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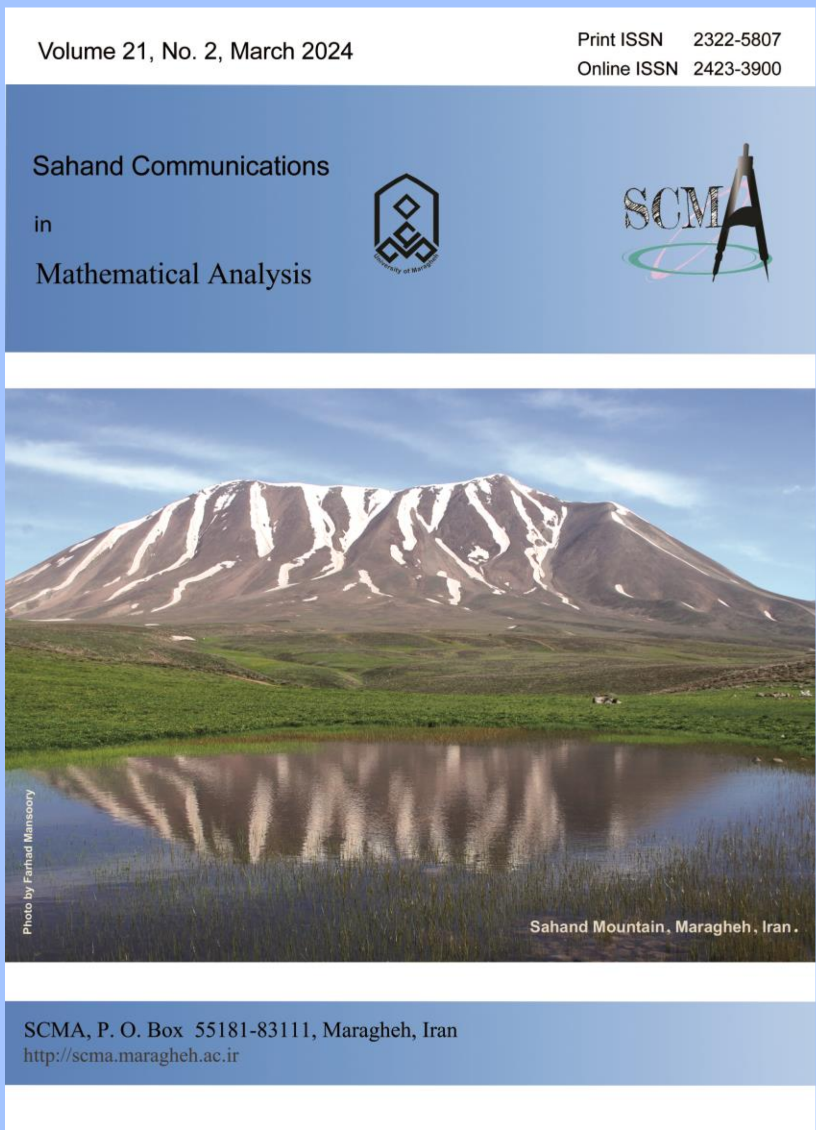

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Beta-Bazilevič Function

Sa'adatul Fitri[1*] and Mohamad Muslikh[2]

Abstract. In this paper, we introduce a relatively new class, $\mathcal{B}_1^{\beta}(\alpha)$, namely the class of Beta-Bazilevič function is generated by the function Bazilevič $\mathcal{B}_1(\alpha)$. We introduce the class in question by constructing the Alpha-Convex function, $\mathcal{M}(\alpha)$, introduced by Miller et al. [9]. Using Lemmas of function with positive real part, we were given a sharp estimate of coefficient problems. The coefficient problems to be solved are the modulus of initial coefficients f, the modulus of inverse coefficients f^{-1}, the modulus of the Logarithmic coefficients $\log \frac{f(z)}{z}$, the Fekete-Szegö problem and the second Hankel determinant problem.

1. Introduction and Definitions

Geometric function theory is a branch of complex analysis that studies the geometric properties of analytic functions. Although this theory appeared in early 20th century, it is still interesting as a field of research nowadays. The base of geometric functions is the theory of the univalent functions. Among interesting topics widely discussed are harmonic theory and quasiconformal mapping, both generalizations of conformal mapping [3].

The first results of the theory of univalent functions were obtained by using the area principle. The collection of all analytic and univalent functions on disk $\mathbb{D} = \{z : |z| < 1\}$, normalized by the conditions $a_0 = 0$ and $a_1 = 1$ that be denoted by S. The functions $f \in \mathcal{S}$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

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In 1916, L. Bieberbach proved that the modulus of the second coefficient for the function f, $|a_2| \leq 2$, using the outer area theorem and suspected that $|a_n| \leq n$ for $n \geq 2$ [4]. These results affect the study of science in the theory of geometric functions. Therefore, it is essential to find new theoretical results in this field with various applications [5].

Several major subclasses of $\mathcal{S}$ have been extensively studied, and more characteristics can be defined, e.g. the classes convex ($\mathcal{C}$), starlike ($\mathcal{S}^*$), close-to-convex ($\mathcal{K}$) and Bazilevič functions. Bazilevič[2] introduced the so-called Bazilevič functions using a differential equation of Löewner and Kufaref in 1923. The classes of Bazilevič functions are denoted by $\mathcal{B}(\alpha, \delta)$ with $\alpha \geq 0$ and δ real. The classes of Bazilevič functions are the largest subclass of $\mathcal{S}$. It is simple to show that $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{B}(\alpha, \delta) \subset \mathcal{S}$.

In recent years, much of attention has been given to the subclasses of Bazilevič functions. The topics discussed are coefficient problems and subordination. Thomas and Marjono[8] studied coefficient problems and subordination in subclass $\mathcal{B}_1(\alpha)$, whose definition mimics that of $\mathcal{U}(\lambda = 1)$. The open problem is when $0 < \lambda < 1$. In addition to the coefficient problem of the function $f \in \mathcal{S}$, several problems can be constructed from the coefficients a_n. Among others are inverse coefficients, logarithmic coefficients, the Fekete-Szegö problem and the second Hankel determinant. Murugusundaramoorthy and Bulboaca studied new subclasses of analytic functions related to a shell-shaped region in 2020[10]. They determined the estimated bounds of the four initial coefficients, the upper bound for the Fekete–Szegö functional and for the Hankel determinant. In 2021, Murugusundaramoorthy, G. et al. examined initial coefficient bounds and Fekete-Szegö inequalities for a subclass of Kamali-type starlike functions connected with the limacon domain of bean shape[11].

We shall be concerned with the subclass $\mathcal{B}_1(\alpha)$ which first introduced by Singh[12].

Definition 1.1. $f \in \mathcal{B}_1(\alpha)$ for $\alpha \geq 0$ if, and only if, for $z \in \mathbb{D}$,

$$\operatorname{Re}\left\{ f'(z) \left(\frac{f(z)}{z} \right)^{\alpha-1} \right\} > 0.$$

In 2020, Fitri and Thomas introduced the class $\mathcal{B}_1^{\gamma}(\alpha)$ of Gamma-Bazilevič functions[6]. They shown that $\mathcal{B}_1^{\gamma}(\alpha)$ is univalent and subset of $\mathcal{B}_1(\alpha)$. Some coefficient problems for this functions are also studied. Definition of Gamma-Bazilevič functions is analogue with Gamma-starlike functions, $\mathcal{M}^{\gamma}$ defined by,

$$\operatorname{Re}\left\{ \left(1 + \frac{zf''(z)}{f'(z)}\right)^{\gamma} \left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma} \right\} > 0,$$

for $z \in \mathbb{D}$ and $\gamma \geq 0$. This class was introduced by Darus and Thomas[4]. Previosly, Miller, Mocanu and Reade[9] have studied the class of Alpha-Convex functions, $\mathcal{M}(\alpha)$ defined by,

$$\operatorname{Re}\left\{\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)\right\}>0,$$

for $z \in \mathbb{D}$ and $\alpha \geq 0$.

The purpose of this paper is to introduce an analogue of $\mathcal{M}(\alpha)$ for Bazilevič functions. The following is the definition in question.

Definition 1.2. Let $\beta \geq 0$ and $0 \leq \alpha \leq 1$, with $f(z) \neq 0$ and $f'(z) \neq 0$. A function $f \in S$ is said to be Beta-Bazilevič functions if for $z \in \mathbb{D}$,

$$\begin{aligned} \operatorname{Re}\Bigg\{\beta\left[\frac{zf'(z)}{f(z)^{1-\alpha}z^{\alpha}}+\frac{zf''(z)}{f'(z)}+(\alpha-1)\left(\frac{zf'(z)}{f(z)}-1\right)\right] \\ +(1-\beta)\left[\frac{zf'(z)}{f(z)^{1-\alpha}z^{\alpha}}\right]\Bigg\} \\ >0. \end{aligned} \tag{1.2}$$

We denote this class by $\mathcal{B}_1^{\beta}(\alpha)$.

Then $\mathcal{B}_1^{0}(\alpha)$ is the class of Bazilevič functions[12], and $\mathcal{B}_1^{\gamma}(0)$ is the class of Alpha-Convex functions $\mathcal{M}(\alpha)$[9]. We also note that the case $\alpha = 1$ and $\gamma = 0$ corresponds to the class $\mathcal{R}$ of functions who's derivative has positive real part. The case $\alpha = 0$ and $\beta = 0$ corresponds to the starlike functions and when $\alpha = 0$ and $\beta = 1$ to convex functions[5].

Definition 1.3. A function $f \in S$ is said to be starlike functions if for $z \in \mathbb{D}$,

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0.$$

Definition 1.4. A function $f \in S$ is said to be convex functions if for $z \in \mathbb{D}$,

$$\operatorname{Re}\left\{\frac{1+zf"(z)}{f'(z)}\right\}>0.$$

2. Preliminaries

In this section, we recall the class $\mathcal{P}$ of functions with positive real part in $\mathbb{D}$, so that $p \in \mathcal{P}$,

$$p(z)=1+\sum_{n=1}^{\infty}c_n z^n, \tag{2.1}$$

if, and only if, $\operatorname{Re}[p(z)] > 0$ for $z \in \mathbb{D}$.

We shall use the following lemmas concerning the coefficients of $p \in \mathcal{P}$.

Lemma 2.1 ([7]). *If $p \in \mathcal{P}$, then for some complex valued x with $|x| \leq 1$, and some complex valued ζ with $|\zeta| \leq 1$,*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right),$$
$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)\zeta.$$

Lemma 2.2 ([1]). *If $p \in \mathcal{P}$, then $|c_n| \leq 2$ for $n \geq 1$, and*

$$\left|c_2 - \frac{\mu}{2}c_1^2\right| \leq \max\{2,\ 2|\mu - 1|\} = \begin{cases} 2, & 0 \leq \mu \leq 2, \\ 2|\mu - 1|, & \textit{elsewhere.} \end{cases}$$

Lemma 2.3 ([1]). *Let $p \in \mathcal{P}$. If $0 \leq B \leq 1$ and $B(2B - 1) \leq D \leq B$, then*

$$\left|c_3 - 2Bc_1c_2 + Dc_1^3\right| \leq 2.$$

Lemma 2.4 ([1]). *Let $p \in \mathcal{P}$. If $0 \leq \lambda \leq 1$, then*

$$|c_3 - 2\lambda c_1 c_2 + \lambda c_1^3| \leq 2.$$

3. Initial Coefficients

We first determine expressions for a_2, a_3 and a_4 in terms of the coefficients of $p \in \mathcal{P}$. It follows from (1.2) that we get the equivalen form,

$$\begin{aligned} \beta&\left[\frac{zf'(z)}{f(z)^{1-\alpha}z^{\alpha}} + \frac{zf''(z)}{f'(z)} + (\alpha - 1)\left(\frac{zf'(z)}{f(z)} - 1\right)\right] \\ &+ (1 - \beta)\left[\frac{zf'(z)}{f(z)^{1-\alpha}z^{\alpha}}\right] \\ &= p(z), \end{aligned} \tag{3.1}$$

with $p \in \mathcal{P}$ and $z \in \mathbb{D}$.

By equating of coefficients in (3.1) gives

$$\begin{aligned} a_2 &= \frac{c_1}{(1+\alpha)(1+\beta)}, \\ a_3 &= \frac{1}{(2+\alpha)(1+2\beta)}\left(c_2 - \frac{(\alpha + \alpha^2 - 6\beta - 2\alpha\beta - 2)}{2(1+\alpha^2)(1+\beta)^2}c_1^2\right), \\ a_4 &= \frac{c_3}{(3+\alpha)(1+3\beta)} \\ &\quad - \frac{(2\alpha + \alpha^2 - 15\beta - 3\alpha\beta - 3)c_1c_2}{(1+\alpha)(2+\alpha)(3+\alpha)(1+\beta)(1+2\beta)(1+3\beta)} \\ &\quad - \frac{Vc_1^3}{6(1+\alpha)^3(2+\alpha)(3+\alpha)(1+\beta)^3(1+2\beta)(1+3\beta)}, \end{aligned} \tag{3.2}$$

where

$$V = 13\alpha + 2\alpha^2 - 7\alpha^3 - 36\beta + 83\alpha\beta + 76\alpha^2\beta$$

$$+ 19\alpha^3\beta + 2\alpha^4\beta - 102\beta^2 - 36\alpha\beta^2 - 6\alpha^2\beta^2 - 6.$$

Theorem 3.1. *If $f \in \mathcal{B}_1^\beta(\alpha)$ then*

$$
\begin{aligned}
|a_2| &\le \frac{2}{(1+\alpha)(1+\beta)}\ ;\ for\ \beta \ge 0\ and\ 0 \le \alpha \le 1,\\
|a_3| &\le \frac{2}{(2+\alpha)(1+2\beta)}\ ;\ for\ \beta = 0\ and\ \alpha = 1,\\
&\le \frac{2(3+8\beta+\beta^2+\alpha^2\gamma(2+\beta)+\alpha(1+6\beta+2\beta^2))}{(1+\alpha)^2(2+\alpha)(1+\beta)^2(1+2\beta)}\ ;\\
&\quad for\ \beta > 0\ and\ 0 \le \alpha \le 1,\\
&\le \frac{2(3+8\beta+\beta^2+\alpha^2\gamma(2+\beta)+\alpha(1+6\beta+2\beta^2))}{(1+\alpha)^2(2+\alpha)(1+\beta)^2(1+2\beta)}\ ;\\
&\quad for\ \beta = 0\ and\ 0 \le \alpha < 1,\\
|a_4| &\le \frac{2V_1}{3(1+\alpha)^3(2+\alpha)(3+\alpha)(1+\beta)^3(1+2\beta)(1+3\beta)}\ ;\\
&\quad for\ \beta > 1\ and\ 0 \le \alpha \le 1,\\
&\le \frac{2V_1}{3(1+\alpha)^3(2+\alpha)(3+\alpha)(1+\beta)^3(1+2\beta)(1+3\beta)}\ ;\\
&\quad for\ 0 \le \beta \le 1\ and\ 0 \le \alpha \le \alpha_1(\beta),\\
&\le \frac{2}{(3+\alpha)(1+3\beta)}\ ;\ for\ 0 \le \beta \le 1\ and\ \alpha_1(\beta) < \alpha \le 1,\\
&\le \frac{2V_1}{3(1+\alpha)^3(2+\alpha)(3+\alpha)(1+\beta)^3(1+2\beta)(1+3\beta)}\ ;\\
&\quad for\ 0 \le \beta < 1\ and\ \alpha = 0,
\end{aligned}
$$

where,

$$
\begin{aligned}
V_1 = &\left(36+228\beta+456\beta^2+132\beta^3+12\beta^4\right)\\
&+ \alpha^4\left(1-\beta+21\beta^2+21\beta^3+6\beta^4\right)\\
&+ \alpha^3\left(5+7\beta+147\beta^2+123\beta^3+30\beta^4\right)\\
&+ \alpha^2\left(11-85\beta+495\beta^2+315\beta^3+54\beta^4\right)\\
&+ \alpha\left(19+185\beta+681\beta^2+345\beta^3+42\beta^4\right),
\end{aligned}
$$

and $\alpha_1(\beta)$ is the positive root of the equations

$$
\begin{aligned}
12\beta^4 &+ 42\beta^3 - 48\beta^2 - 6\beta + \left(34+188\beta+153\beta^2+147\beta^3+42\beta^4\right)x\\
&+ \left(19+211\beta+237\beta^2+189\beta^3+54\beta^4\right)x^2\\
&+ \left(8+94\beta+135\beta^2+105\beta^3+30\beta^4\right)x^3\\
&+ \left(1+17\beta+27\beta^2+21\beta^3+6\beta^4\right)x^4
\end{aligned}
$$

$$= 0.$$

Proof. It is easy to proof the first innequality $|a_2|$ with fact $|c_1| \leq 2$.

Let

$$|a_3| = \frac{1}{(2+\alpha)(1+2\beta)} \left| c_2 - \frac{(\alpha + \alpha^2 - 6\beta - 2\alpha\beta - 2)}{2(1+\alpha^2)(1+\beta)^2} c_1^2 \right|.$$

From 2.2, we get

$$\mu = \frac{(\alpha + \alpha^2 - 6\beta - 2\alpha\beta - 2)}{(1+\alpha^2)(1+\beta)^2},$$

such that $\left| c_2 - \frac{\mu}{2} c_1^2 \right| \leq 2$ provided $\beta = 0$ and $\alpha = 1$. Applying Lemma 2.2 to get the inequality for $|a_3|$.

To prove the first and fourth inequality for a_4, we use fact that the coefficients of c_3, c_1c_2 and c_1^3 are positive. Next we proof tne second and third for $|a_4|$.

For a_4, write

$$a_4 = \frac{1}{(3+\alpha)(1+3\beta)} \left[\left(c_3 - c_1c_2 + \frac{1}{2}c_1^3 \right) + W_1c_1 \left((c_2 - \frac{1}{2}c_1^2) - W_2c_1^2 \right) \right],$$

with

$$W_1 = \frac{5 + \alpha + 21\beta + 12\alpha\beta + 3\alpha^2\beta + 4\beta^2 + 6\alpha\beta^2 + 2\alpha^2\beta^2}{(1+\alpha)(2+\alpha)(1+\beta)(1+2\beta)},$$

and

$$\begin{aligned} W_2 &= \frac{V}{6(1+\alpha)^3(2+\alpha)(1+\gamma)^3(1+2\gamma)} \\ &= \big\{ \alpha^4(1 + 8\beta + 3\beta^2) + \alpha^3(5 + 34\beta - 6\beta^2 - 9\beta^3) \\ &\quad + \alpha^2(8 + 25\beta - 126\beta^2 - 63\beta^3) - \alpha(-1 + 40\beta + 246\beta^2 + 99\beta^3) \\ &\quad -(15 + 99\beta + 201\beta^2 + 45\beta^3) \big\} \big/ \big\{ 6(1+\alpha)^2(1+\beta)^2 \\ &\quad \big[5 + 21\beta 4\beta^2 + \alpha^2\beta(3+2\beta) + \alpha(1 + 12\beta + 6\beta^2) \big] \big\}. \end{aligned}$$

W_1 and W_2 are positive when $\beta \geq 0$ and $0 < \alpha \leq 1$. According Lemma 2.4, with $\lambda = \frac{1}{2}$, we get

$$\begin{aligned} W_1c_1 \left(\left(c_2 - \frac{1}{2}c_1^2 \right) - W_2c_1^2 \right) &\leq W_1|c_1| \left(2 - \frac{1}{2}|c_1|^2 - W_2|c_1|^2 \right) \\ &:= P(|c_1|). \end{aligned}$$

Let $|c_1| = x$, then

$$P(x) = W_1x \left(2 - \frac{1}{2}x^2 - W_2x^2 \right).$$

Since $P'(x) \geq 0$, then a function $P(x)$ is monotone increasing functions when $0 \leq x \leq 2$ for $\beta \geq 0$ and $0 < \alpha \leq 1$. So that the maximum value is when $x = 2$, which give the second inequality for $|a_4|$,

$$\frac{1}{(3+\alpha)(1+3\beta)}\left(2 + P(2)\right).$$

For the third inequality, we use Lemma 2.1 to write c_2 and c_3 in term of c_1. Next, take $c_1 = c$ with $c \in [0, 2]$, and note that $|\zeta| \leq 1$, we have

$$\begin{aligned}
|a_4| \leq\ & \frac{\Delta_1(\alpha,\beta)c^3}{(12(1+\alpha)^3(2+\alpha)(3+\alpha)(1+\beta)^3(1+2\beta)(1+3\beta))} \\
&+ \frac{\left(5+\alpha+21\beta+12\alpha\beta+3\alpha^2\beta+4\beta^2+6\alpha\beta^2+2\alpha^2\beta^2\right)c\left(4-c^2\right)|x|}{2(1+\alpha)(2+\alpha)(3+\alpha)(1+\beta)(1+2\beta)(1+3\beta)} \\
&+ \frac{c\left(4-c^2\right)|x|^2}{4(3+\alpha)(1+3\beta)} + \frac{2\left(4-c^2\right)\left(1-|x|^2\right)}{4(3+\alpha)(1+3\beta)} \\
&:= \phi(c,|x|),
\end{aligned}$$

with

$$\begin{aligned}
\Delta_1(\alpha,\beta) = {} & 36+19\alpha+11\alpha^2+5\alpha^3+\alpha^4+228\beta+185\alpha\beta+85\alpha^2\beta \\
&+7\alpha^3\beta-\alpha^4\beta+456\beta^2+681\alpha\beta^2+495\alpha^2\beta^2+147\alpha^3\beta^2 \\
&+21\alpha^4\beta^2+132\beta^3+345\alpha\beta^3+315\alpha^2\beta^3+123\alpha^3\beta^3 \\
&+21\alpha^4\beta^3+12\beta^4+42\alpha\beta^4+54\alpha^2\beta^4+30\alpha^3\beta^4+6\alpha^4\beta^4.
\end{aligned}$$

Next we have to find the maximum value of $\phi(c, |x|)$ on $I = [0, 2] \times [0, 1]$.

Assume that there is a critical point at $(c_0, |x_0|)$ inside I. If $\phi'(c, |x|)$ is the derivative of $\phi(c, |x|)$ with respect to $|x|$, then $\phi'(c, |x|) = 0$ contains the expression $4 - c^2$. So that gives a contradiction, and the maximum value must occur on I boundary.

On $c = 0$,

$$\phi(0,|x|) = \frac{2\left(1-|x|^2\right)}{(3+\alpha)(1+3\beta)} \leq \frac{2}{(3+\alpha)(1+3\beta)}.$$

On $c = 2$,

$$\begin{aligned}
\phi(2,|x|) &= \frac{2V}{3(1+\alpha)^3(2+\alpha)(3+\alpha)(1+\beta)^3(1+2\beta)(1+3\beta)} \\
&\leq \frac{2}{(3+\alpha)(1+3\beta)},
\end{aligned}$$

for $0 \leq \beta \leq 1$ and $\alpha_1(\beta) < \alpha \leq 1$.

On $|x| = 0$,

$$\phi(c,0) = \frac{\Delta_1(\alpha,\beta)c^3}{(12(1+\alpha)^3(2+\alpha)(3+\alpha)(1+\beta)^3(1+2\beta)(1+3\beta))}$$

$$\le \frac{2}{(3+\alpha)(1+3\beta)},$$

for $c \in [0,2]$.

Finally when $|x| = 1$,

$$\phi(c,1) \le \frac{2}{(3+\alpha)(1+3\beta)}.$$

The inequality of modulus a_2 is sharp when $c_1 = 2$. The first inequality of modulus a_3 is sharp when $c_1 = 0$ and $c_2 = 2$, and the second and third inequalities for modulus of a_3 are sharp when $c_1 = c_2 = 2$. For modulus of a_4, the third inequality is sharp when $c_1 = 0$ and $c_2 = c_3 = 2$, and the other inequalities are sharp when $c_1 = c_2 = c_3 = 2$. Thus, the proof of Theorem 3.1 is complete. □

4. Fekete-Szegö Theorem

In this section, we proof the Fekete-Szegö functional for $\mathcal{B}_1^\beta(\alpha)$, which extend the result on class $B_1(\alpha)$[12].

Theorem 4.1. *Let $f \in \mathcal{B}_1^\beta(\alpha)$. Then for $\nu \in \mathbb{R}$*

$$|a_3 - \nu a_2^2| \le$$
$$\begin{cases} \dfrac{2T}{(1+\alpha)^2(2+\alpha)(1+\beta)^2(1+2\beta)} & if\ \nu \le \nu_1(\alpha,\beta), \\ \dfrac{2}{(2+\alpha)(1+2\beta)} & if\ \nu_1(\alpha,\beta) \le \nu \le \nu_2(\alpha,\beta), \\ -\dfrac{2T}{(1+\alpha)^2(2+\alpha)(1+\beta)^2(1+2\beta)} & if\ \nu \ge \nu_2(\alpha,\beta), \end{cases}$$

where,

$$\nu_1(\alpha,\beta) = \frac{2-\alpha-\alpha^2+6\beta+2\alpha\beta}{4+2\alpha+8\beta+4\alpha\beta},$$
$$\nu_2(\alpha,\beta) = \frac{4+3\alpha+\alpha^2+10\beta+10\alpha\beta+4\alpha^2\beta+2\beta^2+4\alpha\beta^2+2\alpha^2\beta^2}{4+2\alpha+8\beta+4\alpha\beta}.$$

and

$$T = 3+\alpha+8\beta+6\alpha\beta+2\alpha^2\beta+\beta^2+2\alpha\beta^2+\alpha^2\beta^2-4\nu-2\alpha\nu-8\beta\nu-4\alpha\beta\nu.$$

The inequalities is sharp. In particular,

$$|a_3 - a_2^2| \le \frac{2}{(2+\alpha)(1+2\beta)}.$$

Proof. We now apply Lemma 2.2. From (3.2), we get

$$|a_3 - \nu a_2^2| = \frac{2}{(2+\alpha)(1+2\gamma)}\left|c_2 - \frac{\mu}{2}c_1^2\right|,$$

with

$$\mu = \frac{\left(-2+\alpha+\alpha^2-6\beta-2\alpha\beta+4\nu+2\alpha\nu+8\beta\nu+4\alpha\beta\nu\right)}{(1+\alpha)^2(1+\beta)^2}.$$

To prove the second inequality, we use the fact that $\mu \in [0,2]$ for $0 \geq \beta$ and $0 \leq \alpha \leq 1$, when $\nu_1(\alpha,\beta) \leq \nu \leq \nu_2(\alpha,\beta)$. For the first inequality, we use the fact that $\mu - 1$ is positive, and the last inequality for $\mu - 1$ is negative. The second inequality is sharp by choosing $c_1 = 0$ and $c_2 = 2$. At the same time, the other inequalities are sharp if $c_1 = c_2 = 2$. $\square$

5. Logarithmic Coefficients

The Logarithmic Coefficients γ_n of the function $f(z)$ for $z \in \mathbb{D}$ is defined as follows.

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z_n. \tag{5.1}$$

Differentiating the equation (5.1) and equating the coefficients, we get

$$\begin{aligned} \gamma_1 &= \frac{1}{2}a_2 \\ \gamma_2 &= \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right) \\ \gamma_3 &= \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right). \end{aligned} \tag{5.2}$$

Subtituting (3.2) into the expression of (5.2) to obtain

$$\begin{aligned} \gamma_1 &= \frac{c_1}{2(1+\alpha)(1+\beta)} \\ \gamma_2 &= \frac{1}{2(2+\alpha)(1+2\beta)}\left(c_2 - \frac{\left(2\alpha+\alpha^2-2\beta\right)c_1^2}{2(1+\alpha)^2(1+\beta)^2}\right) \\ \gamma_3 &= \frac{1}{2(3+\alpha)(1+3\beta)}\left(c_3 - \frac{\left(3\alpha+\alpha^2-6\beta\right)c_1c_2}{(1+\alpha)(2+\alpha)(1+\beta)(1+2\beta)}\right. \\ &\quad - \frac{\left(-6\alpha^2-5\alpha^3-\alpha^4+6\beta+30\alpha\beta+18\alpha^2\beta\right)c_1^3}{3(1+\alpha)^3(2+\alpha)(1+\beta)^3(1+2\beta)} \\ &\quad \left. - \frac{\left(5\alpha^3\beta+\alpha^4\beta-6\beta^2+6\alpha\beta^2\right)c_1^3}{3(1+\alpha)^3(2+\alpha)(1+\beta)^3(1+2\beta)}\right). \end{aligned} \tag{5.3}$$

For the function $f \in \mathcal{B}_1^\beta(\alpha)$, we have a sharp bound of $|\gamma_n|$ when $n = 1,2,3$. Note that the $|\gamma_1|$ and $|\gamma_2|$ hold for all $0 \geq \beta$ and $0 \leq \alpha \leq 1$.

Theorem 5.1. *Let $f \in \mathcal{B}_1^{\beta}(\alpha)$, then*

$$
\begin{aligned}
|\gamma_1| &\le \frac{1}{(1+\alpha)(1+\beta)} \;;\; \beta \ge 0 \text{ and } 0 \le \alpha \le 1,\\
|\gamma_2| &\le \frac{1}{(2+\alpha)(1+2\beta)} \;;\; 0 \le \beta \le \frac{1}{2}(2\alpha+\alpha^2) \text{ and } 0 \le \alpha \le 1,\\
&\le \frac{1+4\beta+4\alpha\beta+2\alpha^2\beta+\beta^2+2\alpha\beta^2+\alpha^2\beta^2}{(1+\alpha)^2(2+\alpha)(1+\beta)^2(1+2\beta)} \;;\; \beta \ge \frac{1}{2}(2\alpha+\alpha^2)\\
&\qquad\qquad \text{and } 0 \le \alpha \le 1,\\
|\gamma_3| &\le \frac{1}{(3+\alpha)(1+3\beta)} \;;\; 0 \le \beta \le \beta_1(\alpha) \text{ and } 0 \le \alpha \le 1,\\
&\le \frac{G}{(3(1+\alpha)^3(2+\alpha)(3+\alpha)(1+\beta)^3(1+2\beta)(1+3\beta))} \;;\; \beta \ge 1 \text{ and}\\
&\qquad\qquad 0 \le \alpha \le 1.
\end{aligned}
$$

Where,

$$
\begin{aligned}
G = \big(&6+3\alpha+9\alpha^2+5\alpha^3+\alpha^4+42\beta+21\alpha\beta+15\alpha^2\beta-5\alpha^3\beta\\
&-\alpha^4\beta+150\beta^2+291\alpha\beta^2+273\alpha^2\beta^2+105\alpha^3\beta^2+21\alpha^4\beta^2\\
&+78\beta^3+219\alpha\beta^3+225\alpha^2\beta^3+105\alpha^3\beta^3+21\alpha^4\beta^3\\
&+12\beta^4+42\alpha\beta^4+54\alpha^2\beta^4+30\alpha^3\beta^4+6\alpha^4\beta^4\big),
\end{aligned}
$$

and $\beta_1(\alpha)$ is positive root of the equation

$$
\begin{aligned}
&-18\alpha-48\alpha^2-38\alpha^3-14\alpha^4-2\alpha^5\\
&\quad+\left(60+42\alpha-165\alpha^2-185\alpha^3-77\alpha^4-11\alpha^5\right)x\\
&\quad+\left(276+546\alpha+165\alpha^2-107\alpha^3-77\alpha^4-11\alpha^5\right)x^2\\
&\quad+\left(240+546\alpha+282\alpha^2-12\alpha^3-42\alpha^4-6\alpha^5\right)x^3\\
&\quad+\left(72+180\alpha+144\alpha^2+36\alpha^3\right)x^4\\
&=0.
\end{aligned}
$$

Proof. It is known that $|c_1| \le 2$, so the inequality $|g_1| \le \dfrac{1}{(1+\alpha)(1+\beta)}$ is proved. This inequality is sharp when $c_1 = 2$.

The result for $|\gamma_2|$, is obtained from the Fekete-Szegö theorem when $\nu = 1/2$. We apply the second inequality from Theorem 4.1 for the first inequality and the first inequality of Theorem 4.1 for the second inequality. The first inequality is sharp when $c_2 = 2$ and $c_1 = 0$, and the second inequality is sharp by selecting $c_1 = 2 = c_2$.

Next, we use Lemma 2.3 to prove the first inequality of $|\gamma_3|$. Let,

$$B = \frac{(4 + 6\alpha + 2\alpha^2 - 3\lambda + 2\alpha\lambda + \alpha^2\lambda)}{4(1+\alpha)(2+\alpha)},$$

and

$$\begin{aligned} D = \big(&12 + 42\alpha + 54\alpha^2 + 30\alpha^3 + 6\alpha^4 - 18\lambda - 24\alpha\lambda + 12\alpha^2\lambda \\ &+ 24\alpha^3\lambda + 6\alpha^4\lambda + 6\lambda^2 - 13\alpha\lambda^2 + 6\alpha^4\lambda + 6\lambda^2 - 13\alpha\lambda^2 \\ &-2\alpha^2\lambda^2 + 7\alpha^3\lambda^2 + 2\alpha^4\lambda^2\big) /(24(1+\alpha)^3(2+\alpha)). \end{aligned}$$

Since $0 \le B \le 1$, and $B(2B-1) \le D \le B$, are satisfied when $0 \le \beta \le \beta_1(\alpha)$ and $0 \le \alpha \le 1$, then the first inequality for $|\gamma_3|$ is proven. For the second inequality, rewrite γ_3 as follows,

$$\gamma_3 = \frac{1}{2(3+\alpha)(1+3\beta)} \left(c_3 - 2Bc_1c_2 + Bc_1^3 + (D-B)c_1^3\right).$$

Since $D - B \ge 0$ when $\beta \ge 1$ and $0 \le \alpha \le 1$, by using Lemma 2.4, we get the second inequality. Note that there is a open problem for case $\beta_1(\alpha) \le \beta \le 1$ and $0 \le \alpha \le 1$.

The first inequality of $|\gamma_3|$ is sharp when $c_3 = 2$ and $c_1 = c_2 = 0$. While the second inequality is sharp by choosing $c_3 = c_2 = c_1 = 2$. □

6. Inverse Coefficients

Suppose that $\mathcal{B}_1^{\beta}(\alpha)^{-1}$ is a collection of the inverse functions f^{-1} of $\mathcal{B}_1^{\beta}(\alpha)$, then we can write

$$f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + \ldots$$

valid in some disc $|\omega| \le r_o(f)$.

Since $f(f^{-1}(\omega)) = \omega$, then

$$\begin{aligned} A_2 &= -a_2, \\ A_3 &= 2a_2^2 - a_3, \\ A_4 &= -5a_3^2 + 5a_2a_3 - a_4. \end{aligned} \tag{6.1}$$

The following theorem contains sharp bounds for modulus of A_2 and A_3.

Theorem 6.1. *Let $f \in \mathcal{B}_1^{\beta}(\alpha)$ and $f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + \ldots$, then*

$$\begin{aligned} |A_2| &\le \frac{2}{(1+\alpha)(1+\beta)} \;;\; \beta \ge 0 \textit{ and } 0 \le \alpha \le 1, \\ |A_3| &\le \frac{2}{(2+\alpha)(1+2\beta)} \;;\; 0 \le \beta \le \beta_1 \textit{ and } \alpha_1(\beta) \le \alpha \le 1, \\ &\le \frac{2}{(2+\alpha)(1+2\beta)} \;;\; \beta \ge \beta_1 \textit{ and } 0 \le \alpha \le 1, \end{aligned}$$

$$\leq \frac{A}{(1+\alpha)^2(2+\alpha)^2(1+\beta)^2(1+2\beta)^2} \ ; \ 0 \leq \beta \leq \beta_1 \ and \ 0 \leq \alpha \leq \alpha_1(\beta),$$

where,

$$\begin{aligned} A = {} & 8 - 6\alpha^2 - 2\alpha^3 + 4\beta - 28\alpha\beta - 32\alpha^2\beta - 8\alpha^3\beta - 20\beta^2 \\ & - 50\alpha\beta^2 - 40\alpha^2\beta^2 - 10\alpha^3\beta^2 - 8\beta^3 - 20\alpha\beta^3 - 16\alpha^2\beta^3 - 4\alpha^3\beta^3, \end{aligned}$$

$\beta_1 = 0.196...$ *is positive root of* $4x^3 + 10x^2 + 3x - 1 = 0$*, and* $\alpha_1(\beta)$ *is positive root of the equation*

$$\begin{aligned} & - 4 - 2\beta + 10\beta^2 + 4\beta^3 + \left(14\beta + 25\beta^2 + 10\beta^3\right) x \\ & \quad + \left(3 + 16\beta + 20\beta^2 + 8\beta^3\right) x^2 + \left(1 + 4\beta + 5\beta^2 + 2\beta^3\right) x^3 \\ & = 0. \end{aligned}$$

Proof. We again use the coefficient expression in (3.2). Since

$$(1+\alpha)(1+\beta)a_2 = c_1$$

and $|c_1| \leq 2$, the first inequality is proven.

Next, by substituting the expression (3.2) into the expression (6.1), we get

$$|A_3| = \frac{1}{(2+\alpha)(1+2\beta)} \left| c_2 - \frac{6+\alpha^2+10\beta+\alpha(5+\beta)}{2(1+\alpha)^2(2+\alpha)(1+\beta)^2(1+2\beta)^2} c_1^2 \right| .$$

Let

$$\mu = \frac{6+\alpha^2+10\beta+\alpha(5+\beta)}{(1+\alpha)^2(2+\alpha)(1+\beta)^2(1+2\beta)^2},$$

such that $0 \leq \mu \leq 2$ provided $0 \leq \beta \leq \beta_1$ and $\alpha_1(\beta) \leq \alpha \leq 1$, and when $\beta \geq \beta_1$ and $0 \leq \alpha \leq 1$. By using Lemma 2.2 gives the required inequalities for modulus of A_3.

The inequality for modulus of A_2 will be sharp when $c_1 = 2$. The first and second inequalities for modulus of A_3 are sharp by selecting $c_1 = 0$ and $c_2 = -2$, and the third inequality is sharp when $c_2 = c_1 = 2$. □

7. Second Hankel Determinant Problems

The Hankel determinant of the function f for $q \geq 1$ and $n \geq 1$, defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix} \tag{7.1}$$

We prove the following theorem in case $q = n = 2$ for the Beta-Bazilevič function.

Theorem 7.1. *Let $f \in \mathcal{B}_1^{\beta}(\alpha)$, then for $\beta \geq 0$, and $0 < \alpha \leq 1$,*

$$H_2(2) = \left|a_2 a_4 - a_3^2\right| \leq \frac{4}{(2+\alpha)^2(1+2\beta)^2}. \tag{7.2}$$

The inequality is sharp. In particular, $f(z) = z + \dfrac{2z^3}{(2+\alpha)(1+2\beta)}$ is an example of a function that satisfies the sharp condition.

Proof. By substituting the coefficient on (3.2) into (7.2) and simplifying, we get

$$\begin{aligned} H_2(2) = &-\Big(\Big(\Big(6\alpha^3\left(-1+5\beta+8\beta^2\right)+\alpha^4\left(-1+5\beta+8\beta^2\right)\\ &+3\alpha^2\left(-3+31\beta+52\beta^2+4\beta^3\right)\\ &+12\left(1+11\beta+23\beta^2+13\beta^3\right)\\ &+4\alpha\left(1+43\beta+82\beta^2+30\beta^3\right)\Big)c_1^4\Big)/\\ &\Big(12(1+\alpha)^3(2+\alpha)^2(3+\alpha)(1+\beta)^4(1+2\beta)^2(1+3\beta)\Big)\Big)\\ &+\frac{\beta(6+4\alpha+\alpha^2+6\beta)c_1^2c_2}{(1+\alpha)(2+\alpha)^2(3+\alpha)(1+\beta)^2(1+2\beta)^2(1+3\beta)}\\ &-\frac{c_2^2}{(2+\alpha)^2(1+2\beta)^2}+\frac{c_1c_3}{(1+\alpha)(3+\alpha)(1+\beta)(1+3\beta)}. \end{aligned} \tag{7.3}$$

We again use Lemma 2.1, and write $c_1 := c$ with $0 \leq c \leq 2$. Next, take modulus and noting that $|\eta| \leq 1$ to obtain the following espression.

(7.4)

$$\begin{aligned} H_2(2) \leq &\Big(\Big(\Big(6\alpha^3\left(1+\beta+7\beta^2+15\beta^3+9\beta^4+3\beta^5\right)\\ &+\alpha^4\left(1+\beta+7\beta^2+15\beta^3+9\beta^4+3\beta^5\right)\\ &+3\left(-3-25\beta-34\beta^2+18\beta^3+37\beta^4+7\beta^5\right)\\ &+3\alpha^2\left(4+6\beta+51\beta^2+111\beta^3+64\beta^4+16\beta^5\right)\\ &+\alpha\left(2-34\beta+80\beta^2+360\beta^3+258\beta^4+54\beta^5\right)\Big)c^4\Big)/\\ &\Big(12(1+\alpha)^3(2+\alpha)^2(3+\alpha)(1+\beta)^4(1+2\beta)^2(1+3\beta)\Big)\Big)\\ &+\frac{\left(1+\left(11+4\alpha+\alpha^2\right)\beta+\left(17+4\alpha+\alpha^2\right)\beta^2\right)c^2|\zeta|\left(4-c^2\right)}{2(1+\alpha)(2+\alpha)^2(3+\alpha)(1+\beta)^2(1+2\beta)^2(1+3\beta)}\\ &+\frac{\left(7+4\alpha+\alpha^2\right)\beta^3c^2|\zeta|\left(4-c^2\right)}{2(1+\alpha)(2+\alpha)^2(3+\alpha)(1+\beta)^2(1+2\beta)^2(1+3\beta)} \end{aligned}$$

$$
\begin{aligned}
&+\frac{c^2|\zeta|^2\left(4-c^2\right)}{4(1+\alpha)(3+\alpha)(1+\beta)(1+3\beta)}+\frac{|\zeta|^2\left(4-c^2\right)^2}{4(2+\alpha)^2(1+2\beta)^2}\\
&+\frac{c\left(4-c^2\right)\left(1-|\zeta|^2\right)}{2(1+\alpha)(3+\alpha)(1+\beta)(1+3\beta)}\\
&:=\psi(\alpha,\beta,|\zeta|,c).
\end{aligned}
$$

Since the derivative of $\psi(\alpha,\lambda,|\zeta|,c)$ with respect to $|\zeta|$ is positive, then $\psi(\alpha,\lambda,|\zeta|,c)$ will reach the maximum value when $|\zeta|=1$. Then from (7.4), we have

(7.5)

$$
\begin{aligned}
H_2(2)\leq &\left(\left(\left(6\alpha^3\left(1+\beta+7\beta^2+15\beta^3+9\beta^4+3\beta^5\right)\right.\right.\right.\\
&+\alpha^4\left(1+\beta+7\beta^2+15\beta^3+9\beta^4+3\beta^5\right)\\
&+3\left(-3-25\beta-34\beta^2+18\beta^3+37\beta^4+7\beta^5\right)\\
&+3\alpha^2\left(4+6\beta+51\beta^2+111\beta^3+64\beta^4+16\beta^5\right)\\
&\left.\left.+\alpha\left(2-34\beta+80\beta^2+360\beta^3+258\beta^4+54\beta^5\right)\right)c^4\right)/\\
&\left.\left(12(1+\alpha)^3(2+\alpha)^2(3+\alpha)(1+\beta)^4(1+2\beta)^2(1+3\beta)\right)\right)\\
&+\frac{\left(1+\left(11+4\alpha+\alpha^2\right)\beta+\left(17+4\alpha+\alpha^2\right)\beta^2\right)c^2\left(4-c^2\right)}{2(1+\alpha)(2+\alpha)^2(3+\alpha)(1+\beta)^2(1+2\beta)^2(1+3\beta)}\\
&+\frac{\left(7+4\alpha+\alpha^2\right)\beta^3c^2\left(4-c^2\right)}{2(1+\alpha)(2+\alpha)^2(3+\alpha)(1+\beta)^2(1+2\beta)^2(1+3\beta)}\\
&+\frac{c^2\left(4-c^2\right)}{4(1+\alpha)(3+\alpha)(1+\beta)(1+3\beta)}-\frac{\left(4-c^2\right)^2}{4(2+\alpha)^2(1+2\beta)^2}\\
:=&\,\psi_1(\alpha,\beta,c).
\end{aligned}
$$

Next, we must determine the maximum value of $\psi_1(\alpha,\beta,c)$, when $0\leq c\leq 2$. Suppose the derivative of $\psi_1(\alpha,\beta,c)$ with respect to c is $\psi_1'(\alpha,\beta,c)$. According to basic calculus theory, $\psi_1'(\alpha,\beta,c)=0$ has 3 roots. But only one root that satisfies that is $c=0$. The other two roots are not real roots. Then, we only investigate the value of $\psi_1(\alpha,\beta,0)$ and $\psi_1(\alpha,\beta,2)$. Since $\psi_1(\alpha,\beta,0)\geq\psi_1(\alpha,\beta,2)$ when $\beta\geq 0$ and $0<\alpha\leq 1$, then $\psi_1(\alpha,\beta,0)$ is a sharp bound for $H_2(2)$. The inequality will be sharp by choosing $c_1=c_3=0$ and $c_2=2$ on (7.3). $\square$

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