. Let $\mathcal{X}$ be a Hilbert module over a pro-C^*-algebra $\mathcal{A}$ and $\{\xi_i\}_i$ be a standard frame of multipliers for $\mathcal{X}$. The invertible positive and bounded element L in $Hom^*_{\mathcal{A}}(\mathcal{X})$, such that $\sum_i \xi_i \cdot \langle \xi_i, L(\xi)\rangle_{M(\mathcal{X})} = \xi$ for all $\xi \in$ is called the frame operator associated with the standard frame of multipliers $\{\xi_i\}_i$.

Proposition 2.9 ([1])**.** . *Let $\mathcal{X}$ be a Hilbert module over pro-C^*-algebra $\mathcal{A}$ and T be an invertible element in $Hom^*_{\mathcal{A}}(\mathcal{X})$ such that both are uniformly bounded. Then for each $\xi \in \mathcal{X}$,*

$$\|T^{-1}\|_\infty^{-2} \langle \xi, \xi\rangle \leq \langle T\xi, T\xi\rangle \leq \|T\|_\infty^2 \langle \xi, \xi\rangle.$$

Theorem 2.10 ([11])**.** *Let $T \in Hom^*_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$. If T has closed range then:*

(i) *$Ker(T)$ is a complemented submodule of $\mathcal{X}$;*
(ii) *$Ran(T)$, the range of T, is a complemented submodule of $\mathcal{Y}$.*

Proposition 2.11 ([17])**.** *Let $T : \mathcal{X} \to \mathcal{X}$ be an operator, then the following statements are equivalent:*

(i) *T is a positive element in $Hom^*_{\mathcal{A}}(\mathcal{X})$*
(ii) *for any element $\xi \in \mathcal{X}$ the inequality $\langle T\xi, \xi\rangle \geq 0$ holds, i.e. this element is positive in $\mathcal{A}$.*

Lemma 2.12 ([10])**.** *Let $\mathcal{A}$ be a pro-C^*-algebra, suppose $\alpha, \beta \in \mathcal{A}^+$ be such that $\alpha \leq \beta$. Then $p(\alpha) \leq p(\beta)$ and $\lambda^*\alpha\lambda \leq \lambda^*\beta\lambda$ for each $\lambda \in \mathcal{A}$.*

3. Douglas' Factorization Theorem in Hilbert Pro–C^*–Module

We start by defining the bounded generalized inverse module map. We give an equivalent characterization of a bounded generalized inverse $\mathcal{A}$-module map, which is the primary tool to obtain Douglas' type factor decomposition theorem of some uniformly bounded module maps.

Definition 3.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert pro-C^*-modules over a pro-C^*-algebra $\mathcal{A}$, $T \in Hom^*_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$. If there exists a $T^\dagger \in Hom^*_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ such that

(1) $TT^\dagger T = T$;
(2) $T^\dagger TT^\dagger = T^\dagger$;
(3) $\left(TT^\dagger\right)^* = TT^\dagger$;
(4) $\left(T^\dagger T\right)^* = T^\dagger T$,

then $T^\dagger$ is called a bounded generalized inverse module map of T.

Remark 3.2. According to the conditions of the previous definition, we can easily deduce that $T^\dagger$ is unique. In fact suppose that T has another generalized inverse which we denote by T'. The definition 3.1 implies $Ran\left(TT^\dagger\right) = Ran(T) = Ran\left(TT'\right)$ and $Ran\left(T^\dagger T\right) = Ran\left(T^*\right) = Ran\left(T'T\right)$. Note that $TT^\dagger, T^\dagger T, TT', T'T$ are all projections, hence $TT^\dagger = TT'$ and $T^\dagger T = T'T$. As a consequence,, $T^\dagger = T^\dagger TT^\dagger = T^\dagger TT' = T'TT' = T'$.

Theorem 3.3. *Let $T \in Hom^*_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ and let $\mathcal{X}, \mathcal{Y}$ be Hilbert pro-C^*-modules over a pro-C^*-algebra $\mathcal{A}$. Then the following statements are equivalent:*

(1) *T has a generalized inverse module map in $Hom^*_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$,*
(2) *$Ran(T)$ is a closed submodule in $\mathcal{Y}$.*

Proof. (1) $\Rightarrow$ (2) is easy to prove, see the Remark 3.2.

(2) $\Rightarrow$ (1). Let $Ran(T)$ be closed. By Theorem 2.10, $Ker(T)$ and $Ran(T)$ are both complemented submodules in $\mathcal{X}$ and in $\mathcal{Y}$,

respectively, then $\mathcal{X} = Ker(T) \oplus Ran\,(T^*)$ and $\mathcal{Y} = Ker\,(T^*) \oplus Ran(T)$. Define a map $T^\dagger : \mathcal{Y} \mapsto \mathcal{X}$ which is linear by

$$T^\dagger \xi = \begin{cases} \left(T \mid Ker(T)^\perp\right)^{-1} \xi, & \xi \in Ran(T), \\ 0, & \xi \in Ker\,(T^*)\,. \end{cases}$$

and define a linear map $\left(T^\dagger\right)^* : \mathcal{X} \mapsto \mathcal{Y}$ by

$$\left(T^\dagger\right)^* \xi = \begin{cases} \left(T^* \mid Ker\,(T^*)^\perp\right)^{-1} \xi, & \xi \in Ran\,(T^*)\,, \\ 0, & \xi \in Ker(T). \end{cases}$$

To prove that $T^\dagger$is the generalized inverse module map of T, we should prove that $T^\dagger \in Hom^*_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$, equivalently, $\left\langle T^\dagger \xi, \eta \right\rangle = \left\langle \xi, \left(T^\dagger\right)^* \eta \right\rangle, \xi \in \mathcal{Y}, \eta \in \mathcal{X}$. The verification of this identity is uncomplicated utilizing the orthogonal direct sum decompositions. It's also easy to verify the conditions $(1)-(4)$ of Definition 3.1 □

Theorem 3.4. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module over a pro-C^*-algebra $\mathcal{A}$. Let $T, L \in Hom^*_{\mathcal{A}}(\mathcal{X})$. If $Ran(L)$ is closed, then the following statements are equivalent:*

(1) *$Ran(T) \subseteq Ran(L)$.*
(2) *$TT^* \leq \alpha^2 LL^*$ for some $\alpha \geq 0$.*
(3) *There exists $U \in Hom^*_{\mathcal{A}}(\mathcal{X})$ uniformly bounded such that $T = LU$.*

Proof. (3) $\Rightarrow$ (1) is obvious.

(1) $\Rightarrow$ (3) Suppose that (1) holds. For every $\xi \in \mathcal{X}$, we have $T(\xi) \in Ran(L)$. Since L has closed range, then by Theorem 2.10 $Ker(L)$ is a complemented submodule, which it results that there exists a unique $\eta \in KerL^\perp$ such tat $T(\xi) = L(\eta)$. Let's define the map U as follow $U : U(\xi) = \eta$. Observe the fact that then $U(\xi a) = \eta a$ indicates that U is a module map, for each $a \in \mathcal{A}$. The construction of U gives $T = LU$. We will next show that $U \in Hom^*_{\mathcal{A}}(\mathcal{X})$. By Theorem 3.3 L has the generalized inverse module map $L^\dagger \in Hom^*_{\mathcal{A}}(\mathcal{X})$. Since $T = LU$ then $L^\dagger T = L^\dagger LU$. Note that $kerL^\perp = Ran\,(L^*)$, hence $L^\dagger LU(\xi) = L^\dagger L(\eta) = \eta = U(\xi)$, for every $\xi \in \mathcal{X}$, that is, $L^\dagger LU = U$. According to the above proof we have $U = L^\dagger T \in Hom^*_{\mathcal{A}}(\mathcal{X})$. In fact, $U^* = T^* \left(L^\dagger\right)^*$.

(3) $\Rightarrow$ (2) Suppose that (3) holds. then there exists $U \in Hom^*_{\mathcal{A}}(\mathcal{X})$ such that $T = LU$, $TT^* = LUU^*U^* \leq \|U\|^2_\infty LL^*$, so $TT^* \leq \lambda^2 LL^*$, where $\lambda = \|U\|_\infty$.

(2) $\Rightarrow$ (3) Let be U_1 a module map defined as follow $U_1 : L^*\mathcal{X} \to T^*\mathcal{X}$ such that $U_1L^*(\xi) = T^*(\xi), \forall \xi \in \mathcal{X}$. By (2) we have that

$$\begin{aligned}\bar{p}_{\mathcal{X}}(U_1(L^*\xi))^2 &= \bar{p}_{\mathcal{X}}(T^*)^2 \\ &= p(\sqrt{\langle TT^*\xi, \xi\rangle}) \\ &\leq p(\sqrt{\lambda^2 \langle LL^*\xi, \xi\rangle}) \\ &= \lambda^2 \bar{p}_{\mathcal{X}}(L^*\xi)^2.\end{aligned}$$

As well as U_1 is well defined and uniformly bounded. Since $R(L)$ is closed and also $R(L^*)$ [11, Theorem 3.2.4]. Out of the observation above, setting $U_1\xi = 0, \xi \in (L^*\mathcal{X})^{\perp}$, it is simple to verify that U_1 is bounded in $\mathcal{X}$ and from the construction of U_1 we have $U_1L^* = T^*$. Notify that $L^*(L^*)^{\dagger} L^* = L^*$, so $U_1L^*(\xi) = U_1L^*\left((L^*)^{\dagger} L^*(\xi)\right) = T^*\left((L^*)^{\dagger} L^*(\xi)\right)$, from which it results that $T^* = T^*(L^*)^{\dagger} L^*$, i.e, $T = LL^{\dagger}T$ (in this case by using the fact $\left(L^{\dagger}\right)^* = (L^*)^{\dagger}$), from which $T = LU$, where $U = L^{\dagger}T$. ☐

Theorem 3.5. *Let $T, L \in Hom^*_{\mathcal{A}}(\mathcal{X})$ be two uniformly bounded operators, then*

$$Ran(T) + Ran(L) = Ran(\sqrt{TT^* + LL^*}).$$

Proof. Let $M = \begin{pmatrix} T & -L \\ 0 & 0 \end{pmatrix}$ on $\mathcal{X} \oplus \mathcal{X}$. Then we have

$$\begin{aligned}(Ran(T) + Ran(L)) \oplus \{0\} &= Ran(M) \\ &= Ran(MM^*)^{1/2} \\ &= Ran\begin{pmatrix} (TT^* + LL^*)^{1/2} & 0 \\ 0 & 0 \end{pmatrix} \\ &= Ran\left(\sqrt{TT^* + LL^*}\right) \oplus \{0\}.\end{aligned}$$

Which gives

$$Ran(T) + Ran(L) = Ran(\sqrt{TT^* + LL^*}).$$ ☐

Corollary 3.6. *Let $T, L_1, L_2 \in Hom^*_{\mathcal{A}}(\mathcal{X})$. The statements below are equivalent:*

(1) *$Ran(T) \subset Ran(L_1) + Ran(L_2)$.*
(2) *$TT^* \leq \alpha^2 (L_1L_1^* + L_2L_2^*)$ for some constant $\alpha > 0$.*
(3) *There are two uniformly bounded operators U and V such that $T = L_1U + L_2V$.*

Proof. By using Theorems 3.3 and 3.5 we observe that the conditions (1) and (2) are equivalent.

Since it is evident that the condition (3) implies (1), it is enough to prove that the statement (1) implies (3).

Now if we assume that $M = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} L_1 & L_2 \\ 0 & 0 \end{pmatrix}$, the assertion (1) implies $Ran(M) \subset Ran(N)$. Therefore $M = NU$ for some 2×2 matrix U and then Theorem 3.3 implies (3).

□

Lemma 3.7. *Let $\mathcal{A}$ be a pro-C^*-algebra and $\alpha, \beta \in \mathcal{A}^+$ be such that*

$$p(\alpha\lambda) \leq p(\beta\lambda), \quad \text{for all } \lambda \in \mathcal{A}^+.$$

Then $\alpha^2 \leq \beta^2$.

Proof. Without loss of generality, assume that $p(\alpha) \leq 1$ and $p(\beta) \leq 1$. Suppose that the inequality $\alpha^2 \leq \beta^2$ is not true. Then there exists $x_0 \in \operatorname{sp}\left(\alpha^2 - \beta^2\right)$ such that $x_0 > 0$, where $\operatorname{sp}\left(\alpha^2 - \beta^2\right)$ denotes the spectrum of $\alpha^2 - \beta^2$. It follows that

$$m = \max\left\{x : x \in \operatorname{sp}\left(\alpha^2 - \beta^2\right)\right\} > 0$$

Let f be any continuous real-valued function defined on the real line such that

$$\begin{cases} 0 \leq f(t) \leq 1, & \text{for all } t \in \mathbb{R}, \\ f(t) = 0, & \text{for all } t \in \left(-\infty, \frac{m}{2}\right], \\ f(t) = 1, & \text{for all } t \in [m, +\infty). \end{cases}$$

Choose $\lambda = f\left(\alpha^2 - \beta^2\right)$. Then $\lambda \in \mathcal{A}^+$since f is non-negative. By use of the functional calculus, we have

$$\begin{aligned} p(\lambda\left(\alpha^2 - \beta^2\right)\lambda) &= \max\left\{xf(x)^2 : x \in \operatorname{sp}\left(\alpha^2 - \beta^2\right)\right\} \\ &= m \\ &> 0 \end{aligned} \tag{3.1}$$

Since $x > \frac{m}{2}$ whenever $f(x) \neq 0$, it follows again by use of the functional calculus that

$$\lambda\left(\alpha^2 - \beta^2\right)\lambda \geq \frac{m}{2}\lambda^2. \tag{3.2}$$

Note that $\lambda\beta^2\lambda$ is self-adjoint, so there exists a state ρ acting on $\mathcal{A}$ such that $\phi\left(\lambda\beta^2\lambda\right) = p(\lambda\beta^2\lambda)$. Then by 3.2 and the assumption $p(\beta) \leq 1$, we have

$$\begin{aligned} p(\beta\alpha^2\beta) &\geq \phi\left(\lambda\alpha^2\lambda\right) \\ &\geq \phi\left(\lambda\beta^2\lambda\right) + \frac{m}{2}\phi\left(\lambda^2\right) \end{aligned}$$

$$\geq \frac{m+2}{2}\phi\left(\lambda\beta^2\lambda\right)$$
$$= \frac{m+2}{2}p(\lambda\beta^2\lambda)$$

This shows that if $\lambda\beta^2\lambda \neq 0$, then $p(\lambda\alpha^2\lambda) > p(\lambda\beta^2\lambda)$. On the other hand, if $\lambda\beta^2\lambda = 0$, then it follows from 3.1 that $p(\lambda\alpha^2\lambda) > 0$. So in either case, we have

$$\begin{aligned} p(\alpha\lambda)^2 &= p(\lambda\alpha^2\lambda) \\ &> p(cb^2c) \\ &= p(\beta\lambda)^2, \end{aligned}$$

in contradiction to the assumption that $p(\alpha\lambda) \leq p(\beta\lambda)$. □

Theorem 3.8. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules. Given operators T and L from $Hom^*_{\mathcal{A}}(\mathcal{X},\mathcal{Y})$, the subsequent two statements are equivalent:*

(i) $TT^* \leq LL^*$;

(ii) $\bar{p}_{\mathcal{X}}(T^*\xi) \leq \bar{p}_{\mathcal{X}}(L^*\xi)$ *for all* $\xi \in \mathcal{Y}$.

Proof. $(i) \Rightarrow (ii)$ follows from Proposition 2.11 and Lemma 2.12.

$(ii) \Rightarrow (i)$ let $\xi \in \mathcal{Y}$ and $\alpha = \langle TT^*\xi, \xi\rangle$ and $\beta = \langle LL^*\xi, \xi\rangle$. Then $\alpha, \beta \in \mathcal{A}^+$, for any $\lambda \in \mathcal{A}^+$ it is established that

$$\begin{aligned} p\left(\alpha^{\frac{1}{2}}\lambda\right)^2 &= p\left(\left(\alpha^{\frac{1}{2}}\lambda\right)^* \alpha^{\frac{1}{2}}\lambda\right) \\ &= p(\lambda^*\alpha\lambda) \\ &= p(\lambda\alpha\lambda) \\ &= p(\lambda\langle TT^*\xi, \xi\rangle\lambda) \\ &= p(\langle TT^*(\xi\lambda), (\xi\lambda)\rangle) \\ &\leq p(\langle LL^*(\xi\lambda), (\xi\lambda)\rangle) \\ &= p\left(\beta^{\frac{1}{2}}\lambda\right)^2 \end{aligned}$$

As a result, by Lemma 3.7, we have $\alpha \leq \beta$ and so

$$\langle TT^*\xi, \xi\rangle \leq \langle LL^*\xi, \xi\rangle$$

Therefore by Proposition 2.11, that $TT^* \leq LL^*$. □

4. Atomic System in $Hom^*_{\mathcal{A}}(\mathcal{X})$

Next, we introduce the concept of atomic systems for operators in Hilbert pro-C^*-modules and we establish some results.

Definition 4.1. Let $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$, we say that $\{\xi_i\}_{i=1}^{\infty}$ is an atomic system for K if :

i) The serie $\sum_i m_i \xi_i$ converges for all $m = (m_i) \in \mathcal{H}_\mathcal{A}$;

ii) There exists $C > 0$ such that for every $x \in \mathcal{X}$ there exists $m_x = (m_i) \in \mathcal{H}_\mathcal{A}$ such that $\langle m_x, m_x \rangle_{\mathcal{H}_\mathcal{A}} \le C \langle x, x \rangle_\mathcal{X}$ and $Kx = \sum_i m_i \xi_i$.

Proposition 4.2. *Suppose that $\{\xi_i\}_{i=1}^\infty$ is a Bessel sequence in $\mathcal{X}$. Then $\{\xi_i\}_{i=1}^\infty$ is an atomic system for L, where L is the frame operator.*

Proof. We have the module morphism

$$\theta : \mathcal{X} \to \mathcal{H}_\mathcal{A} \text{ defined by } \theta x = \{\langle x, \xi_i \rangle\}_{i=1}^\infty,$$

and its adjoint

$$\theta : \mathcal{H}_\mathcal{A} \to \mathcal{X} \text{ defined by } \theta\left(\{c_i\}_{i=1}^\infty\right) = \sum_{i=1}^\infty c_i \xi_i.$$

Let $L = \theta^* \theta$. Then

$$L : \mathcal{X} \to \mathcal{X}, \qquad Lx = \sum_{i=1}^\infty \langle x, \xi_i \rangle \xi_i$$

Let $a_\xi = \{a_i\} = \{\langle \xi, \xi_i \rangle\}_{i=1}^\infty \in \mathcal{H}_\mathcal{A}$

Now

$$\begin{aligned} \bar{p}_\mathcal{X}(a_\xi)^2 =& p(\langle a_\xi, a_\xi \rangle) \\ =& p\left(\sum_{i \in I} \langle \xi, \xi_i \rangle_{M(\mathcal{X})} \langle \xi_i, \xi \rangle_{M(\mathcal{X})}\right) \\ \le& D \bar{p}_\mathcal{X}(\xi)^2 \end{aligned}$$

Then, $\{\xi_i\}_{i=1}^\infty$ is an atomic system for L. □

Lemma 4.3. *Let $\mathcal{A}$ be a unital pro-C^*-algebra and $\{\xi_i\}_{j \in I}$ be a sequence of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{X}$ over $\mathcal{A}$. Then $\{\xi_i\}_{i \in I}$ is a Bessel sequence with bound D if and only if*

$$p\left(\sum_{i \in I} \langle x, \xi_i \rangle \langle \xi_i, x \rangle\right) \le D \bar{p}_\mathcal{X}(\xi)^2$$

for all $\xi \in \mathcal{X}$.

Proof. " $\Rightarrow$ " Evident

" $\Leftarrow$ " Let T be a linear operator defined as follow $T : \mathcal{X} \to \mathcal{H}_\mathcal{A}$

$$T\xi = \sum_{i \in I} \langle \xi, \xi_i \rangle e_i, \quad \forall \xi \in \mathcal{X}.$$

Then

$$\bar{p}_\mathcal{X}(T\xi)^2 = p(\langle T\xi, T\xi \rangle)$$

$$= p\left(\sum_{i\in I}\langle\xi,\xi_i\rangle\langle\xi_i,\xi\rangle\right) \\ \leq D\bar{p}_{\mathcal{X}}(\xi)^2,$$

which implies that T is bounded.

It is clear that T is $\mathcal{A}$-linear. Then we have

$$\langle T\xi, T\xi\rangle \leq D\langle\xi,\xi\rangle$$

equivalently, $\sum_{i\in I}\langle\xi,\xi_i\rangle\langle\xi_i,\xi\rangle \leq D\langle\xi,\xi\rangle$, as desired. □

Theorem 4.4. *If $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$, then there exists an atomic system for the operator K.*

Proof. Let $\{\xi_i\}_{i\in I}$ be a standard normalized tight frame for $\mathcal{X}$. Since

$$\xi = \sum_{i\in I}\langle\xi,\xi_i\rangle\xi_i, \quad \xi\in\mathcal{X},$$

we have

$$K\xi = \sum_{i\in I}\langle\xi,\xi_i\rangle K\xi_i, \quad \xi\in\mathcal{X}.$$

For $\xi\in\mathcal{H}$, putting $a_{i,\xi} = \langle\xi,\xi_i\rangle$ and $f_i = K\xi_i$ for all $i\in I$, we get

$$\begin{aligned}\sum_{i\in I}\langle\xi,f_i\rangle\langle f_i,\xi\rangle &= \sum_{i\in I}\langle\xi,K\xi_i\rangle\langle K\xi_i,\xi\rangle \\ &= \sum_{i\in I}\langle K^*\xi,\xi_i\rangle\langle\xi_i,K^*\xi\rangle \\ &= \langle K^*\xi,K^*\xi\rangle \\ &\leq \|K^*\|_\infty^2\langle\xi,\xi\rangle.\end{aligned}$$

Therefore $\{f_i\}_{i\in I}$ is a Bessel sequence for $\mathcal{H}$ and we conclude that the series $\sum_{i\in I}c_if_i$ converges for all $c = \{c_i\}_{i\in I}\in\ell^2(A)$ by Lemma 4.3. We also have

$$\begin{aligned}\sum_{i\in I}a_{i,\xi}a_{i,\xi}^* &= \sum_{i\in I}\langle\xi,\xi_i\rangle\langle\xi_i,\xi\rangle \\ &= \langle\xi,\xi\rangle,\end{aligned}$$

which completes the proof. □

Our next result deals with Bessel sequences.

Theorem 4.5. *Let $\{\xi_i\}_{i\in I}$ be a Bessel sequence for $\mathcal{X}$ and $K\in Hom^*_{\mathcal{A}}(\mathcal{X})$. Suppose that $T\in Hom^*_{\mathcal{A}}(\mathcal{X},\mathcal{H}_{\mathcal{A}})$ is given by $T(\xi) = \{\langle\xi,\xi_i\rangle\}_{i\in I}$ and $\overline{Ran(T)}$ is orthogonally complemented. Then the following statements are equivalent:*

(1) $\{\xi_i\}_{i\in I}$ *is an atomic system for* K;
(2) *there exists two positives values* C *and* D *such that*

$$C^{-1}\left(\bar{p}_{\mathcal{X}}\left(K^{*}\xi\right)\right)^{2}\leq p\left(\sum_{i\in l}\langle\xi,\xi_i\rangle_{M(\mathcal{X})}\langle\xi_i,\xi\rangle_{M(\mathcal{X})}\right)\leq D\left(\bar{p}_{\mathcal{X}}(\xi)\right)^{2};$$

(3) *There exists* $Q\in Hom^{*}_{\mathcal{A}}(\mathcal{X},\mathcal{H}_{\mathcal{A}})$ *such that* $K=T^{*}Q$.

Proof. (1) $\Rightarrow$ (2) For each $\xi\in\mathcal{X}$

$$\left(\bar{p}_{\mathcal{X}}\left(K^{*}\xi\right)\right)^{2}=\left(\sup\left\{p(\langle\xi,Ky\rangle):\bar{p}_{\mathcal{X}}(y)\leq 1\right\}\right)^{2}$$

and $Ky=\sum_i m_i\xi_i$. Then

$$\begin{aligned}\left(\bar{p}_{\mathcal{X}}\left(K^{*}\xi\right)\right)^{2}&=\left(\sup\left\{p\left(\left\langle\xi,\sum_{i\in l}\xi_i m_i\right\rangle\right):\bar{p}_{\mathcal{X}}(y)\leq 1\right\}\right)^{2}\\&=\left(\sup\left\{p\left(\sum_{i\in I}m_i^{*}\langle\xi_i,\xi\rangle_{M(\mathcal{X})}\right):\bar{p}_{\mathcal{X}}(y)\leq 1\right\}\right)^{2}\\&\leq\sup_{\bar{p}_{\mathcal{X}}(y)\leq 1}\left\{p\left(\sum_{i\in I}m_i^{*}m_i\right)p\left(\sum_{i\in l}\langle\xi,\xi_i\rangle_{M(\mathcal{X})}\langle\xi_i,\xi\rangle_{M(\mathcal{X})}\right)\right\}\\&\leq Cp\left(\sum_{i\in I}\langle\xi,\xi_i\rangle_{M(\mathcal{X})}\langle\xi_i,\xi\rangle_{M(\mathcal{X})}\right)\end{aligned}$$

Then

$$C^{-1}\left(\bar{p}_{\mathcal{X}}\left(K^{*}\xi\right)\right)^{2}\leq p\left(\sum_{i\in l}\langle\xi,\xi_i\rangle_{M(\mathcal{X})}\langle\xi_i,\xi\rangle_{M(\mathcal{X})}\right)$$

Since $\{\xi_i\}_{i=1}^{\infty}$ is Bessel sequence, then there exists positive value D such that

$$p\left(\sum_{i\in l}\langle\xi,\xi_i\rangle_{M(\mathcal{X})}\langle\xi_i,\xi\rangle_{M(\mathcal{X})}\right)\leq D\left(\bar{p}_{\mathcal{X}}(\xi)\right)^{2}$$

Then

$$C^{-1}\left(\bar{p}_{\mathcal{X}}\left(K^{*}\xi\right)\right)^{2}\leq p\left(\sum_{i\in l}\langle\xi,\xi_i\rangle_{M(\mathcal{X})}\langle\xi_i,\xi\rangle_{M(\mathcal{X})}\right)\leq D\left(\bar{p}_{\mathcal{X}}(\xi)\right)^{2}.$$

(2) $\Rightarrow$ (3) Since $\{\xi_i\}_{i\in I}$ is a Bessel sequence, we get $T^{*}e_i=\xi_i$, where $\{e_i\}_{i\in I}$ is the standard orthonormal basis for $\mathcal{H}_{\mathcal{A}}$. Therefore

$$C^{-1}\left(\bar{p}_{\mathcal{X}}\left(K^{*}\xi\right)\right)^{2}\leq p\left(\sum_{i\in l}\langle\xi,\xi_i\rangle_{M(\mathcal{X})}\langle\xi_i,\xi\rangle_{M(\mathcal{X})}\right)$$

$$= p\left(\sum_{i\in I} \langle \xi, T^*e_i\rangle \langle T^*e_i, \xi\rangle\right)$$
$$= p\left(\sum_{i\in I} \langle T\xi, e_i\rangle \langle e_i, T\xi\rangle\right)$$
$$= \bar{p}_{\mathcal{X}}(T\xi)^2, \quad \xi \in \mathcal{X}.$$

Theorem 3.4 yields that there exists $Q \in Hom^*_{\mathcal{A}}(\mathcal{X}, \mathcal{H}_{\mathcal{A}})$ such that $K = T^*Q$.

(3) $\Rightarrow$ (1) For every $\xi \in \mathcal{X}$, we have

$$Q\xi = \sum_{i\in I} \langle Q\xi, e_i\rangle e_i$$

Thus

$$T^*Q\xi = \sum_{i\in I} \langle Q\xi, e_i\rangle T^*e_i, \quad \xi \in \mathcal{X}$$

Let $c_i = \langle Q\xi, e_i\rangle$, so for all $\xi \in \mathcal{X}$ we get

$$\sum_{i\in I} c_i c_i^* = \sum_{i\in I} \langle Q\xi, e_i\rangle \langle e_i, Q\xi\rangle$$
$$= \langle Q\xi, Q\xi\rangle$$
$$\leq \|Q\|_\infty^2 \langle \xi, \xi\rangle$$

Since $\{\xi_i\}_{i\in I}$ is a Bessel sequence for $\mathcal{X}$, we get that $\{\xi_i\}_{i\in I}$ is an atomic system for K. ☐

Corollary 4.6. *Let $\{\xi_i\}_{i\in I}$ be a frame of multiplier for $\mathcal{X}$ with bounds $C, D > 0$ and $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$. Then $\{\xi_i\}_{i\in I}$ is an atomic system for K with bounds $\frac{1}{C^{-1}\|K\|^2}$ and D.*

Proof. Let S be the frame operator of $\{\xi_i\}_{i\in I}$.We show that condition (2) of the Theorem 4.5 is verified. Since $\left\{S^{-1}\xi_i\right\}_{i\in I}$ is a frame for $\mathcal{X}$ with bounds $D^{-1}, C^{-1} > 0$ and $x = \sum_{n\in J} \langle x, f_n\rangle S^{-1}f_n$ for all $x \in \mathcal{H}$, we get

$$(\bar{p}_{\mathcal{X}}(K^*\xi))^2 = (\sup\{p(\langle \xi, Ky\rangle) : \bar{p}_{\mathcal{X}}(y) \leq 1\})^2$$

And $Ky = \sum_i m_i\xi_i$.

$$\bar{p}_{\mathcal{X}}(K^*\xi)^2 = (\sup\{p(\langle \xi, Ky\rangle) : \bar{p}_{\mathcal{X}}(y) \leq 1\})^2$$
$$= \left(\sup\left\{p\left\langle\left(\sum_{n\in J} \langle \xi, f_n\rangle K^*S^{-1}f_n, y\right)\right\rangle : \bar{p}_{\mathcal{X}}(y) \leq 1\right\}\right)^2$$

$$= \left(\sup\left\{p\sum_{n\in J}\langle\xi, f_n\rangle\left\langle K^*S^{-1}f_n, y\right\rangle : \bar{p}_{\mathcal{X}}(y)\leq 1\right\}\right)^2$$
$$\leq \sup_{\bar{p}_{\mathcal{X}}(y)\leq 1}\left\{p\left(\sum_{n\in J}\left\langle Ky, S^{-1}f_n\right\rangle\left\langle S^{-1}f_n, Ky\right\rangle\right)\right.$$
$$\left.p\left(\sum_{i\in I}\langle\xi,\xi_i\rangle_{M(\mathcal{X})}\langle\xi_i,\xi\rangle_{M(\mathcal{X})}\right)\right\}$$
$$\leq \sup_{\bar{p}_{\mathcal{X}}(y)\leq 1} C^{-1}p\left(\sum_{i\in I}\langle\xi,\xi_i\rangle_{M(\mathcal{X})}\langle\xi_i,\xi\rangle_{M(\mathcal{X})}\right)\bar{p}_{\mathcal{X}}(Ky)^2$$
$$\leq C^{-1}\bar{p}_{\mathcal{X}}(K)^2 p\left(\sum_{i\in I}\langle\xi,\xi_i\rangle_{M(\mathcal{X})}\langle\xi_i,\xi\rangle_{M(\mathcal{X})}\right)$$

Hence $\{\xi_i\}_{i\in I}$ is an atomic system for K. □

Theorem 4.7. *Let $K_1, K_2 \in Hom^*_{\mathcal{A}}(\mathcal{X})$. If $\{\xi_i\}_{i\in I}$ is an atomic system for K_1 and K_2 and α,β are scalars, then $\{\xi_i\}_{i=1}^{\infty}$ is an atomic system for $\lambda K_1 + \gamma K_2$ and K_1K_2.*

Proof. $\{\xi\}_{i=1}^{\infty}$ is an atomic system for K_1 and K_2, then there are two positives constantes $0 < \lambda_n \leq \gamma_n < \infty$ $(n = 1, 2)$ such that

$$\lambda_n\left(\bar{p}_{\mathcal{X}}\left(K_n^*x\right)\right)^2 \leq p\left(\sum_{i\in l}\langle x,\xi_i\rangle_{M(\mathcal{X})}\langle\xi_i,x\rangle_{M(\mathcal{X})}\right) \leq \gamma_n\left(\bar{p}_{\mathcal{X}}(x)\right)^2, \quad \text{for all } x\in\mathcal{X}. \tag{4.1}$$

We have

$$\left(\bar{p}_{\mathcal{X}}\left(\alpha K_1 + K_2\right)^* x\right)^2$$
$$= p\left(\left\langle\left(\alpha K_1 + K_2\right)^* x, \left(\alpha K_1 + K_2\right)^* x\right\rangle\right)$$
$$= p\left(\left\langle\left(\alpha K_1\right)^* x, \left(\alpha K_1\right)^* x\right\rangle + \left\langle\left(\beta K_2\right)^* x, \left(\beta K_2\right)^* x\right\rangle\right)$$
$$\leq p(\alpha^2)Cp\left(\sum_{i\in I}\langle x,\xi_i\rangle_{M(\mathcal{X})}\langle\xi_i,x\rangle_{M(\mathcal{X})}\right)$$
$$+ p(\beta^2)Dp\left(\sum_{i\in I}\langle x,\xi_i\rangle_{M(\mathcal{X})}\langle\xi_i,x\rangle_{M(\mathcal{X})}\right)$$
$$\leq \max\left(p(\alpha^2)C, p(\beta^2)D\right)$$
$$\times p\left(\sum_{i\in I}\langle x,\xi_i\rangle_{M(\mathcal{X})}\langle\xi_i,x\rangle_{M(\mathcal{X})}\right)$$

by setting $M = \max(p(\alpha^2)C, p(\beta^2)D)$ It follows that

$$M^{-1}\left(\bar{p}_{\mathcal{X}}\left(\alpha K_1 + \beta K_2\right)^* x\right)^2 \le p\left(\sum_{i\in I}\langle x, \xi_i\rangle_{M(\mathcal{X})}\langle \xi_i, x\rangle_{M(\mathcal{X})}\right)$$

Hence $\{\xi_i\}_{i=1}^{\infty}$ satisfies the lower frame condition.

And from inequalities 4.1 , we get

$$p\left(\sum_{i\in l}\langle x, \xi_i\rangle_{M(\mathcal{X})}\langle \xi_i, x\rangle_{M(\mathcal{X})}\right) \le \frac{\gamma_1+\gamma_2}{2}\left(\bar{p}_{\mathcal{X}}(x)\right)^2, \quad \text{for all } x \in \mathcal{X}$$

Hence $\{\xi_i\}_{i=0}^{\infty}$ is an atomic system for $\alpha K_1 + \beta K_2$. □

5. K-frames in Hilbert Pro-C^*-Module

Now, we define the K-frame in Hilbert pro-C^*-modules, and show that under some conditions, every ordinary multiplier frame is a K-frame. Then, we use Douglas' factorization theorem to demonstrate some K-frames properties. We finally show the relationship between Bessel sequences and K-frame in pro-C^*-modules.

Definition 5.1. Let $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$. $\{\xi_i\}_{i=0}^{\infty}$ is a sequence in $\mathcal{X}$, $\{\xi_i\}_{i=0}^{\infty}$ is called K-frame for $\mathcal{X}$ if there existe two positive constants A and B such that

$$A\langle K^*\xi, K^*\xi\rangle_{\mathcal{X}} \le \sum_{i\in I}\langle \xi, \xi_i\rangle_{M(\mathcal{X})}\langle \xi_i, \xi\rangle_{M(\mathcal{X})} \le B\langle \xi, \xi\rangle_{\mathcal{X}}, \quad \forall \xi \in \mathcal{X}$$

A and B are respectively called lower and upper bounds for K-frame $\{\xi_i\}_{i\in I}$

Definition 5.2. Let $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$ be a bounded operator. A sequence $\{\xi_i\}_{i\in I}$ in $\mathcal{X}$ is said to be a tight K-frame with bound A if

$$A\langle K^*\xi, K^*\xi\rangle_{\mathcal{X}} = \sum_{i\in I}\langle \xi, \xi_i\rangle_{M(\mathcal{X})}\langle \xi_i, \xi\rangle_{M(\mathcal{X})}, \quad \text{for all } \xi \in \mathcal{X}$$

When $A = 1$, it is called a Parseval K-frame .

Theorem 5.3. *Let K be an invertible element in $Hom^*_{\mathcal{A}}(\mathcal{X})$ such that both are uniformly bounded with $\|K\|_{\infty}^2$. Then every ordinary frame of multiplier is K-frame for $\mathcal{X}$.*

Proof. Suppose that $\{\xi_i\}_{i=0}^{\infty}$ is frame for $\mathcal{X}$ then there two constants A and B such that

$$A\langle \xi, \xi\rangle_{\mathcal{X}} \le \sum_{i\in I}\langle \xi, \xi_i\rangle_{M(\mathcal{X})}\langle \xi_i, \xi\rangle_{M(\mathcal{X})} \le B\langle \xi, \xi\rangle_{\mathcal{X}}, \quad \forall \xi \in \mathcal{X} \tag{5.1}$$

For K an invertible element in $Hom^*_{\mathcal{A}}(\mathcal{X})$, we have $\langle K^*\xi, K^*\xi\rangle_{\mathcal{X}} \leq \|K\|^2_\infty \langle \xi,\xi\rangle_{\mathcal{X}}$. Then $\frac{1}{\|K\|^2_\infty}\langle K^*\xi, K^*\xi\rangle_{\mathcal{X}} \leq \langle \xi,\xi\rangle_{\mathcal{X}}$, for all $\xi \in \mathcal{X}$. From the inequality 5.1, we have

$$\begin{aligned} A\frac{1}{\|K\|^2_\infty}\langle K^*\xi, K^*\xi\rangle_{\mathcal{X}} &\leq A\langle \xi,\xi\rangle_{\mathcal{X}} \\ &\leq \sum_{i\in I}\langle \xi,\xi_i\rangle_{M(\mathcal{X})}\langle \xi_i,\xi\rangle_{M(\mathcal{X})} \\ &\leq B\langle \xi,\xi\rangle_{\mathcal{X}} \end{aligned}$$

Hence $\{\xi_i\}_{i=0}^{\infty}$ is K-frame for $\mathcal{X}$. □

Proposition 5.4. *Let $\{\xi_i\}_{i=1}^{\infty}$ be a K-frame for $\mathcal{X}$. Let $L \in \mathcal{X}$ be a bounded uniformly operator with $Ran(L) \subseteq Ran(K)$. Then $\{f_i\}_{i=1}^{\infty}$ is a L-frame for $\mathcal{X}$.*

Proof. Let $\{\xi_i\}_{i=1}^{\infty}$ be K-frame for $\mathcal{X}$. Then there are positive constants A and B such that

$$A\langle K^*\xi, K^*\xi\rangle_{\mathcal{X}} \leq \sum_{i\in I}\langle \xi,\xi_i\rangle_{M(\mathcal{X})}\langle \xi_i,\xi\rangle_{M(\mathcal{X})} \leq B\langle \xi,\xi\rangle_{\mathcal{X}}, \quad \forall \xi \in \mathcal{X} \tag{5.2}$$

Since $Ran(L) \subseteq Ran(K)$, by Douglas' theorem 3.4, there exists $\alpha > 0$ such that $LL^* \leq \alpha^2 KK^*$. From the inequality (5.2), we have

$$\begin{aligned} \frac{A}{\alpha^2}\langle L^*\xi, L^*\xi\rangle_{\mathcal{X}} &\leq A\langle K^*\xi, K^*\xi\rangle_{\mathcal{X}} \\ &\leq \sum_{i\in I}\langle x,\xi_i\rangle_{M(\mathcal{X})}\langle \xi_i,x\rangle_{M(\mathcal{X})} \\ &\leq B\langle \xi,\xi\rangle_{\mathcal{X}}, \quad \forall \xi \in \mathcal{X}. \end{aligned}$$

Therefore $\{\xi_i\}_{i=1}^{\infty}$ is a L-frame for $\mathcal{X}$. □

Theorem 5.5. *Let $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$ and $\{\xi_i\}_{i\in I}$ be a Bessel sequence for $\mathcal{X}$. Suppose that $T \in Hom^*_{\mathcal{A}}(\mathcal{X}, \mathcal{H}_{\mathcal{A}})$ is given by $T(\xi) = \{\langle \xi,\xi_i\rangle\}_{i\in I}$ and $\overline{Ran(T)}$ is orthogonally complemented. Then $\{\xi_i\}_{i\in I}$ is a K-frame for $\mathcal{X}$ if and only if there exists a linear bounded operator $L : \mathcal{H}_{\mathcal{A}} \to \mathcal{X}$ such that $Le_i = \xi_i$ and $Ran(K) \subseteq Ran(L)$, where $\{e_i\}_i$ is the orthonormal basis for $\mathcal{H}_{\mathcal{A}}$.*

Proof. Suppose that $\{\xi_i\}_{i\in I}$ is K-frame. Then $A\bar{p}_{\mathcal{X}}(K^*\xi)^2 \leq \bar{p}_{\mathcal{X}}(T\xi)^2$ for all $\xi \in \mathcal{X}$. By Theorem 3.8, there is $\alpha > 0$ such that

$$KK^* \leq \alpha T^*T.$$

Setting $T^* = L$, we get $KK^* \leq \alpha LL^*$ and therefore $Ran(K) \subseteq Ran(L)$. Conversely, since $Ran(K) \subseteq Ran(L)$, by Theorem 3.4 there exists $\alpha > 0$

such that $KK^* \leq \alpha LL^*$. Therefore

$$\begin{aligned}\frac{1}{\alpha}\langle K^*\xi, K^*\xi\rangle &\leq \langle L^*\xi, L^*\xi\rangle \\ &= \sum_{i\in I}\langle \xi, \xi_i\rangle\langle \xi_i, \xi\rangle, \quad \xi \in \mathcal{X}\end{aligned}$$

Then $\{\xi_i\}_{i\in I}$ is K-frame for $\mathcal{X}$. □

Proposition 5.6. *Let $\{\xi_i\}_{i\in I}$ be a Bessel sequence of $\mathcal{X}$, $\{\xi_i\}_{i\in I}$ is a K-frame with bounds $A, B > 0$ if and only if $L \geq AKK^*$, where L is the frame operator for $\{\xi_i\}_{i\in I}$.*

Proof. A sequence $\{\xi_i\}_{i\in I}$ is a K-frame for $\mathcal{X}$ if and only if

$$\begin{aligned}\langle AKK^*\xi, \xi\rangle &= A\langle K^*\xi, K^*\xi\rangle \\ &\leq \sum_{i\in I}\langle \xi, \xi_i\rangle_{M(\mathcal{X})}\langle \xi_i, \xi\rangle_{M(\mathcal{X})} \\ &= \langle L\xi, \xi\rangle \\ &\leq B\langle \xi, \xi\rangle .\end{aligned}$$

Then $\{\xi_i\}_{i\in I}$ is a K-frame if and only if $L \geq AKK^*$. □

Theorem 5.7. *Let $\{\xi_i\}_{i\in I}$ be a Bessel sequence in $\mathcal{X}$. Then $\{\xi_i\}_{i\in I}$ is a K-frame for $\mathcal{X}$ if and only if $K = L^{1/2}L$, for some $U \in Hom^*_{\mathcal{A}}(\mathcal{X})$.*

Proof. Suppose $\{\xi_i\}_{i\in I}$ is a K-frame, by Proposition 5.6 there exist two positives constantes A and B

$$AKK^* \leq L^{1/2}L^{1/2^*}$$

Therefore by Douglas' theorem 3.4, $K = L^{1/2}U$, for some L bounded in $Hom^*_{\mathcal{A}}(\mathcal{X})$.

Conversly, let $K = L^{1/2}U$, for some L bounded in $Hom^*_{\mathcal{A}}(\mathcal{X})$. Then by Douglas' factorization theorem, $L^{1/2}$ majorizes K^*. Then there is a positive number A such that $L \geq A^2KK^*$. Therefore by Proposition 5.6 $\{\xi_i\}_{i\in I}$ is a K-frame for $\mathcal{X}$. □

Example 5.8. Suppose that $\{u_i\}_{i=1}^{\infty}$ is an orthonormal basis in $\mathcal{H}_{\mathcal{A}}$. Define operators L and K on $\mathcal{H}_{\mathcal{A}}$ by $Lu_i = u_{i-1}$ for $i > 1$ and $Lu_1 = 0$ and $Ku_i = u_{i+1}$ respectively. It is clear that $\{Ku_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}_{\mathcal{A}}$. Suppose $\{Ku_i\}_{i\in I}$ is a L-frame. Then by Proposition 5.6, there exists $A > 0$ such that $KK^* \geq ALL^*$. Hence by Douglas, theorem, $Ran(L) \subseteq Ran(K)$. But this is contradiction to $Ran(L) \nsubseteq Ran(K)$, since $u_1 \in Ran(L)$ but $u_1 \notin Ran(K)$.

Theorem 5.9. *Let $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$ be an uniformly bounded operator such that $Ran(K)$ is closed. The frame operator of a K-frame is invertible on the subspace $Ran(K)$ of $\mathcal{X}$.*

Proof. Suppose $\{\xi_i\}_{i\in I}$ is a K-frame for $\mathcal{X}$. Then there exists $A>0$ such that

$$A\left\langle K^*\xi, K^*\xi\right\rangle_{\mathcal{X}} \leq \sum_{i\in I}\left\langle \xi,\xi_i\right\rangle_{M(\mathcal{X})}\left\langle \xi_i,\xi\right\rangle_{M(\mathcal{X})} \tag{5.3}$$

Since $Ran(K)$ is closed, then $KK^\dagger\xi=\xi$, for all $\xi\in Ran(K)$. That is,

$$KK^\dagger\Big|_{Ran(K)} = I_{Ran(K)}$$

we have $I^*_{Ran(K)} = \left(K^\dagger|_{Ran(K)}\right)^* K^*$. For any $\xi\in Ran(K)$, we obtain

$$\begin{aligned}\langle\xi,\xi\rangle &= \left\langle \left(K^\dagger\Big|_{R(K)}\right)^* K^*\xi, \left(K^\dagger\Big|_{R(K)}\right)^* K^*\xi\right\rangle\\ &\leq \left\|K^\dagger\right\|_\infty^2\cdot\langle K^*\xi,K^*\xi\rangle\end{aligned}$$

Therefore

$$\langle K^*\xi,K^*\xi\rangle \geq \left\|K^\dagger\right\|_\infty^{-2}\cdot\langle\xi,\xi\rangle$$

In combination with 5.3, we obtain

$$\begin{aligned}\sum_{i\in I}\left\langle \xi,\xi_i\right\rangle_{M(\mathcal{X})}\left\langle \xi_i,\xi\right\rangle_{M(\mathcal{X})} &\geq A\left\langle K^*\xi,K^*\xi\right\rangle_{\mathcal{X}}\\ &\geq A\left\|K^\dagger\right\|_\infty^{-2}\cdot\langle\xi,\xi\rangle, \quad \text{for all } \xi\in \mathrm{Ran}(K).\end{aligned}$$

Hence, by the definition of K-frame, we get

$$\begin{aligned}A\left\|K^\dagger\right\|_\infty^{-2}\cdot\langle\xi,\xi\rangle_{\mathcal{X}} &\leq \sum_{i\in I}\left\langle \xi,\xi_i\right\rangle_{M(\mathcal{X})}\left\langle \xi_i,\xi\right\rangle_{M(\mathcal{X})}\\ &\leq B\left\langle\xi,\xi\right\rangle_{\mathcal{X}}\end{aligned}$$

Therefore

$$\begin{aligned}A\left\|K^\dagger\right\|_\infty^{-2}\cdot\langle\xi,\xi\rangle_{\mathcal{X}} &\leq \langle S\xi,S\xi\rangle_{\mathcal{X}}\\ &\leq B\left\langle\xi,\xi\right\rangle_{\mathcal{X}}\end{aligned}$$

And so $S: Ran(K)\to Ran(S)$ is a bounded linear operator and invertible on $Ran(K)$. □

Theorem 5.10. *Let $K\in Hom^*_{\mathcal{A}}(\mathcal{X})$ be uniformly bounded such that $Ran(K)$ is dense. Let $\{\xi_i\}_{i\in I}$ be a K-frame and $T\in Hom^*_{\mathcal{A}}(\mathcal{X})$ be uniformly bounded such that $Ran(T)$ is closed. If $\{T\xi_i\}_{i\in I}$ is a K-frame for $\mathcal{X}$, then T is surjective.*

Proof. Let's assume that $\{T\xi_i\}_{i\in I}$ is a K-frame for $\mathcal{X}$ with frame bounds A and B. Then for any $\xi \in \mathcal{X}$,

$$A\langle K^*\xi, K^*\xi\rangle \leq \sum_{i\in I} \langle \xi, T\xi_i\rangle_{M(\mathcal{X})} \langle T\xi_i, \xi\rangle_{M(\mathcal{X})} \leq B\langle \xi, \xi\rangle . \tag{5.4}$$

Since $Ran(K)$ is dense, $\mathcal{X} = \overline{Ran(K)}$, so K^* is injective. Then from 5.4, T^* is injective since $Ker(T^*) \subseteq Ker(K^*)$. Moreover, $Ran(T) = Ker(T^*)^{\perp} = \mathcal{X}$. Thus T is surjective. □

Theorem 5.11. *Let $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$ be uniformly bounded and let $\{\xi_i\}_{i\in I}$ be a K-frame for $\mathcal{X}$. If $T \in Hom^*_{\mathcal{A}}(\mathcal{X})$ is uniformly bounded and has a closed range with $TK = KT$, then $\{T\xi_i\}_{i\in I}$ is a K-frame for $Ran(T)$.*

Proof. As T has a closed range, it has the generalized inverse $T^\dagger$ such that $TT^\dagger = I$. Now $I = I^* = T^{\dagger^*}T^*$. Then for each $\xi \in Ran(T), K^*\xi = T^{\dagger^*}T^*K^*\xi$, so we have

$$\begin{aligned}\langle K^*\xi, K^*\xi\rangle &= \left\langle T^{\dagger^*}T^*K^*\xi, T^{\dagger^*}T^*K^*\xi\right\rangle \\ &\leq \left\|T^{\dagger^*}\right\|^2_\infty \langle T^*K^*\xi, T^*K^*\xi\rangle\end{aligned}$$

Therefore

$$\left\|T^{\dagger^*}\right\|^{-2}_\infty \langle K^*\xi, K^*\xi\rangle \leq \langle T^*K^*\xi, T^*K^*\xi\rangle .$$

Now for each $\xi \in Ran(T)$,

$$\begin{aligned}\sum_{i\in I} \langle \xi, T\xi_i\rangle_{\mathcal{X}} \langle T\xi_i, \xi\rangle_{\mathcal{X}} &= \sum_{i\in I} \langle T^*\xi, \xi_i\rangle_{\mathcal{X}} \langle \xi_i, T^*\xi\rangle_{\mathcal{X}} \\ &\geq A\langle K^*T^*\xi, K^*T^*\xi\rangle \\ &= A\langle T^*K^*\xi, T^*K^*\xi\rangle \\ &\geq A\left\|T^{\dagger^*}\right\|^{-2}_\infty \langle K^*\xi, K^*\xi\rangle .\end{aligned}$$

Since $\{\xi_i\}_{i\in I}$ is a Bessel sequence with bound B, for each $\xi \in Ran(T)$, we have

$$\begin{aligned}\sum_{i\in I} \langle \xi, T\xi_i\rangle_{M(\mathcal{X})} \langle T\xi_i, \xi\rangle_{M(\mathcal{X})} &= \sum_{i\in I} \langle T^*\xi, \xi_i\rangle_{M(\mathcal{X})} \langle \xi_i, T^*\xi\rangle_{M(\mathcal{X})} \\ &\leq B\langle T^*\xi, T^*\xi\rangle \\ &\leq B\|T\|^2_\infty \langle \xi, \xi\rangle .\end{aligned}$$

Therefore $\{T\xi_i\}_{i\in I}$ is a K-frame for $Ran(T)$. □

Theorem 5.12. *Let $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$ be uniformly bounded such that $Ran(K)$ is dense. Let $\{\xi_i\}_{i\in I}$ be a K-frame and let $T \in Hom^*_{\mathcal{A}}(\mathcal{X})$ be* *uniformly bounded such that $Ran(T)$ is closed. If $\{T\xi_i\}_{i\in I}$ and $\{T^*\xi_i\}_{i\in I}$ are K-frames for $\mathcal{X}$, then T is invertible.*

Proof. Suppose that $\{T\xi_i\}_{i\in I}$ is a K-frame for $\mathcal{X}$ with frame bounds A_1 and B_1. Then for any $\xi \in \mathcal{X}$,

$$A_1 \langle K^*\xi, K^*\xi\rangle \leq \sum_{i\in I} \langle \xi, T\xi_i\rangle_{M(\mathcal{X})} \langle T\xi_i, \xi\rangle_{M(\mathcal{X})} \leq B_1 \langle \xi, \xi\rangle \tag{5.5}$$

As $Ran(K)$ is dense, K^* is injective. Then from 5.5, T^* is injective since $Ker\,(T^*) \subseteq Ker\,(K^*)$. Moreover $Ran(T) = Ker\,(T^*)^{\perp} = \mathcal{X}$. Then T is surjective.

Suppose $\{T^* f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{X}$ with frame bounds A_2 and B_2. Then for any $\xi \in \mathcal{X}$,

$$A_2 \langle K^*\xi, K^*\xi\rangle \leq \sum_{i\in I} \langle \xi, T^*\xi_i\rangle_{M(\mathcal{X})} \langle T^*\xi_i, \xi\rangle_{M(\mathcal{X})} \leq B_2 \langle \xi, \xi\rangle \tag{5.6}$$

As K has a dense range, K^* is injective. Then from 5.6, T is injective since $Ker(T) \subseteq Ker\,(K^*)$. Therefore T is bijective. Using the Bounded Inverse Theorem, T is invertible. □

Theorem 5.13. *Let $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$ and let $\{\xi_i\}_{i\in I}$ be a K-frame for $\mathcal{X}$ and let $T \in Hom^*_{\mathcal{A}}(\mathcal{X})$ be uniformly bounded and be a co-isometry with $TK = KT$. Then $\{T\xi_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{X}$.*

Proof. Suppose $\{\xi_i\}_{i\in I}$ is a K-frame for $\mathcal{X}$. For every $\xi \in \mathcal{X}$

$$\begin{aligned}
\sum_{i\in I} \langle \xi, T\xi_i\rangle_{\mathcal{X}} \langle T\xi_i, \xi\rangle_{\mathcal{X}} &= \sum_{i\in I} \langle T^*\xi, \xi_i\rangle_{\mathcal{X}} \langle \xi_i, T^*\xi\rangle_{\mathcal{X}} \\
&\geq A \langle K^*T^*\xi, K^*T^*\xi\rangle \\
&= A \langle T^*K^*\xi, T^*K^*\xi\rangle \\
&= A \langle K^*\xi, K^*\xi\rangle
\end{aligned}$$

It is obvious that $\{T\xi_i\}_{i\in I}$ is a Bessel sequence. Since $\{\xi_i\}_{i\in I}$ is a Bessel sequence, for each $\xi \in \mathcal{X}$ we have

$$\begin{aligned}
\sum_{i\in I} \langle \xi, T\xi_i\rangle_{\mathcal{X}} \langle T\xi_i, \xi\rangle_{\mathcal{X}} &= \sum_{i\in I} \langle T^*\xi, \xi_i\rangle_{\mathcal{X}} \langle \xi_i, T^*\xi\rangle_{\mathcal{X}} \\
&\leq B \|T\|_{\infty}^2 \langle \xi, \xi\rangle .
\end{aligned}$$

Therefore $\{T\xi_i\}_{i\in I}$ is a K-frame for $\mathcal{X}$. □

6. Sums of K-Frames

In the following section we show that the sums of K-frames under some conditions is again a K-frame in Hilbet pro-C^*-modules.

Theorem 6.1. *Let $\{\xi_i\}_{i\in I}$ and $\{\eta_i\}_{i\in I}$ be two K-frames for $\mathcal{X}$ and let L_1 and L_2 be respectively their corresponding operators in Proposition 5.6. If $L_1L_2^*$ and $L_2L_1^*$ are positive operators and $Ran\,(L_1) + Ran\,(L_2)$ is closed, then $\{\xi_i + \eta_i\}_{i\in I}$ is a K-frame for $\mathcal{X}$.*

Proof. Suppose that $\{\xi_i\}_{i\in I}$ and $\{\eta_i\}_{i\in I}$ are two K-frames for $\mathcal{X}$. By the assumption, there are two bounded operators L_1 and L_2 such that $L_1u_i = \xi_i$, $L_2u_i = \eta_i$ and $Ran(K) \subseteq Ran\,(L_1)\,, Ran(K) \subseteq Ran\,(L_2)$, we denote by $\{u_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}_\mathcal{A}$

So $Ran(K) \subseteq Ran\,(L_1) + Ran\,(L_2)$, by Corollary 3.6

$$KK^* \leq \alpha^2\,(L_1L_1^* + L_2L_2^*)\,, \quad \text{for some } \alpha > 0$$

For every $\xi \in \mathcal{X}$, we have

$$\begin{aligned}
\sum_{i\in I} \langle \xi, \xi_i + \eta_i\rangle \langle \xi_i + \eta_i, \xi\rangle &= \sum_{i\in I} \langle (L_1^* + L_2^*)\xi, u_i\rangle \langle u_i, (L_1{}^* + L_2^*)\xi\rangle \\
&= \sum_{i\in I} \langle (L_1 + L_2)^*\xi, u_i\rangle \langle u_i, (L_1 + L_2)^*\xi\rangle \\
&= \langle (L_1 + L_2)^*\xi, (L_1 + L_2)^*\xi\rangle \\
&= \langle L_1{}^*\xi, L_1{}^*\xi\rangle + \langle L_1{}^*\xi, L_2{}^*\xi\rangle \\
&\quad + \langle L_2{}^*\xi, L_1{}^*\xi\rangle + \langle L_2{}^*\xi, L_2{}^*\xi\rangle \\
&\geq \langle (L_1L_1{}^* + L_2L_2{}^*)\xi, \xi\rangle \\
&\geq \frac{1}{\alpha^2}\left(\langle KK^*\xi, \xi\rangle\right). \\
&\geq \frac{1}{\alpha^2}\left(\langle K^*\xi, K^*\xi\right\rangle.
\end{aligned}$$

Therefore $\{\xi_i + \eta_i\}_{i\in I}$ is a K-frame for $\mathcal{X}$. □

Corollary 6.2. *Let $\{\xi_i\}_{i\in I}$ and $\{\eta_i\}_{i\in I}$ be K-frames for $\mathcal{X}$ with frame operators L_1 and L_2 respectively. Then $K = L_1^{1/2}T_1 + L_2^{1/2}T_2$, for some bounded operators T_1 and T_2 in $Hom^*_\mathcal{A}(\mathcal{X})$.*

Proof. Since $\{\xi_i\}_{i\in I}$ and $\{\eta_i\}_{i\in I}$ are K-frames for $\mathcal{X}$, by Proposition 5.6, there are positive constants A_1 and A_2 such that

$$L_1 \geq A_1KK^*, \qquad L_2 \geq A_2KK^*$$

Then by Douglas' theorem, we have

$$Ran(K) \subseteq Ran\left(L_1^{1/2}\right)$$

and

$$Ran(K) \subseteq Ran\left(L_2^{1/2}\right).$$

Hence $Ran(K) \subseteq Ran\left(S_1^{1/2}\right) + Ran\left(S_2^{1/2}\right)$. Hence by Corollary 3.6, there exists two bounded operators T_1, T_2 in $Hom_{\mathcal{A}}^*(\mathcal{X})$ such that $K = L_1^{1/2}T_1 + L_2^{1/2}T_2$. □

Theorem 6.3. *Suppose that $\{\xi_i\}_{i\in I}$ is a K-frame for $\mathcal{X}$ such that L is its frame operator and let T be a positive operator. Then $\{\xi_i + T\xi_i\}_{i\in I}$ is a K-frame. Furthermore $\{\xi_i + T^n f_i\}_{i\in I}$ is a K-frame for $\mathcal{X}$ for any natural number n.*

Proof. Let $\{\xi_i\}_{i\in I}$ be a K-frame for $\mathcal{X}$. Then by Proposition 5.6, there is $A > 0$ such that $L \geq AKK^*$. $(I+T)L(I+T)^*$ is the frame operator for $\{\xi_i + T\xi_i\}_{i\in I}$ for the raison that for each $\xi \in \mathcal{X}$,

$$\begin{aligned}\sum_{i\in I} \langle \xi, (\xi_i + T\xi_i)\rangle (\xi_i + T\xi_i) &= (I+T)\sum_{i\in I} \langle \xi, (I+T)\xi_i\rangle \xi_i \\ &= (I+T)L(I+T)^*\xi\end{aligned}$$

In addition, we have

$$\begin{aligned}(I+T)L(I+T)^* &= L + LT^* + TL + TLT^* \\ &\geq L \\ &\geq AKK^*\end{aligned}$$

By Propostion 5.6, we can say that $\{\xi_i + T\xi_i\}_{i\in I}$ is a K-frame for $\mathcal{X}$. For every given natural number n, the frame operator for $\{\xi_i + T^n\xi_i\}_{i\in I}$ is $(I+T^n)L(I+T^n)^* \geq L$. Therefore $\{\xi_i + T^n\xi_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{X}$. □

Corollary 6.4. *Let $\{\xi_i\}_{i\in I}$ be a K-frame for $\mathcal{X}$ and let L be its frame operator. Suppose that $\{I_1, I_2\}$ is a partition of $\mathbb{N}$. For $j = 1, 2$, let L_j be the frame operator for the Bessel sequence $\{\xi_i\}_{i\in I_j}$. Then*

$$\{\xi_i L_1^m \xi_i\}_{i\in I_1} \cup \{\xi_i + L_2^n \xi_i\}_{i\in I_2}$$

is a K-frame for $\mathcal{X}$ for every $m, n \in \mathbb{N}$.

Proof. For every $m \in \mathbb{N}$, L^m can be defined as follow

$$L^m\xi = \sum_{i\in I_j} \left\langle \xi, L^{\frac{m-1}{2}}\xi_i \right\rangle L^{\frac{m-1}{2}}\xi_i$$

For each $\xi \in \mathcal{X}$,

$$\sum_{i\in I_1} \langle \xi, \xi_i + L_1^m\xi_i\rangle (\xi_i + L_1^m\xi_i) = (I + L_1^m)\sum_{i\in I_1} \langle \xi, \xi_i + L_1^m\xi_i\rangle \xi_i$$

$$
\begin{aligned}
&= (I + L_1^m) \sum_{i\in I_1} \langle \xi, (I + L_1^m)\, \xi_i\rangle\, \xi_i \\
&= (I + L_1^m)\, L_1\, (I + L_1^m)^*\, \xi \\
&= (I + L_1^m) \left(L_1 + L_1^{(1+m)}\right) \xi \\
&= \left(L_1 + 2L_1^{(1+m)} + L_1^{(1+2m)}\right) \xi.
\end{aligned}
$$

Therefore $L_1 + 2L_1^{(1+m)} + L_1^{(1+2m)}$ and $L_2 + 2L_2^{(1+n)} + L_2^{(1+2n)}$ are the frame operators for $\{\xi_i + L_1^m \xi_i\}_{i\in I_1}$ and $\{\xi_i + L_2^n \xi_i\}_{i\in I_2}$ respectively. Consider that L_0 is the frame operator for $\{\xi_i + L_1^m\xi_i\}_{i\in I_1} \cup \{\xi_i + L_2^n\xi_i\}_{i\in I_2}$.

As $\{\xi_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{X}$, then there exists $A > 0$ such that $L \geq AKK^*$ and $L_0 \geq L_1 + L_2 = L \geq AKK^*$. From which $\{\xi_i + L_1^m\xi_i\}_{i\in I_1} \cup \{\xi_i + L_2^n\xi_i\}_{i\in I_2}$ is a K-frame for $\mathcal{X}$. □

Theorem 6.5. *Let $\{\xi_i\}_{i\in I}$ and $\{\eta_i\}_{i\in I}$ be Parseval K-frames for $\mathcal{X}$, with synthesis operators L_1 and L_2 respectively. If $L_1 L_2^* = 0$ then $\{\xi_i + \eta_i\}_{i\in I}$ is a 2-tight K-frame for $\mathcal{X}$.*

Proof. Suppose $\{\xi_i\}_{i\in I}$ and $\{\eta_i\}_{i\in I}$ are two Parseval K-frames for $\mathcal{X}$. Then there are transform operators $L_1, L_2 \in Hom^*_{\mathcal{A}}(\mathcal{X})$ such that $L_1 e_i = \xi_i$ and $L_2 e_i = \eta_i$ with $Ran(K) = Ran(L_1), Ran(K) = Ran(L_2)$ respectively. For each $\xi \in \mathcal{X}$, we have

$$
\begin{aligned}
\sum_{i\in I} \langle \xi, \xi_i + \eta_i\rangle_{M(\mathcal{X})} \langle \xi_i + \eta_i, \xi\rangle_{M(\mathcal{X})} &= \langle (L_1 + L_2)^*\, \xi, (L_1 + L_2)^*\, \xi\rangle_{\mathcal{X}} \\
&= \langle L_1^*\xi, L_1^*\xi\rangle_{\mathcal{X}} + \langle L_2 L_1^*\xi, \xi\rangle_{\mathcal{X}} \\
&\quad + \langle L_1 L_2^*\xi, L_1 L_2^*\xi\rangle_{\mathcal{X}} + \langle L_2^*\xi, L_2^*\xi\rangle_{\mathcal{X}} \\
&= \langle L_1^*\xi, L_1^*\xi\rangle_{\mathcal{X}} + \langle L_2^*\xi, L_2^*\xi\rangle_{\mathcal{X}} \\
&= 2\, \langle K^*\xi, K^*\xi\rangle_{\mathcal{X}}
\end{aligned}
$$
□

Theorem 6.6. *Let $\{\xi_i\}_{i\in I}$ and $\{\eta_i\}_{i\in I}$ be K-frames for $\mathcal{X}$ and let L_1 and L_2 be synthesis operators for sequences $\{\xi_i\}_{i\in I}$ and $\{\eta_i\}_{i\in I}$ respectively, such that $L_1 L_2^* = 0$ and let $T_j \in Hom^*_{\mathcal{A}}(\mathcal{X})$ an operator uniformly bounded with $R(L_j) \subseteq R(T_j L_j)$, for $j = 1, 2$. Then $\{T_1\xi_i + T_2\eta_i\}_{i\in I}$ is a K-frame for $\mathcal{X}$.*

Proof. Suppose that $\{\xi_i\}_{i\in I}$ and $\{\eta_i\}_{i\in I}$ are two K-frames for $\mathcal{X}$. Then by Theorem 5.5, there exists an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ in $\mathcal{H}_{\mathcal{A}}$ such that $L_1 e_i = \xi_i, L_2 e_i = \eta_i$ and $R(K) \subseteq R(L_1), R(K) \subseteq R(L_2)$. For each $\xi \in \mathcal{X}$

$$
\sum_{i\in I} \langle \xi, T_1\xi_i + T_2\eta_i\rangle_{M(\mathcal{X})} \langle \xi, T_1\xi_i + T_2\eta_i\rangle_{M(\mathcal{X})}
$$

$$
\begin{aligned}
&= \sum_{i\in I} \langle \xi, T_1T_1L_1e_i + T_2L_2e_i\rangle_{M(\mathcal{X})} \langle \xi, T_1L_1e_i + T_2L_2e_i\rangle_{M(\mathcal{X})} \\
&= \langle (T_1L_1 + T_2L_2)^* \xi, (T_1L_1 + T_2L_2)^* \xi\rangle \\
&= \langle (T_1L_1)^* \xi, (T_1L_1)^* \xi\rangle + \langle T_2L_2L_1^*T_1^*\xi, \xi\rangle \\
&\quad + \langle T_1L_1L_2^*T_2^*\xi, \xi\rangle + \langle (T_2L_2)^* \xi, (T_2L_2)^* \xi\rangle \\
&= \langle (T_1L_1)^* \xi, (T_1L_1)^* \xi\rangle + \langle (T_2L_2)^* \xi, (T_2L_2)^* \xi\rangle
\end{aligned}
$$

We have that $Ran(K) \subseteq Ran(L_j) \subseteq Ran(T_jL_j)$ for $j = 1, 2$. So by Douglas' factorization theorem, for each $j = 1, 2$, there exists $\alpha_j > 0$ such that

$$KK^* \leq \alpha_j (T_jL_j)(T_jL_j)^*$$

Then from the above inequality, for each $\xi \in \mathcal{X}$

$$
\begin{aligned}
&\sum_{i\in I} \langle \xi, T_1\xi_i + T_2\eta_i\rangle_{M(\mathcal{X})} \langle \xi, T_1\xi_i + T_2\eta_i\rangle_{M(\mathcal{X})} \\
&\quad = \langle (T_1L_1)^* \xi, (T_1L_1)^* \xi\rangle + \langle (T_2L_2)^* \xi, (T_2L_2)^* \xi\rangle \\
&\quad \geq \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right) \langle K^*\xi, K^*\xi\rangle
\end{aligned}
$$

Hence $\{T_1\xi_i + T_2\eta_i\}_{i\in I}$ is a K-frame for $\mathcal{X}$. □

Acknowledgment. The authors thank the area editor and referees for their valuable comments and suggestions.

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[1]LaSMA Laboratory Department of Mathematics Faculty of Sciences, Dhar El Mahraz University Sidi Mohamed Ben Abdellah, Fez, Morocco.
Email address: mohamed.rossafi@usmba.ac.ma

[2]Laboratory of Partial Differential Equations, Spectral Algebra and Geometry, Department of Mathematics, Faculty Of Sciences, University of Ibn Tofail, Kenitra, Morocco.
Email address: roumaissae.eljazzar@uit.ac.ma

[3] Department of Mathematics, University of Central Florida, Orlando, FL., 32816, USA.
Email address: ram.mohapatra@ucf.edu