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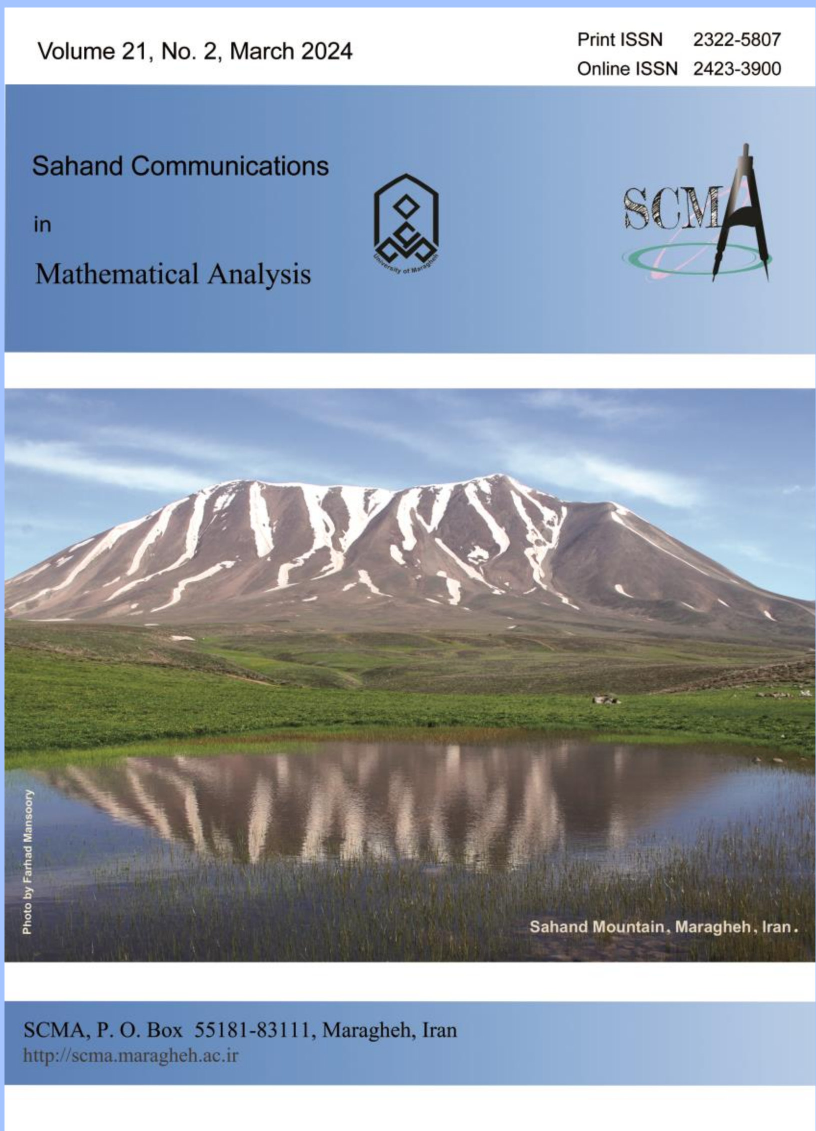

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Generalized Difference Lacunary Weak Convergence of Sequences

Bibhajyoti Tamuli[1 *] and Binod Chandra Tripathy[2]

Abstract. In this paper, we introduce the concept of generalized difference lacunary weak convergence of sequences. Using the concept of difference operator, we have introduced some new classes of sequences. We investigated several of its algebraic and topological properties, such as solidness, symmetry and monotone. We gave appropriate examples and detailed discussions to validate our established failure instances and definitions. Further, we have established some inclusion relations of the introduced sequence spaces with other sequence spaces, in particularly with the weak Cesàro summable sequences.

1. Introduction

The initial work on lacunary sequences was done by Freedman et al. [10]. They studied Cesàro summable sequences strongly lacunary convergent sequences considering a general lacunary sequence θ and established a relation between classes of the two types of sequences. Further lacunary sequences have been studied by Tripathy and Baruah [17], Ercan et al. [6], Gumus [11], Dowari and Triptahy [2–5], Haloi et al. [12], Esi, Tripathy and Sarma [7], Tripathy and Et [15], Tripathy and Dutta [16] and Et, Mohiuddine and Sengül [9].

Banach [1] proposed the concept of weak convergence, which is quite interesting but restricted in several ways. Tripathy and Mahanta have studied Vector-Valued sequence spaces [18] and many others in recent years. Many of the results associated with these concepts are generally,

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only true for separable space. Our main interest here would be to investigate the concepts of generalized difference lacunary weak convergence of sequences. In this paper ω, ℓ_∞, c and c_0 denote the set of all bounded, convergent and null sequences of real or complex numbers.

2. Definition and Preliminaries

A sequence of positive integer $\theta = (k_r)$ is called lacunary if $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$ and $q_r = \frac{k_r}{K_{r-1}}$, for $r \in \mathbb{N}$. In this paper, the intervals determined by θ will denoted by $I_r = (k_{r-1}, k_r]$. Let X be a normed space, then the set of all bounded linear functional, $BL(X,K)$, where K is the field of scalars, is called an algebraic dual of X denoted by X'.

Freedman et al. [10] first defined the space N_θ of lacunary strongly convergent sequence as

$$N_\theta = \left\{ x : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

Kizmaz [13] introduced the concept of difference sequence. Then Esi, Tripathy and Sarma [7] introduced the generalized difference sequence spaces as follows:

Let $m, n \geq 0$ be fixed integers,

$$Z(\Delta_n^m) = \{x = (x_k) \in \omega : \Delta_n^m x = (\Delta_n^m x_k) \in Z\}$$

for $Z = \ell_\infty$, c and c_0; where $\Delta_n^m x_k = \Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+n}$ and $\Delta_n^0 x_k = x_k$, for all $k \in \mathbb{N}$. This generalized difference operator has the binomial representation shown below:

$$\Delta_n^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+nv}, \quad \text{for all } k \in \mathbb{N}.$$

For $m = 1$ and $n = 1$, these spaces represent the spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [13]. For $n = 1$, these spaces represent the spaces $\ell_\infty(\Delta^m), c(\Delta^m)$ and $c_0(\Delta^m)$ introduced and studied by Et and Colak [8]. For $m = 1$, these spaces represent the spaces $\ell_\infty(\Delta_n), c(\Delta_n)$ and $c_0(\Delta_n)$ introduced and studied by Tripathy and Esi [14].

The sequence spaces $Z(\Delta_n^m)$ for $Z = \ell_\infty, c$ and c_0 are Banach spaces, by the norm

$$\|x\|_w = \sum_{i=1}^{mn} |x_i| + \sup_k |\Delta_n^m x_k|, \quad \text{for } m \geq 1, n \geq 1.$$

Definition 2.1. In a norm space X a sequence (x_n) is said to be weakly convergent if there exists an element $x \in X$ such that

$$\lim_{n\to\infty} f\left(x_n - x\right) = 0, \quad \text{for all } f \in X'.$$

Definition 2.2. In a norm space X a sequence (x_n) is said to be be lacunary weakly convergent to $L \in X$ if

$$\lim_{n\to\infty} \frac{1}{h_r} \sum_{k\in I_r} f\left(x_k - L\right) = 0,$$

for each $f \in X'$, where X' is the algebraic dual of X. In this paper lacunary weak convergent is denoted by $\mathcal{D}_\theta$.

Definition 2.3. A sequence (x_k) is said to be lacunary weakly generalized difference convergent to L, if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} f\left(\Delta_n^m x_k - L\right) = 0.$$

Definition 2.4. A sequence (x_k) in normed linear space X is said to be weakly Cesàro summable to $L \in X$, if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f\left(x_i - L\right) = 0.$$

Definition 2.5. A sequence space $E \subset \omega$ is said to be solid if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ for all sequence of scalar (α_k) with $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$.

Definition 2.6. Let $K = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subset \mathbb{N}$. Let $(x_n) \in \omega$, then the $K-$step space of the sequence space $E \subset \omega$ is define by

$$\lambda_K^E = \left\{(x_{k_i}) \in \omega : (x_k) \in E\right\}.$$

Definition 2.7. A canonical pre-image (y_n) of sequence $(x_n) \in E$, where $K-$step space λ_K^E is considered, is defined by

$$y_n = \begin{cases} x_n, & \text{if } n \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.8. If the sequence space $E \subset \omega$ contains all pre-imeges of its step spaces then E is said to be monotone.

Definition 2.9. If $(x_n) \in E$ implies $(x_\pi(n))$ belongs to E, where π is a permutation of $\mathbb{N}$, then a sequence space E is said to be symmetric.

Remark 2.10. A sequence space E is solid implies E monotone, but not necessarily conversely.

3. Main Result

We give our main result in this part. Let θ be a lacunary sequence and (x_k) be any non-zero sequence. Here we define some classes of sequences

$$[\mathcal{D}_\theta, \Delta_n^m]_0 = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} f(\Delta_n^m x_k) = 0 \right\};$$

$$[\mathcal{D}_\theta, \Delta_n^m]_1 = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} f(\Delta_n^m x_k - L) = 0 \right\};$$

$$[\mathcal{D}_\theta, \Delta_n^m]_\infty = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} f(\Delta_n^m x_k) < \infty \right\};$$

$$(\sigma_1)_w(\Delta_n^m) = \left\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(\Delta_n^m x_i - L) = 0 \right\};$$

$$[(AC_\theta)_w, \Delta_n^m] = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} f(\Delta_n^m x_{k+i} - L) = 0, \right.$$
$$\left. \text{uniformly in } \; i \in \mathbb{N} \right\}.$$

Theorem 3.1. *The classes* $[\mathcal{D}_\theta, \Delta_n^m]_0$, $[\mathcal{D}_\theta, \Delta_n^m]_1$ *and* $[\mathcal{D}_\theta, \Delta_n^m]_\infty$ *are linear spaces.*

Proof. We establish the result for $[\mathcal{D}_\theta, \Delta_n^m]_0$. The other cases will follow similarly.

Let $(x_k), (y_k) \in [\mathcal{D}_\theta, \Delta_n^m]_0$. Then we have

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} f\left(\Delta_n^m x_k\right) = 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} f\left(\Delta_n^m y_k\right) = 0.$$

Now for $\alpha, \beta \in \mathbb{C}$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} f\left(\Delta_n^m(\alpha x_k + \beta y_k)\right) = \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} f\left(\alpha\Delta_n^m(x_k) + \beta\Delta_n^m(y_k)\right)$$
$$\leq |\alpha| \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} f\left(\Delta_n^m(x_k)\right)$$

$$+ |\beta| \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} f\left(\Delta_n^m(y_k)\right)$$

$$\to 0, \text{ as } r \to \infty$$

Hence $(\alpha x_k+\beta y_k) \in [\mathcal{D}_\theta, \Delta_n^m]_0$. Therefore $[\mathcal{D}_\theta, \Delta_n^m]_0$ is a linear space. □

Theorem 3.2. *The classes* $[\mathcal{D}_\theta, \Delta_n^m]_0, [\mathcal{D}_\theta, \Delta_n^m]_1$ *and* $[\mathcal{D}_\theta, \Delta_n^m]_\infty$ *are normed linear spaces, normed by*

$$\|(x_k)\|_w = \sum_{i=1}^{mn} |f(x_i)| + \sup_r \frac{1}{h_r} \sum_{k\in I_r} |f(\Delta_n^m x_k)|$$

for $m \geq 1, n \geq 1$.

Theorem 3.3. *Let* X *be a Banach space then* $[\mathcal{D}_\theta, \Delta_n^m]_0, [\mathcal{D}_\theta, \Delta_n^m]_1$ *and* $[\mathcal{D}_\theta, \Delta_n^m]_\infty$ *are Banach spaces, normed by*

$$\|(x_k)\|_w = \sum_{i=1}^{mn} |f(x_i)| + \sup_r \frac{1}{h_r} \sum_{k\in I_r} |f(\Delta_n^m x_k)| \tag{3.1}$$

for $m \geq 1, n \geq 1$.

Proof. Let $f \in X'$, then by the linearity of f, it can be verified that $\|.\|_w$ defined on $[\mathcal{D}_\theta, \Delta_n^m]_{\mathcal{G}}, \mathcal{G} = 0, 1, \infty$ is a norm. We now show that these sequence spaces are complete concerning the norm given by (3.1). Consider the Cauchy sequence (x^s) in $[\mathcal{D}_\theta, \Delta_n^m]_\infty$, where $(x^s) = (x_i^s) = (x_1^s, x_2^s, x_3^s, ...) \in [\mathcal{D}_\theta, \Delta_n^m]_\infty$ for each $s \in \mathbb{N}$. Then

$$\left\|x^s - x^t\right\|_w = \sum_{i=1}^{mn} \left|f(x_i^s - x_i^t)\right| + \sup_r \frac{1}{h_r} \sum_{k\in I_r} \left|f\left(\Delta_n^m(x_k^s - x_k^t)\right)\right| \to 0,$$

as $s, t \to \infty$ for $m \geq 1, n \geq 1; mn = n$ for $m = 1$ and $mn = m$ for $n = 1$.

Hence we obtain $\left|x_k^s - x_k^t\right| \to 0$ as $s, t \to \infty$ for each $k \in \mathbb{N}$.

Therefore, $(x_k^s) = (x_1^s, x_2^s, x_3^3, \ldots)$ is a Cauchy sequence in $\mathbb{C}$, the set of complex numbers.

Since, $\mathbb{C}$ is complete, it is converges, then

$$\lim_{s\to\infty} x_k^s = x_k$$

Say, for each $k \in \mathbb{N}$. Since (x^s) is a Cauchy sequence,there exist $n_0 = n_0(\varepsilon)$ for each $\varepsilon > 0$, such that

$$\left\|x^s - x^t\right\|_w < \varepsilon$$

for all $s, t \geq n_0$. Hence

$$\sum_{i=1}^{n} \left|f(x_i^s - x_i^t)\right| < \varepsilon$$

and

$$\left|f\left(\Delta_n^m(x_k^s - x_k^t)\right)\right| = \left|\sum_{v=0}^{m}(-1)^v \binom{m}{v}\left(x_{k+nv}^s - x_{k+nv}^t\right)\right| < \varepsilon,$$

for all $k \in \mathbb{N}$ and for all $s, t \geq n_0$.

We have f a linear function on X, so taking limit as $t \to \infty$, in the above two inequalities, we have

$$\lim_{t\to\infty} \sum_{i=1}^{n} \left|f(x_i^s - x_i^t)\right| = \sum_{i=1}^{n} |f(x_i^s - x_i)| < \varepsilon$$

and

$$\lim_{t\to\infty} \left|f\left(\Delta_n^m(x_i^s - x_i^t)\right)\right| = \left|f\left(\Delta_n^m(x_i^s - x_i)\right)\right| < \varepsilon$$

for all $s \geq n_0$. This implies that $\|x^s - x\|_w < 2\varepsilon \ \forall \ s \geq n_0$, That is $x^s - x,$ as $s \to \infty$, where $x = (x_k)$. Also, since

$$\begin{aligned}
|f(\Delta_n^m x_k)| &= \left|f\left(\sum_{v=0}^{m}(-1)^v \binom{m}{v}(x_{k+nv})\right)\right| \\
&= \left|f\left(\sum_{v=0}^{m}(-1)^v \binom{m}{v}\left(x_{k+nv} - x_{k+nv}^{n_0} + x_{k+nv}^{n_0}\right)\right)\right| \\
&\leq \left|f\left(\sum_{v=0}^{m}(-1)^v \binom{m}{v}\left(x_{k+mv}^{n_0} - x_{k+mv}\right)\right)\right| \\
&\quad + \left|f\left(\sum_{v=0}^{m}(-1)^v \binom{m}{v}\left(x_{k+nv}^{n_0}\right)\right)\right| \\
&\leq \|x^{n_0} - x\|_w + \left\|\Delta_n^m x_k^{n_0}\right\| \\
&= O(1)
\end{aligned}$$

Hence $x \in [\mathcal{D}_\theta, \Delta_n^m]$. Therefore $[\mathcal{D}_\theta, \Delta_n^m]$ is a Banach space. Similarly it can be established for the other spaces.

Without proof we state the following result. □

Proposition 3.4. *The spaces* $[\mathcal{D}_\theta, \Delta_n^m]_0, [\mathcal{D}_\theta, \Delta_n^m]_1$ *and* $[\mathcal{D}_\theta, \Delta_n^m]_\infty$ *are BK space.*

Theorem 3.5. $(\sigma_1)_w(\Delta_n^m) \subset [\mathcal{D}_\theta, (\Delta_n^m)]$ *if and only if* $\liminf_r q_r > 1$.

Proof. Let $\liminf_r q_r > 1$, $\exists\ \delta > 0$ such that, $1 + \delta \leqslant q_r$, $\forall\ r \geqslant 1$. Then for the sequence $x \in (\sigma_1)_w(\Delta_n^m)$ we have,

$$\tau_r = h_r^{-1} \sum_{i=1}^{k_r} f(\Delta_n^m x_i) - h_r^{-1} \sum_{i=1}^{k_{r-1}} f(\Delta_n^m x_i)$$

$$= \frac{k_r}{h_r}\left(k_r^{-1}\sum_{i=1}^{k_r} f(\Delta_n^m x_i\right) - \frac{k_{r-1}}{h_r}\left(k_{r-1}^{-1}\sum_{i=1}^{k_{r-1}} f(\Delta_n^m x_i)\right)$$

Since, $h_r = k_r - k_{r-1}$, for each $k \in \mathbb{N}$, so we have

$$\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta}, \qquad \frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}.$$

The term $k_r^{-1}\sum_{i=1}^{k_r} f(\Delta_n^m x_i)$ and $k_{r-1}^{-1}\sum_{i=1}^{k_{r-1}} f(\Delta_n^m x_i)$ both converges to 0 and it follows that τ_r converges to 0, that is, $x \in [\mathcal{D}_\theta, (\Delta_n^m)]$.

Therefore, $(\sigma_1)_w(\Delta_n^m) \subset [\mathcal{D}_\theta, \Delta_n^m]$.

Conversely, Let $\liminf_r q_r = 1$. Since θ is lacunary sequence, we choose a subsequence (k_{r_j}) of θ satisfying $\frac{k_{r_j}}{k_{r_j-1}} < 1 + \frac{1}{j}$ and $\frac{k_{r_j-1}}{k_{r-1}} > j$, where $r_j \geq r_{j-1} + 2$.

Define $x = (x_i)$ by

$$\Delta_n^m x_i = \begin{cases} e, & \text{if } i \in I_{rj}, \text{ for some } j = 1, 2, \ldots; \\ \bar{\theta}, & \text{otherwise.} \end{cases}$$

Where e is the identity element of X and $\bar{\theta}$ is the zero element of X.

Then, for any real L,

$$h_{r_j}{}^{-1}\sum_{I_{r_j}} f(\Delta_n^m x_i - L) = f(1-L), \quad \text{for } j = 1, 2, \ldots$$

and

$$h_r^{-1}\sum_{I_r} f(\Delta_n^m x_i - L) = f(L), \quad \text{for } r \neq r_j.$$

Since $x \notin [\mathcal{D}_\theta, \Delta_n^m]$. However, x is weakly Cesàro summable, if t is any sufficiently large integer, so we can find the unique j for which $k_{r_j-1} < t \leq k_{r_{j+1}-1}$ and

$$t^{-1}\sum_{i=1}^{t} f(\Delta_n^m x_i) \leq \frac{k_{r_j-1} + h_{r_j}}{k_{r_j-1}} \leq \frac{2}{j}.$$

Since, $t \to \infty$ implies that $j \to \infty$. Hence, we have $x \in (\sigma_1)_w(\Delta_n^m)$. $\square$

Theorem 3.6. $[\mathcal{D}_\theta, \Delta_n^m] \subset (\sigma_1)_w(\Delta_n^m)$ *if and only if* $\limsup_r q_r < \infty$.

Proof. Let $\limsup_r q_r < \infty$, then $\exists\, Z > 0$ such that $q_r < Z$ for all $r \geq 1$. Considering $x \in [\mathcal{D}_\theta, \Delta_n^m]$ and $\varepsilon > 0$ we can then find $R > 0$ and $K > 0$ such that $\sup_{i \geq R} \tau_i < \varepsilon$ and $\tau_i < K$, for all $i = 1, 2, \ldots$ If t is any integer with $k_{r-1} < t \leq k_r$, where $r > R$, we can write.

$$t^{-1}\sum_{i=1}^{t} f(\Delta_n^m x_i) \leq k_{r-1}^{-1}\sum_{i=1}^{k_r} f(\Delta_n^m x_i)$$

$$
\begin{aligned}
&= k_{r-1}^{-1}\left(\sum_{I_1} f(\Delta_n^m x_i) + \sum_{I_2} f(\Delta_n^m x_i) + \cdots + \sum_{I_r} f(\Delta_n^m x_i)\right)\\
&= \frac{k_1}{k_{r-1}}\tau_1 + \frac{k_2-k_1}{k_{r-1}}\tau_2 + \frac{k_R-k_{R-1}}{k_{r-1}}\tau_R + \frac{k_{R+1}-k_R}{k_{r-1}}\tau_{R+1}\\
&\quad + \cdots + \frac{k_r-k_{r-1}}{k_{r-1}}\tau_r\\
&\le \left(\sup_{i\ge 1}\tau_i\right)\frac{k_r}{k_{r-1}} + \left(\sup_{i\ge R}\tau_i\right)\frac{k_r-k_R}{k_{r-1}}\\
&< K\frac{k_R}{k_{r-1}} + \varepsilon Z.
\end{aligned}
$$

Since, $k_{r-1}\to\infty$, as $t\to\infty$, it follows that $t^{-1}\sum_{i=1}^{t} f(\Delta_n^m x_i)\to 0$. i.e., $x\in(\sigma_1)_w(\Delta_n^m)$.

Suppose $\limsup_r q_r = \infty$. To prove the result we are to find a sequence x such that $x\in[\mathcal{D}_\theta,\Delta_n^m]$ and $x\notin(\sigma_1)_w(\Delta_n^m)$. First, choose a subsequence. (k_{r_j}) of θ so that $q_{r_j} > j$ and then define $x=(x_i)$

$$
x_i = \begin{cases} e, & \text{if } i\in K_{r_j-1} < i \le 2K_{r_j-1}, \text{ for some } j=1,2,\ldots;\\ \bar{\theta}, & \text{otherwise.}\end{cases}
$$

Where e is the identity element of X and $\bar{\theta}$ is the zero element of X.

Then $\tau_{r_j} = \frac{k_{r_j-1}}{k_{r_j}-k_{r_j-1}} < \frac{1}{j-1}$, and if $r\ne r_j, \tau_r=\bar{\theta}$.

Thus $x\in[\mathcal{D}_\theta,\Delta_n^m]$. For the sequence $x=(x_i)$ above and $i=1,\ldots,k_{r_J}$,

$$
\begin{aligned}
k_{r_j}^{-1}\sum_i f(\Delta_n^m x_i - 1) &\ge k_{r_j}^{-1}(k_{r_j} - 2k_{r_j-1})\\
&= 1 - \frac{2k_{r_j-1}}{k_r}\\
&> 1 - \frac{2}{j}
\end{aligned}
$$

which converges to 1 and, for $i=1,\ldots,2k_{r_j-1}$,

$$
\begin{aligned}
\frac{1}{2k_{r_j-1}}\sum_i f(\Delta_n^m x_i) &\ge \frac{k_{r_j-1}}{2k_{r-j-1}}\\
&= \frac{1}{2}
\end{aligned}
$$

and it follows that $x\notin(\sigma_1)_w(\Delta_n^m)$. □

We state the following theorem without proof given the previous two theorems.

Theorem 3.7. *$[\mathcal{D}_\theta, \Delta_n^m] = (\sigma_1)_w(\Delta_n^m)$ if and only if $1 < \liminf_r q_r < \infty$.*

Result 3.8. The class of sequences $[\mathcal{D}_\theta, \Delta_n^m]_\mathcal{G}, \mathcal{G} = 0, 1, \infty$ are not solid in general.

To support of the above result we prove the following example.

Example 3.9. We prove for $[\mathcal{D}_\theta, \Delta]_\infty$. Let $m = n = 1$, then consider $X = \mathbb{R}$ and consider the norm $|.|$., the function $f(x) = x, \quad$ for all $x \in \mathbb{R}$. Consider the sequence (x_k) which is defined as $x_k = k, \quad$ for all $k \in \mathbb{N}$. Then

$$\frac{1}{h_r} \sum_{k \in I_r} |\Delta x_k| = 1.$$

Consider the sequence of scalar (α_k) defined by $\alpha_k = (-1)^k$ then we get

$$\Delta(\alpha x_k) = (-1)^k (2k + 1).$$

We have,

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} |\Delta x_k| &= \frac{1}{h_r} \sum_{k \in I_r} \Big| (-1)^k (2k + 1) \Big| \\ &= \frac{1}{h_r} \sum_{k \in I_r} (2k + 1) \to \infty, \text{ as } r \to \infty. \end{aligned}$$

Hence, $(\alpha_k x_k) \notin [\mathcal{D}_\theta, \Delta]_\infty$.

Therefore, the space $[\mathcal{D}_\theta, \Delta]_\infty$ is not solid.

Similar proof can be provided to show that the other sequence classes are not solid.

Combining Remark 2.10 and Result 3.8 we give the following result.

Result 3.10. The class of sequences $[\mathcal{D}_\theta, \Delta_n^m]_\mathcal{G}, \mathcal{G} = 0, 1, \infty$ are not monotone.

Result 3.11. The class of sequences $[\mathcal{D}_\theta, \Delta_n^m]_\mathcal{G}, \mathcal{G} = 0, 1, \infty$ are not symmetric.

In order to justify the above result we provide the following example.

Proof. We establish this for $[\mathcal{D}_\theta, \Delta_n^m]_1$. Consider the lacunary sequence $\theta = 2^r$ for $m = n = 1$.Consider the sequence (x_k) defined by $x_k = k$, for all $k \in \mathbb{N}$. If (x_k) is defined as the following, then let (y_k) be the rearrangement of (x_k), which is defined as follows

$$(y_k) = (x_1, x_2, x_4, x_3, x_9 \cdots)$$

Then $(y_k) \notin [\mathcal{D}_\theta, \Delta_n^m]_1$.

Similarly it can be shown that the other classes of sequence are also not symmetric. □

Theorem 3.12. $(AC_\theta)_w(\Delta_n^m) \subset [\mathcal{D}_\theta, \Delta_n^m]$ *and the inclusion are strict.*

Proof. Let $x \in (AC_\theta)_w(\Delta_n^m)$ and $\varepsilon > 0,$ there exists $N > 0$ and L such that

$$\frac{1}{h_r}\sum_{k\in I_r} f(\Delta_n^m x_{k+i} - L) < \varepsilon, \quad \text{for } n > N, r = 1, 2, 3, \dots$$

Now we can select $H > 0$ for the lacunary sequence θ so that $r \geq N$ and thus $\tau_r < \varepsilon$.

Thus $x \in \mathcal{D}_\theta$. Therefore to find a sequence in $\mathcal{D}_\theta$ but not in $(AC_\theta)_w(\Delta_n^m)$ we define $x = (x_i)$ by

$$x_i = \begin{cases} e, & \text{if, for some } k_{r-1} < i \leq k_{r-1} + \sqrt{h_r}; \\ \bar{\theta}, & \text{otherwise.} \end{cases}$$

Where e is the identity element of X and $\bar{\theta}$ is the zero element of X.

But, $\tau_r = \frac{1}{h_r}\sum_{I_r} f(x_i) = \frac{1}{h_r}[\sqrt{h_r}] = \frac{1}{\sqrt{h_r}}$ which converges to 0 as $r \to \infty$. Hence the sequence (x_i) is an oscillatory sequence with repect to n. So it is not weakly almost convergent to 0. □

Theorem 3.13. *For* $m \geq 1$ *and* $n \geq 1, [\mathcal{D}_\theta, \Delta_n^{m-1}]_\mathcal{G} \subset [\mathcal{D}_\theta, \Delta_n^m]_\mathcal{G}$ *for* $\mathcal{G} = 0, 1, \infty$. *In general,* $[\mathcal{D}_\theta, \Delta_n^i]_\mathcal{G} \subset [\mathcal{D}_\theta, \Delta_n^m]_\mathcal{G}$ *for* $\mathcal{G} = 0, 1, \infty$ *and* $i = 0, 1, \dots, m-1$. *The inclusion are strict.*

Proof. Let $(x_k) \in [\mathcal{D}_\theta, \Delta_n^{m-1}]_0$.

Then we have,

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} f(\Delta_n^{m-1}x_k) = 0 \tag{3.2}$$

Now,

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} f(\Delta_n^m x_k) = \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} f\left(\Delta_n^{m-1}x_k - \Delta_n^{m-1}x_{k+1}\right)$$
$$\leq \left(\frac{1}{h_r}\sum_{k\in I_r} f\left(\Delta_n^{m-1}x_k\right) - \frac{1}{h_r}\sum_{k\in I_r} f\left(\Delta_n^{m-1}x_{k+1}\right)\right)$$

as $r \to \infty$ we have

$$\frac{1}{h_r}\sum_{k\in I_r} f(\Delta_n^m x_k) = 0, \quad \text{by (3.2)}$$

which implies $(x_k) \in [\mathcal{D}_\theta, \Delta_n^{m-1}]_0$.

The other cases will follow similarly.

Proceeding inductively we have,$[\mathcal{D}_\theta, \Delta_m^i]_\mathcal{G} \subset [\mathcal{D}_\theta, \Delta_n^m]_\mathcal{G}$, for $\mathcal{G} = 0, 1, \infty$ and $i = 0, 1, \dots, n-1$.

The inclusion is strict follows from the following example. □

Example 3.14. We consider a lacunary sequence $\theta = 2^r$ and consider a non zero sequence $(x_k) = (k_{n-1})$. Then $\Delta_n^m(x_k) = 0, \Delta_n^m x_k = \sum_{v=0}^{m-1}(-1)^v \binom{m-1}{v} x_{k+nv}$, for all $k \in \mathbb{N}$. Therefore $(x_k) \in [\mathcal{D}_\theta, \Delta_n^m]$ but $(x_k) \notin [\mathcal{D}_\theta, \Delta_n^{m-1}]_0$.

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