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New Subclass of Convex Functions Concerning Infinite Cone

Fatolah Hasanvand[1], Shahram Najafzadeh[2*] and Ali Ebadian[3]

Abstract. We introduce a new subclass of convex functions as follows:

$$\mathcal{K}_{IC} := \left\{ f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|f'(z) - 1\right|, \quad |z| < 1 \right\},$$

where $\mathcal{A}$ denotes the class of analytic and normalized functions in the unit disk $|z| < 1$. Some properties of this particular class, including subordination relation, integral representation, the radius of convexity, rotation theorem, sharp coefficients estimate and Fekete-Szegö inequality associated with the k-th root transform, are investigated.

1. Introduction

For $r > 0$, let $\Delta_r := \{z \in \mathbb{C} : |z| < r\}$ and let $\Delta_1 = \Delta$ be the open unit disk on the complex plane. We denote by $\mathcal{A}$ the class of all analytic and normalized functions Δ. Indeed any $f \in \mathcal{A}$ has the following representation

$$f(z) = z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n + \cdots, \quad (z \in \Delta). \tag{1.1}$$

Also, we denote by $\mathcal{S} \subset \mathcal{A}$, the class of univalent functions. For each univalent function f of the form (1.1), the k–th root transform is defined by

$$F(z) = \left[f\left(z^k\right)\right]^{1/k} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}, \quad (z \in \Delta). \tag{1.2}$$

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Further, we denote by $\mathcal{P}$ the well-known class of analytic functions p with $p(0) = 1$ and $\mathrm{Re}(p(z)) > 0$ $(z \in \Delta)$. If f and g are two of the functions $\mathcal{A}$, then we say that f is subordinate to g, written $f(z) \prec g(z)$, if there exists a Schwarz function w such that $f(z) = g(w(z))$ for all $z \in \Delta$. In particular, if the function g is univalent Δ, then we have the following equivalence:

$$f(z) \prec g(z) \quad \Leftrightarrow \quad (f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta)).$$

Next, we say that a function $f \in \mathcal{A}$ belongs to the class $\mathcal{Q}(\alpha)$ if it satisfies the following condition

$$\mathrm{Re}\left(\frac{f(z)}{z}\right) > \alpha, \quad (0 \leq \alpha < 1, z \in \Delta).$$

Also, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}(\alpha)$ if f satisfies

$$\mathrm{Re}\left(f'(z)\right) > \alpha, \quad (0 \leq \alpha < 1, z \in \Delta).$$

We note that $f \in \mathcal{R}(\alpha)$ if and only if $zf' \in \mathcal{Q}(\alpha)$. It is well-known that any $f \in \mathcal{R}(\alpha)$ is univalent in Δ (see [12] or [20]).

Robertson in [14] introduced the family $\mathcal{K}(\alpha)$ of functions f convex of order α, $0 \leq \alpha < 1$, that satisfy the inequality

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (z \in \Delta).$$

As usual, we write $\mathcal{K}(0) \equiv \mathcal{K}$ and denotes the class of convex functions. It is necessary to point out that there are several well–known subclasses of $\mathcal{K}$. For example, Ruscheweyh [16] defined the subclass $\mathcal{D}$ of $\mathcal{K}$ including all functions f for which

$$\mathrm{Re}(f'(z)) \geq \left|zf''(z)\right|, \quad (z \in \Delta).$$

Also, Goodman (see [4]) introduced the family $\mathcal{UCV} \subset \mathcal{K}$ of uniformly convex functions f having the property that for every circular arc γ contained in Δ with center also in Δ, the image arc $f(\gamma)$ is a convex arc. He then gave the two–variable characterization

$$\mathrm{Re}\left(1 + \frac{(z-\zeta)f''(z)}{f'(z)}\right) > 0, \quad ((z, \zeta) \in \Delta \times \Delta).$$

Afterwards, Ma and Minda [8, 9] and Rønning [15] independently found a more applicable one-variable characterization for $\mathcal{UCV}$, namely

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|, \quad (z \in \Delta).$$

Finally, in 2015, Sokół and Nunokawa (see [18]) introduced and studied a certain subclass of convex functions as follows

$$\mathcal{MN} := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right| \right\}.$$

They obtained the order of starlikeness and strong starlikeness for the class $\mathcal{MN}$. However, in this paper, by Goodman's idea and motivated by the above classes, we introduce a new subclass of convex functions as follows

$$\mathcal{K}_{IC} := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|f'(z) - 1\right| \right\}.$$

It is clear that $\mathcal{K}_{IC} \subset \mathcal{K}$. Letting

$$1 + \frac{zf''(z)}{f'(z)} = u_1 + iv_1, \qquad f'(z) = u_2 + iv_2,$$

where $u_i, v_i \in \mathbb{R}$ $(i = 1, 2)$ and $f \in \mathcal{K}_{IC}$, we conclude that $u_1 > |u_2 + iv_2 - 1|$ or equivalently $u_1^2 - \left(u_2^2 - 1\right)^2 - v_2^2 > 0$. It is clear that $u_1^2 - \left(u_2^2 - 1\right)^2 - v_2^2 > 0$ is infinite cone. Thus the class $\mathcal{K}_{IC}$ is related to the infinite cone.

The following lemmas will be useful.

Lemma 1.1 (Generalization of Nunokawa's lemma [13]). *Let $p(z)$ be an analytic function in $|z| < 1$ of the form*

$$p(z) = 1 + \sum_{n=m}^{\infty} c_n z^n, \quad (c_m \neq 0),$$

with $p(z) \neq 0$ in $|z| < 1$. If there exists a point z_0, $|z_0| < 1$, such that

$$|\arg\{p(z)\}| < \frac{\pi\varphi}{2} \text{ for } |z| < |z_0|$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi\varphi}{2}$$

for some $\varphi > 0$. Then we have

$$\frac{z_0 p'(z_0)}{p(z)} = il\varphi,$$

where

$$l \geq \frac{m}{2}\left(a + \frac{1}{a}\right) \geq m \text{ when } \arg\{p(z_0)\} = \frac{\pi\varphi}{2} \tag{1.3}$$

and

$$l \leq -\frac{m}{2}\left(a + \frac{1}{a}\right) \leq -m \text{ when } \arg\{p(z_0)\} = -\frac{\pi\varphi}{2} \tag{1.4}$$

where

$$\{p(z_0)\}^{1/\varphi} = \pm ia, \quad a > 0.$$

In complex analysis, the Fekete-Szegö inequality is an inequality for the coefficients of univalent analytic functions found by Fekete-Szegö [3] related to the Bieberbach conjecture. By applying this inequality, we can find many interesting geometric properties of the defined subclasses. The Fekete-Szegö inequality states that if

$$f(z) = z + \sum_{k=2}^{+\infty} a_k z^k,$$

is a univalent function on the open unit disk and $0 \le \lambda < 1$, then

$$|a_3 - \lambda a_2^2| \le 1 + 2\exp\left(\frac{-2\lambda}{(1-\lambda)}\right).$$

See [5, 10].

Lemma 1.2. ([7]) *Let the function $g(z)$ given by*

$$g(z) = 1 + c_1 z + c_2 z^2 + \cdots,$$

be in the class $\mathcal{P}$. Then, for any complex number μ

$$|c_2 - \mu c_1^2| \le 2\max\{1, |2\mu - 1|\}.$$

The result is sharp.

In the present paper, we study some interesting properties of the class $\mathcal{K}_{IC}$. All of our results are the best.

2. Main Results

The first result is the following, which plays a vital role in order to prove other results. We need the following definition.

Definition 2.1. The univalent function q is called a dominance of the solution of a subordination equation if $p \prec q$ for all p satisfies the equation. Also, a dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominant q is said to be the best dominant.

Theorem 2.2. *Let the function $f \in \mathcal{A}$ belongs to the class $\mathcal{K}_{IC}$. Then we have*

$$f'(z) \prec \frac{1}{1-z}, \quad (z \in \Delta). \tag{2.1}$$

The function $\frac{1}{1-z}$ is the best dominante.

Proof. Let $f \in \mathcal{A}$. If we define $p(z) := f'(z)$, then p is analytic function in Δ and $p(0) = 1$. A simple calculation gives us

$$1 + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)}, \quad (z \in \Delta). \tag{2.2}$$

Since $f \in \mathcal{K}_{IC}$, by using (2.2) we have

$$\begin{aligned} \mathrm{Re}\left(1 + \frac{zp'(z)}{p(z)}\right) &> |p(z) - 1| \\ &> \mathrm{Re}(1 - p(z)), \end{aligned} \tag{2.3}$$

or

$$\mathrm{Re}\left(p(z) + \frac{zp'(z)}{p(z)}\right) > 0, \quad (z \in \Delta). \tag{2.4}$$

By the subordination principle, the above inequality (2.4) is equivalent to the following Briot–Boquet differential subordination

$$p(z) + \frac{zp'(z)}{p(z)} \prec \frac{1+z}{1-z} \qquad (z \in \Delta). \tag{2.5}$$

Applying Theorem 3.3d, [11, p. 109], we conclude that

$$p(z) \prec q(z) \prec \frac{1+z}{1-z},$$

where $q(z)$ is the univalent solution of the differential equation

$$q(z) + \frac{zq'(z)}{q(z)} = \frac{1+z}{1-z}, \quad (z \in \Delta). \tag{2.6}$$

Also $q(z)$ is the best dominant of (2.5). It is easy to see that the solution of the differential equation (2.6) is equal to

$$q(z) = \left(\int_0^1 \left(\frac{1-z}{1-tz}\right)^2 dt\right)^{-1} = \frac{1}{1-z} \tag{2.7}$$

and concluding the proof. □

Ruschewyeh and Singh (see [17]) showed that if q is the analytic solution of (2.6), then

$$\min_{|z|=r} \mathrm{Re}(p(z)) \geq \min_{|z|=r}(q(z)).$$

Because $\mathrm{Re}\{1/(1-z)\} > 1/2$, we have the following.

Corollary 2.3. $\mathcal{K}_{IC} \subset \mathcal{R}(1/2)$. *This means that the members of the class* $\mathcal{K}_{IC}$ *are univalent.*

In the following result, applying Theorem 2.2, we present an integral representation for functions in the class $\mathcal{K}_{IC}$.

Theorem 2.4. *If $f \in \mathcal{K}_{IC}$, then*

$$f(z) = \int_0^z \frac{\mathrm{d}t}{1 - w(t)}, \quad (z \in \Delta), \tag{2.8}$$

where $w(t)$ is Schwarz function.

Proof. The proof is very easy and thus we omit the details. □

Applying formula (2.8) for $w(z) = z$ gives that

$$\begin{aligned} \widetilde{f}(z) &= -\log(1 - z) \\ &= z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^n}{n} + \cdots, \quad (z \in \Delta) \end{aligned} \tag{2.9}$$

which is extremal function for several problems in the class $\mathcal{K}_{IC}$. The Figure 2 shows the image of Δ under the function $\widetilde{f}$. Also we note that $\widetilde{f} \in \mathcal{K}(1/2)$.

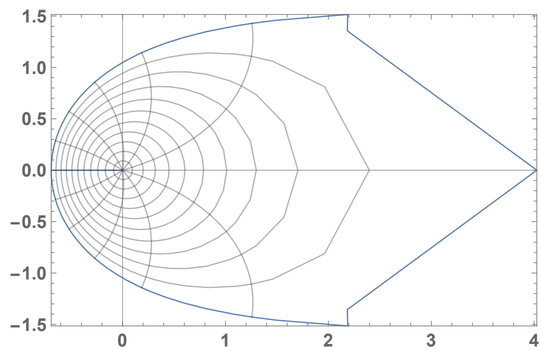

FIGURE 1. The boundary curve of $\widetilde{f}(\Delta)$

In the next theorem we show that there are plenty of functions in the class $\mathcal{K}_{IC}$.

Theorem 2.5. *A function*

$$\ell_\tau(z) = \frac{z}{1 - \tau z}, \quad (z \in \Delta), \tag{2.10}$$

is in the class $\mathcal{K}_{IC}$ if and only if $\tau < 1/2$.

Proof. If $\tau = 0$, then the proof is obvious. Let $\tau \neq 0$. With a simple calculation (2.10) implies that

$$1 + \frac{z\ell_\tau''(z)}{\ell_\tau'(z)} = \frac{1 + \tau z}{1 - \tau z} =: F_\tau(z), \quad (z \in \Delta)$$

and

$$\ell_\tau'(z) - 1 = \frac{1}{(1 - \tau z)^2} - 1 =: G_\tau(z), \quad (z \in \Delta).$$

Thus

$$\mathrm{Re}\left(1+\frac{z\ell_\tau''(z)}{\ell_\tau'(z)}\right)=\mathrm{Re}\left(F_\tau(z)\right),\quad (z\in\Delta)$$

and

$$\left|\ell_\tau'(z)-1\right|=\left|\frac{1}{(1-\tau z)^2}-1\right|.$$

Easily seen that the functions $F_\tau(z)$ and $G_\tau(z)$ do not have any poles in $\overline{\Delta}$ and are analytic in Δ, when $\tau<1/2$. Therefore, looking for the $\min\{\mathrm{Re}(F_\tau(z)) : z\in\Delta\}$ it is sufficient to consider it on the boundary $\partial F_\tau(\Delta)=\{\partial F_\tau(e^{i\theta}) : \theta\in(-\pi,\pi)\}$. Thus we have

$$\mathrm{Re}(F_\tau(z))=\frac{1-\tau^2}{|1-\tau e^{i\theta}|^2}. \tag{2.11}$$

Also

$$\left|\frac{1}{(1-\tau e^{i\theta})^2}-1\right|\geq\frac{1}{|1-\tau e^{i\theta}|^2}-1. \tag{2.12}$$

Now from (2.11) and (2.12) and by definition of $\mathcal{K}_{IC}$ we have

$$\frac{1-\tau^2}{|1-\tau e^{i\theta}|^2}>\frac{1}{|1-\tau e^{i\theta}|^2}-1,$$

or $1>2\tau\cos\theta$ where $-\pi<\theta<\pi$ and concluding the proof. □

The bounds for $|\arg f'(z)|$ constitute the rotation theorem and have been considered by many researchers. They have earned it for some subclasses of univalent functions. For example, if $f\in\mathcal{K}$, then we have $|\arg f'(z)|\leq 2\arcsin(r)$ (see [2]). In the next theorem, we shall consider this problem for the class $\mathcal{K}_{IC}$.

Theorem 2.6. *If the function f belongs to the class $\mathcal{K}_{IC}$, then*

$$|\arg f'(z)|<\frac{\pi}{2},\quad (z\in\Delta). \tag{2.13}$$

The result is sharp.

Proof. Let $f\in\mathcal{K}_{IC}$. Then by definition of $\mathcal{K}_{IC}$ and taking $p(z)=f'(z)$, we have

$$\mathrm{Re}\left(1+\frac{zp'(z)}{p(z)}\right)>|p(z)-1|,\quad (z\in\Delta). \tag{2.14}$$

Suppose that there exists a point z_0, $|z_0|<1$, such that

$$|\arg\{p(z)\}|<\frac{\pi}{2},\quad (|z|<|z_0|)$$

and

$$|\arg\{p(z_0)\}|=\frac{\pi}{2}.$$

Then from Lemma 1.1, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = il,$$

where

$$l \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \text{ when } \arg\{p(z_0)\} = \frac{\pi}{2} \tag{2.15}$$

and

$$l \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \text{ when } \arg\{p(z_0)\} = -\frac{\pi}{2} \tag{2.16}$$

where

$$p(z_0) = \pm ia, \qquad a > 0.$$

In the both cases $\arg\{p(z_0)\} = \frac{\pi}{2}$ and $\arg\{p(z_0)\} = -\frac{\pi}{2}$, we have

$$\operatorname{Re}\left(1 + \frac{z_0 p'(z_0)}{p(z_0)}\right) = 1 + \operatorname{Re}\{il\} = 1 \tag{2.17}$$

and

$$|p(z_0) - 1| = |\pm ia - 1| = \sqrt{a^2+1}, \quad (a > 0). \tag{2.18}$$

Since $1 < \sqrt{a^2+1}$, and by (2.17) and (2.18), we get

$$\operatorname{Re}\left(1 + \frac{z_0 p'(z_0)}{p(z_0)}\right) < |p(z_0) - 1|,$$

which contradicts (2.14). Therefore,

$$|\arg\{p(z)\}| = |\arg f'(z)| < \frac{\pi}{2}, \quad (z \in \Delta).$$

We consider the function $\widetilde{f}(z) = -\log(1-z)$ for the sharpness. It is a simple exercise that

$$\left|\arg \widetilde{f}'(z)\right| = \left|\arg \frac{1}{1-z}\right| < \frac{\pi}{2}$$

and thus, the proof of this theorem is completed. □

3. On Coefficients

In this section first, coefficient estimates of functions which belong to the class $\mathcal{K}_{IC}$ is obtained.

Theorem 3.1. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}_{IC}$, *then*

$$|a_n| \leq \frac{1}{n}, \quad (n \geq 2). \tag{3.1}$$

The result is sharp.

Proof. Let $f \in \mathcal{K}_{IC}$. Then by using the Theorem 2.2, we have

$$f'(z) \prec \frac{1}{1-z}, \quad (z \in \Delta).$$

The above subordination relation gives that

$$\sum_{n=2}^{\infty} n a_n z^{n-1} \prec \sum_{n=2}^{\infty} z^{n-1}.$$

Now, by Rogosinski's theorem, we get $n|a_n| \le 1$ $(n = 2, 3, \ldots)$ and conclude the proof. Also, the result is sharp for the function $\widetilde{f}$ that defined in (2.9). This is the end of proof. □

It is well–known that if f is of the form (1.1) and is a convex univalent function in the unit disk Δ, then $|a_n| \le 1$ $(n = 2, 3, \ldots)$, and the estimate is sharp. Since $\mathcal{K}_{IC} \subset \mathcal{K}$, thus coefficient estimates of the functions which belong to the class $\mathcal{K}_{IC}$, i.e. $1/n$ $(n = 2, 3, \ldots)$ are less than 1, the bound of the coefficients of convex functions.

We consider this problem for functions which are in the class $\mathcal{K}_{IC}$. See [1, 6, 19].

Theorem 3.2. *Let $f \in \mathcal{K}_{IC}$ and F be the k–th root transform of f defined by* (1.2)*. Then, for any complex number μ,*

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \le \max\left\{1, \frac{3}{16k}\left|k - 1 + 2\mu\right|\right\}. \tag{3.2}$$

The result is sharp.

Proof. Let $f \in \mathcal{K}_{IC}$. Then from Theorem 2.2 and by subordination principle, there exists a Schwarz function $w(z)$ such that it satisfies in

$$f'(z) = \frac{1}{1 - w(z)}, \quad (z \in \Delta). \tag{3.3}$$

Define

$$p(z) := \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + \cdots. \tag{3.4}$$

Since $w(z)$ is Schwarz function, thus $p(z) \in \mathcal{P}$. From (3.3), we have

$$1 + 2a_2 z + 3a_3 z^2 + \cdots = 1 + \frac{1}{2} p_1 z + \frac{1}{2} p_2 z^2 + \cdots. \tag{3.5}$$

Equating the coefficients of z and z^2 on both sides of (3.5), we get

$$a_2 = \frac{1}{4} p_1 \tag{3.6}$$

and

$$a_3 = \frac{1}{6} p_2. \tag{3.7}$$

On the other hand, easily seen that for each f given by (1.1), we have

$$F(z) = [f(z^{1/k})]^{1/k} = z + \frac{1}{k}a_2 z^{k+1} + \left(\frac{1}{k}a_3 - \frac{1}{2}\frac{k-1}{k^2}a_2^2\right) z^{2k+1} + \cdots . \tag{3.8}$$

The equations (1.2) and (3.8), give us

$$b_{k+1} = \frac{1}{k}a_2, \qquad b_{2k+1} = \frac{1}{k}a_3 - \frac{1}{2}\frac{k-1}{k^2}a_2^2. \tag{3.9}$$

Substituting (3.6) and (3.7) into (3.9), we get

$$b_{k+1} = \frac{1}{4k}p_1,$$

and

$$b_{2k+1} = \frac{1}{6k}p_2 - \frac{1}{32}\frac{k-1}{k}p_1^2.$$

Therefore

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{1}{6k}\left[p_2 - \frac{1}{2}\left(\frac{3(k-1)}{8k} + \frac{3\mu}{4k}\right)p_1^2\right]. \tag{3.10}$$

Assuming

$$\mu' := \frac{1}{2}\left(\frac{3(k-1)}{8k} + \frac{3\mu}{4k}\right)$$

Moreover, as an application of Lemma 1.2, the inequality (3.2) follows. A simple check implies that the result is sharp for the function's k–th root transforms (2.9). □

The problem of finding sharp upper bounds for the coefficient functional $|a_3 - \mu a_2^2|$ for different subclasses of the normalized analytic function class $\mathcal{A}$ is known as the Fekete-Szegö problem. Moreover, by putting $k = 1$ in Theorem 3.2, we have:

Corollary 3.3 (Fekete-Szegö inequality). *If $f \in \mathcal{K}_{IC}$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \le \max\left\{1, \frac{3|\mu|}{8}\right\}.$$

The result is sharp.

Conclusions

In Geometric Function Theory, many authors have studied and investigated various coefficient functionals of other classes of analytic functions. Using a subordination structure, we achieved the bounds of coefficients and arguments.

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