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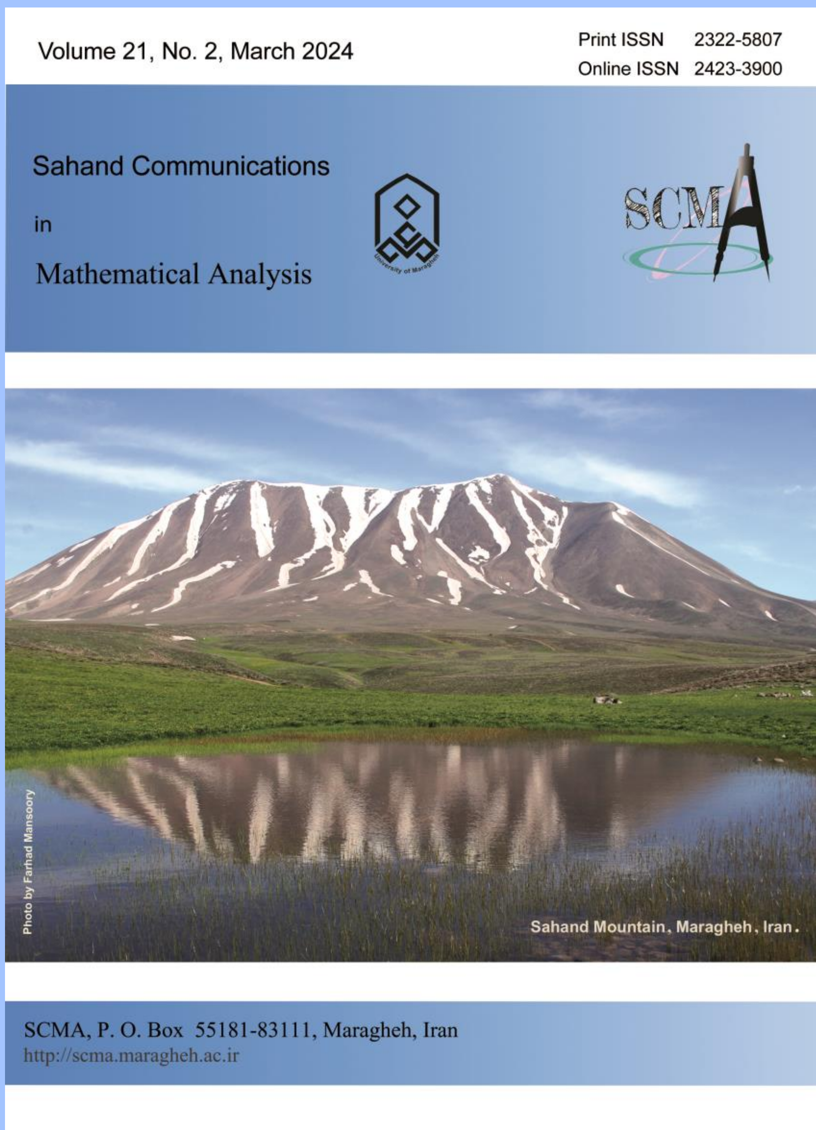

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# Generalization of Banach Contraction Principle for Prešić's Type Mappings in Soft Metric Spaces

Bijender Singh[1], Vizender Singh[2] and Özlem Acar[3*]

Abstract. The objective of this paper is to highlight the idea of $k$-weakly and $2k$-weakly soft compatible mappings and their utilization in proving the main results. For this aim, we establish some fixed point results for the Prešić's type contractive mappings in the context of soft metric spaces, when the set of the parameter is finite. Also we give an example to show that the condition of finiteness on the set of parameter can't be omitted. Some examples are given to support main findings of this article. Finally, an application of a soft version of BCP in iterated soft function systems is established.

## 1. Introduction

The notion of soft-set theory as a technique for approaching complexity and the decision-making problem was established by Russian researcher Molodtsov in 1999 [4]. Soft set theory is used in numerous areas of study, including social sciences, astronomy, chemistry, economics, informatics, and medical sciences. Maji et al. [25, 26] employed the principle of soft sets when making decisions and described several soft set operations. Ali et al. [20] also proposed several soft set operations that have been very effective in the area of soft set theory. Chen et al. [2] have developed some innovative works in the settings of soft set theory.

Soft topological spaces were investigated by Shabir and Naz [22]. Cagman et al. [23] suggested soft topological space theory and achieved

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different characteristics concerning soft topological spaces. Riaz and Fatima [21] employed soft sets, soft elements, and soft points to introduce the ideas of soft dense, non-soft dense sets, soft first category, soft second category, and Baire's soft space for soft metric spaces, as well as construct the soft metric version of Baire's category theorem. Pei and Miao [5] examined the relationship between information technology and soft sets. The essential properties of soft sets and soft real numbers were proposed by Samanta and Das [29]-[31]. Soft elements and soft points in soft sets were investigated, and the concept of soft metric spaces was established. In their soft mappings on soft sets, Samanta and Majumdar [27] investigated the idea of soft groups, images, and inverse images. Ahmad and Kharal [1] have developed soft-class mappings, soft-set images, and inverse soft-set images. Rong investigated the countability of soft topological spaces, soft spaces, and soft Lindolof space in [33], as well as several key conclusions based on these notions.

Roy and Samanta [32] explored some important findings with a soft base and soft sub-base in the theory of topological spaces. In their work on soft continuity, soft openness, and soft closeness, Zorlutuna and Cakir [9] looked into the effect of soft-separation axioms and expanded the pasting lemma about soft set theory. Subhashinin and Sekar investigated numerous intriguing aspects of the soft pre-open set to define the soft pre-topology and soft pre-sub-maximum in [10]. The concept of a soft ideal for a soft topological space was presented by Yildirim et al. [7], and soft I-Baire spaces for a soft topological space were constructed. Many scholars in the last decade have been working on soft topology and soft metric spaces (their work is noted in) [2]-[3],[6],[8],[11]-[19],[27]-[28],[34].

This article is organized as; Section 2 contains some basic of soft set from existing literature of soft metric theory. In Section 3, we proved soft metric version of Banach contraction principle for Prešić's type mappings. In Section 4, an application of a soft version of BCP in iterated soft function systems is established. Finally in Section 5, we conclude our work and provide an idea that some further improvement may be possible.

## 2. Preliminaries

The latter's $X$, $P(X)$, $E$, $SP\left(\tilde{X}\right)$, $\mathbb{R}(A)^*$ and $B(\mathbb{R})$ will be used to signify in this work the universal set, the power set of $X$, the set of parameters, the set of all soft points of $\tilde{X}$, the set of all non-negative reals and the set of all non-empty bounded subsets of $\mathbb{R}$ respectively.

**Definition 2.1** ([4]). Let $X$ be an initial universe and $E$ describes as a non-empty collection of parameters and $\phi \neq A \subseteq E$. Soft set over $A$

corresponds to a pair $(F, A)$ or $F_A$, where $F$ is the mapping from $A$ to $P(X)$ *i.e.* $F : A \to P(X)$. It can be written as

$$(F, A) = \{(\eta, F(\eta)) : \eta \in A, F(\eta) \in P(X)\}.$$

**Definition 2.2** ([5],[7])**.** Let $(F, A_1)$ and $(G, A_2)$ be two soft sets over a universe $X$.

(i) Union is defined as

$$(F, A_1)\tilde{\cup}(G, A_2) = \begin{cases} F(\eta), & \text{if } \eta \in A_1/A_2, \\ G(\eta), & \text{if } \eta \in A_2/A_1, \\ F(\eta) \cup G(\eta), & \text{if } \eta \in A_2 \cap A_1; \end{cases}$$

(ii) Intersection is defined as $(H, J) = (F, A_1)\tilde{\cap}(G, A_2)$, where $J = A_1 \cap A_2 \neq \phi$ and $H(\eta) = F(\eta) \cap G(\eta)$, $\forall \eta \in J$.

**Definition 2.3** ([5],[7])**.** Let $(F, E)$ be a soft set over the universe $X$. Then pair $(F, E)$ is

(i) **Soft null set** if, $F(\eta) = \phi, \forall \eta \in E$;
(ii) **Soft non-null** if, $F(\eta) \neq \phi$, for at least one $\eta \in E$;
(iii) **Absolute soft set** if, $F(\eta) = X, \forall \eta \in E$ and it is denoted by $\tilde{X}$;
(iv) **Soft point** if, $F(\lambda) = \{x\}$ for exactly one $\lambda \in E$, some $x \in X$ and $F(\eta) = \phi$ for all $\eta \in E \setminus \{\lambda\}$ denoted by $(F^x_\lambda, E)$ or simply by $F^x_\lambda$.

Soft set theory's fundamental notions and properties can be found in [4],[7],[15].

**Definition 2.4** ([18])**.** A soft real set is a mapping $F : A \to B(\mathbb{R})$ and it is denoted by $(F, A)$. We suppose that $\hat{a}$, $\hat{b}$, $\hat{c}$ signifing soft real numbers, and $\bar{l}$, $\bar{m}$, $\bar{n}$ signifying the specific kind of soft real numbers such as $\bar{a}(\eta) = a$, for all $\eta \in A$ etc.

**Definition 2.5** ([18])**.** For two soft real numbers $\hat{a}$, $\hat{b}$ we say that

(i) $\hat{a} \tilde{\leq} \hat{b}$ if $\hat{a}(\eta) \leq \hat{b}(\eta)$, $\forall\ \eta \in A$;
(ii) $\hat{a} \tilde{\geq} \hat{b}$ if $\hat{a}(\eta) \geq \hat{b}(\eta)$, $\forall\ \eta \in A$;
(iii) $\hat{a} \tilde{<} \hat{b}$ if $\hat{a}(\eta) < \hat{b}(\eta)$, $\forall\ \eta \in A$;
(iv) $\hat{a} \tilde{>} \hat{b}$ if $\hat{a}(\eta) > \hat{b}(\eta)$, $\forall\ \eta \in A$.

**Remark 2.6** ([15])**.** Recall that if $\hat{r}$ is a soft mapping from a soft set $(F, A)$ to a soft set $(G, B)$, then for each soft point $F^x_\lambda \in (F, A)$ there exists exactly one soft point $G^y_\mu \in (G, B)$ such that $\hat{r}(F^x_\lambda)$=$G^y_\mu$.

**Definition 2.7** ([15])**.** If $\eta \in A$ and $F(\eta) = \{x\} \subset G(\eta)$, then a soft point $F^x_\eta$ is said to belong to a soft set $(G, A)$. We write $F^x_\eta \tilde{\in} (G, A)$.

**Definition 2.8** ([15])**.** Two soft points $F^x_\alpha$ and $F^y_\beta$ are said to be equal if $\alpha = \beta$ and $F(\alpha) = F(\beta)$ i.e. $x = y$. Thus $F^x_\alpha \neq F^y_\beta \Leftrightarrow x \neq y$ or $\alpha \neq \beta$.

We define the soft metric using the idea of soft points as follows:

**Definition 2.9** ([15])**.** Let $X$ be a universe, $A$ be a non-empty subset of parameters and $\tilde{X}$ be the absolute soft set *i.e.*, $F(\lambda) = X$ for all $\lambda \in A$, where $(F, A) = \tilde{X}$. A mapping $d : SP\left(\tilde{X}\right) \times SP\left(\tilde{X}\right) \tilde{\rightarrow} \mathbb{R}(A)^*$ is said to be a soft metric on $\tilde{X}$ if for any $X^x_\lambda, X^y_\mu, X^z_\gamma \in SP\left(\tilde{X}\right)$, the following holds:

(i) $d\left(X^x_\lambda, X^y_\mu\right) \tilde{\geq} \bar{0}$, for all $X^x_\lambda, X^y_\mu \tilde{\in} \tilde{X}$.
(ii) $d\left(X^x_\lambda, X^y_\mu\right) = \bar{0}$ if and only if $X^x_\lambda = X^y_\mu$.
(iii) $d\left(X^x_\lambda, X^y_\mu\right) = d\left(X^y_\mu, X^x_\lambda\right)$, for all $X^x_\lambda, X^y_\mu \tilde{\in} \tilde{X}$.
(iv) $d\left(X^x_\lambda, X^z_\gamma\right) \tilde{\leq} d\left(X^x_\lambda, X^y_\mu\right) + d\left(X^y_\mu, X^z_\gamma\right)$, for all $X^x_\lambda, X^y_\mu, X^z_\gamma \tilde{\in} \tilde{X}$.

$\left(\tilde{X}, d, A\right)$ or simply $(\tilde{X}, d)$ is called soft metric space with the soft set $\tilde{X}$ and a soft metric $d$. The conditions $(i), (ii), (iii)$ and $(iv)$ are referred to as soft metric axioms.

Every crisp metric $d$ on a crisp set $X$ can be extended to a soft metric on the soft set $\tilde{X}$. Also, every parametrized family of crisp metrics can be perceived as a soft metric but any soft metric is not merely a family of crisp metrics.

**Definition 2.10** ([15])**.** Let $(\tilde{X}, d, E)$ be a soft metric space.

(i) **Soft open ball** with center at a soft point $F^x_\lambda$ and radius $\bar{r} \tilde{\geq} \bar{0}$ is

$$B\left(F^x_\lambda, \bar{r}\right) = \left\{F^y_\mu \tilde{\in} \tilde{X} : d\left(F^x_\lambda, F^y_\mu\right) \tilde{<} \bar{r}\right\} \subset SP\left(\tilde{X}\right);$$

(ii) **Soft closed ball** with center at a soft point $F^x_\lambda$ and radius $\bar{r} \tilde{\geq} \bar{0}$ is

$$B\left(F^x_\lambda, \bar{r}\right) = \left\{F^y_\mu \tilde{\in} \tilde{X} : d\left(F^x_\lambda, F^y_\mu\right) \tilde{\leq} \bar{r}\right\} \subset SP\left(\tilde{X}\right).$$

Given a soft metric space $(\tilde{X}, d)$, a collection $\left\{F^{x_\alpha}_{\lambda\alpha}\right\}_{\alpha\in\lambda}$ of soft points in $\tilde{X}$, simply denoted by $\left\{F^x_{\lambda,\alpha}\right\}_{\alpha\in\Lambda}$. In particular, a sequence $\left\{F^{x_n}_{\lambda_n}\right\}_{n\in\mathbb{N}}$ of soft points in $\tilde{X}$ will be denoted by $\left\{F^x_{\lambda,n}\right\}_n$.

**Definition 2.11** ([15])**.** Let $(\tilde{X}, d)$ be a soft metric space and $\left\{F^x_{\lambda,n}\right\}_n$ be a sequence of soft points in $\tilde{X}$, then

(i) sequence $\left\{F^x_{\lambda,n}\right\}_n$ is convergent to $F^y_\mu \in \tilde{X}$ if, for every $\bar{\epsilon} \tilde{>} \bar{0}$, $\exists\ m \in \mathbb{N}$ such that $d\left(F^x_{\lambda,n}, F^y_\mu\right) \tilde{<} \bar{\epsilon}$, whenever $n \geq m$. *i.e.* $d\left(F^x_{\lambda,n}, F^y_\mu\right) \rightarrow \bar{0}$ as $n \rightarrow \infty$;

(ii) sequence $\left\{F^x_{\lambda,n}\right\}_n$ is a Cauchy sequence if, for each $\bar{\epsilon} \tilde{>} \bar{0}, \exists\, m \in \mathbb{N}$ such that $d\left(F^x_{\lambda,i}, F^y_{\mu,j}\right) \tilde{<} \bar{\epsilon},\ \forall i,j \geq m$. *i.e.* $d\left(F^x_{\lambda,i}, F^y_{\mu,j}\right) \rightarrow \bar{0}$ as $i,j \rightarrow \infty$;

(iii) if limit of a sequence $\left\{F^x_{\lambda,n}\right\}_n$ exists then, it is unique;

(iv) every convergent sequence $\left\{F^x_{\lambda,n}\right\}_n$ is a Cauchy sequence in a soft metric space $(\tilde{X}, d)$;

(v) if for every Cauchy sequence $\left\{F^x_{\lambda,n}\right\}_n$ of soft points in $\tilde{X}$ such that $d\left(F^x_{\lambda,n}, F^y_\mu\right) \rightarrow \bar{0}$ as $n \rightarrow \infty$ for some $F^y_\mu$ in $\tilde{X}$ then, the pair $(\tilde{X}, d)$ is a complete metric space.

## 3. Main Results

Abbas [17] generalized the Banach contraction principle for complete metric spaces in to the setting of complete soft metric spaces $(\tilde{X}, d)$ with $A$ is a (non-empty) countable finite set, and it was shown that the restriction of finiteness on $A$ can't be eased. Here in this section we use notation of soft point as $X^x_\lambda$ instead of $F^x_\lambda$.

**Theorem 3.1** ([17])**.** *Assume that* $(\tilde{X}, d)$ *be a complete soft metric space with countable finite set* $A$*. The soft mapping* $T : \tilde{X} \tilde{\rightarrow} \tilde{X}$ *follows*

$$d\left(T\left(X^x_\lambda\right), T(X^y_\mu)\right) \tilde{\leq} \bar{c} d\left(X^x_\lambda, X^y_\mu\right),$$

*for all* $X^x_\lambda, X^y_\mu \in SP\left(\tilde{X}\right)$*, and* $\bar{0} \tilde{\leq} \bar{c} \tilde{<} \bar{1}$*. As such* $T$ *possesses a unique fixed soft point. i.e.* $X^x_\lambda \in SP\left(\tilde{X}\right)$ *such that* $T\left(X^x_\lambda\right) = X^x_\lambda$*.*

**Definition 3.2.** Let $X \neq 0$ and $A$ is finite set, let $T : \tilde{X}^{2k} \tilde{\rightarrow} \tilde{X}$ and $f : \tilde{X} \tilde{\rightarrow} \tilde{X}$. Then $(f, T)$ is called a $2k$-weakly soft compatible pair if $f\left(T\left(X^x_\lambda, X^x_\lambda, \ldots, X^x_\lambda\right)\right) = T\left(f\left(X^x_\lambda\right), f\left(X^x_\lambda\right), \ldots, f\left(X^x_\lambda\right)\right)$, whenever $X^x_\lambda \in \tilde{X}$ is such that $f\left(X^x_\lambda\right) = T\left(X^x_\lambda, X^x_\lambda, \ldots, X^x_\lambda\right)$.

In the following results; we have generalized the Banach contraction principle for Prešić's type contractive mappings [11],[13],[28] in setting of soft metric spaces.

**Theorem 3.3.** *Let* $\left(\tilde{X}, d, A\right)$ *be a soft metric space with finite set* $A$, $k$ *is a positive integer and* $S, T : \tilde{X}^{2k} \tilde{\to} \tilde{X}$ *and* $f : \tilde{X} \tilde{\to} \tilde{X}$ *be a soft mapping satisfying*

$$d\left(S\left(X_{\lambda,1}^{x_1}, X_{\lambda,2}^{x_2}, \ldots, X_{\lambda,2k}^{x_{2k}}\right), T\left(X_{\lambda,2}^{x_2}, X_{\lambda,3}^{x_3}, \ldots, X_{\lambda,2k+1}^{x_{2k+1}}\right)\right) \tilde{\leq} \bar{\alpha} \max\left\{d\left(f\left(X_{\lambda,i}^{x_i}\right), f\left(X_{\lambda,i+1}^{x_{i+1}}\right)\right)\right\}, \tag{3.1}$$

$1 \leq i \leq 2k$ *and for all* $X_{\lambda,1}^{x_1}, X_{\lambda,2}^{x_2}, \ldots, X_{\lambda,2k+1}^{x_{2k+1}}$ *in* $\tilde{X}$,

$$d\left(T\left(X_{\mu,1}^{y_1}, X_{\mu,2}^{y_2}, \ldots, X_{\mu,2k}^{y_{2k}}\right), S\left(X_{\mu,2}^{y_2}, X_{\mu,3}^{y_3}, \ldots, X_{\mu,2k+1}^{y_{2k+1}}\right)\right) \tilde{\leq} \bar{\alpha} \max\left\{d\left(f\left(X_{\mu,i}^{y_i}\right), f\left(X_{\mu,i+1}^{y_{i+1}}\right)\right)\right\}, \tag{3.2}$$

$1 \leq i \leq 2k$ *for all* $X_{\mu,1}^{y_1}, X_{\mu,2}^{y_2}, \ldots, X_{\mu,2k+1}^{y_{2k+1}}$ *in* $\tilde{X}$ *where* $0 \leq \bar{\alpha} < 1$.

$$d\left(S\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right), T\left(X_\gamma^z, X_\gamma^z, \ldots, X_\gamma^z\right)\right) \tilde{<} d\left(f\left(X_\mu^y\right), f(X_\gamma^z)\right) \tag{3.3}$$

*for all* $X_\mu^y, X_\gamma^z \in \tilde{X}$ *with* $X_\mu^y \neq X_\gamma^z$. *Suppose that* $f\left(\tilde{X}\right)$ *is complete and* $(f, T)$ *or* $(f, S)$ *is a* $2k$*-weakly soft compatible pair. Then there is a unique soft point* $X_\mu^y \in \tilde{X}$ *such that*

$$\begin{aligned} f\left(X_\mu^y\right) &= X_\mu^y \\ &= T\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right) \\ &= S\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right). \end{aligned}$$

*Proof.* Suppose that $X_{\lambda,1}^{x_1}, X_{\lambda,2}^{x_2}, \ldots, X_{\lambda,2k}^{x_{2k}}$ are arbitrary points in $\tilde{X}$ and for $n \in \mathbb{N}$, define

$$f\left(X_{\lambda,2n+2k-1}^{x_{2n+2k-1}}\right) = S\left(X_{\lambda,2n-1}^{x_{2n-1}}, X_{\lambda,2n}^{x_{2n}}, \ldots, X_{\lambda,2n+2k-2}^{x_{2n+2k-2}}\right),$$

and

$$f\left(X_{\lambda,2n+2k}^{x_{2n+2k}}\right) = T\left(X_{\lambda,2n}^{x_{2n}}, X_{\lambda,2n+1}^{x_{2n+1}}, \ldots, X_{\lambda,2n+2k-1}^{x_{2n+2k-1}}\right).$$

Next we show that the sequence $\left\{f\left(X_{\lambda,n}^{x_n}\right)\right\}$ is a Cauchy sequence.

For simplicity, set $d_n = d\left(f\left(X_{\lambda,n}^{x_n}\right), f\left(X_{\lambda,n+1}^{x_{n+1}}\right)\right)$. We shall prove by induction that inequality

$$d_n \tilde{\leq} \bar{\mu} \bar{\theta}^n, \tag{3.4}$$

is true for $n \in \mathbb{N}$, where

$$\bar{\mu} = \max\left\{\frac{d_1}{\bar{\theta}^1}, \frac{d_2}{\bar{\theta}^2}, \ldots, \frac{d_{2k}}{\bar{\theta}^{2k}}\right\}, \qquad \bar{\theta} = \bar{\alpha}^{\frac{1}{2k}}.$$

By the definition of $\bar{\mu}$ and $\bar{\theta}$ inequality (3.4) is obviously true for $n \in \{1, 2, \ldots, 2k\}$. Suppose that the following $(2k-1)$ inequalities

$$\begin{aligned}
&d_{2n} \tilde{\preceq} \bar{\mu}\bar{\theta}^{2n},\\
&d_{2n+1} \tilde{\preceq} \bar{\mu}\bar{\theta}^{2n+1},\\
&\quad\vdots\\
&d_{2n+2k-2} \tilde{\preceq} \bar{\mu}\bar{\theta}^{2n+2k-2},
\end{aligned}$$

holds. Then

$$\begin{aligned}
d_{2n+2k-1} &= d\left(f\left(X^{x_{2n+2k-1}}_{\lambda,2n+2k-1}\right), f\left(X^{x_{2n+2k}}_{\lambda,2n+2k}\right)\right)\\
&= d\left(S\left(X^{x_{2n-1}}_{\lambda,2n-1}, X^{x_{2n}}_{\lambda,2n}, \ldots, X^{x_{2n+2k-2}}_{\lambda,2n+2k-2}\right)\right.\\
&\qquad \left., T\left(X^{x_{2n}}_{\lambda,2n}, X^{x_{2n+1}}_{\lambda,2n+1}, \ldots, X^{x_{2n+2k-1}}_{\lambda,2n+2k-1}\right)\right)\\
&\tilde{\preceq} \bar{\alpha} \max\left\{d\left(f\left(X^{x_i}_{\lambda,i}\right), f\left(X^{x_{i+1}}_{\lambda,i+1}\right)\right)\right\}\\
&= \bar{\alpha} \max\left\{d_{2n-1}, d_{2n}, \ldots, d_{2n+2k-2}\right\}\\
&\tilde{\preceq} \bar{\alpha} \max\left\{\bar{\mu}\bar{\theta}^{2n-1}, \bar{\mu}\bar{\theta}^{2n}, \ldots, \bar{\mu}\bar{\theta}^{2n+2k-2}\right\}\\
&= \bar{\alpha}\bar{\mu}\bar{\theta}^{2n-1}\\
&= \bar{\mu}\bar{\theta}^{2n+2k-1}.
\end{aligned}$$

Hence by induction hypothesis the inequality (3.4) is true for each $n \in \mathbb{N}$. Now, for any $n, p \in \mathbb{N}$, we have

$$\begin{aligned}
d\left(f\left(X^{x_n}_{\lambda,n}\right), f\left(X^{x_{n+p}}_{\lambda,n+p}\right)\right) &\tilde{\preceq} d\left(f\left(X^{x_n}_{\lambda,n}\right), f\left(X^{x_{n+1}}_{\lambda,n+1}\right)\right)\\
&\quad + d\left(f\left(X^{x_{n+1}}_{\lambda,n+1}\right), f\left(X^{x_{n+2}}_{\lambda,n+2}\right)\right)\\
&\quad + \cdots + d\left(f\left(X^{x_{n+p-1}}_{\lambda,n+p-1}\right), f\left(X^{x_{n+p}}_{\lambda,n+p}\right)\right)\\
&= d_n + d_{n+1} + \cdots + d_{n+p-1}\\
&\tilde{\preceq} \bar{\mu}\bar{\theta}^{n} + \bar{\mu}\bar{\theta}^{n+1} + \cdots + \bar{\mu}\bar{\theta}^{n+p-1}\\
&= \bar{\mu}\bar{\theta}^{n}\left\{1 + \bar{\theta} + \cdots + \bar{\theta}^{p-1}\right\}\\
&\tilde{<} \frac{\bar{\mu}\bar{\theta}^{n}}{1-\bar{\theta}}.
\end{aligned}$$

As $\bar{\theta} < \bar{1}$, we conclude that the sequence $\{X^{x_n}_{\lambda,n}\}$ is a Cauchy sequence. Since $\tilde{X}$ is complete, then there exists $X^y_\mu$ in $\tilde{X}$ such that

$$X^y_\mu = \lim_{n\to\infty} f\left(X^{x_n}_{\lambda,n}\right).$$

Then for any integer $n \in \mathbb{N}$, we have

$$\begin{aligned}
&d\left(S\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right), f\left(X_{\lambda,2n+2k-1}^{x_{2n+2k-1}}\right)\right) \\
&\quad = d\left(S\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right), S\left(X_{\lambda,2n-1}^{x_{2n-1}}, X_{\lambda,2n}^{x_{2n}}, \ldots, X_{\lambda,2n+2k-2}^{x_{2n+2k-2}}\right)\right) \\
&\quad \tilde{\leq} d\left(S\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right), T\left(X_\mu^y, X_\mu^y, \ldots, X_{\lambda,2n-1}^{x_{2n-1}}\right)\right) \\
&\qquad + d\left(T\left(X_\mu^y, X_\mu^y, \ldots, X_{\lambda,2n-1}^{x_{2n-1}}\right), S\left(X_\mu^y, X_\mu^y, \ldots, X_{\lambda,2n-1}^{x_{2n-1}}, X_{\lambda,2n}^{x_{2n}}\right)\right) \\
&\qquad + d\left(S\left(X_\mu^y, X_\mu^y, \ldots, X_{\lambda,2n}^{x_{2n}}\right), T\left(X_\mu^y, X_\mu^y, \ldots, X_{\lambda,2n}^{x_{2n}}, X_{\lambda,2n+1}^{x_{2n+1}}\right)\right) \\
&\qquad + \cdots + d\left(S\left(X_\mu^y, X_\mu^y, \ldots, X_{\lambda,2n+2k-4}^{x_{2n+2k-4}}\right)\right. \\
&\qquad \left., T\left(X_\mu^y, X_{\lambda,2n-1}^{x_{2n-1}}, \ldots, X_{\lambda,2n+2k-3}^{x_{2n+2k-3}}\right)\right) \\
&\qquad + d\left(T\left(X_\mu^y, X_{\lambda,2n-1}^{x_{2n-1}}, \ldots, X_{\lambda,2n+2k-3}^{x_{2n+2k-3}}\right)\right. \\
&\qquad \left., S\left(X_{\lambda,2n-1}^{x_{2n-1}}, X_{\lambda,2n}^{x_{2n}}, \ldots, X_{\lambda,2n+2k-2}^{x_{2n+2k-2}}\right)\right) \\
&\quad \tilde{\leq} \bar{\alpha} d\left(f\left(X_\mu^y\right), f\left(X_{\lambda,2n-1}^{x_{2n-1}}\right)\right) \\
&\qquad + \bar{\alpha} \max\left\{d\left(f\left(X_\mu^y\right), f\left(X_{\lambda,2n-1}^{x_{2n-1}}\right)\right), d\left(f\left(X_{\lambda,2n-1}^{x_{2n-1}}\right), f\left(X_{\lambda,2n}^{x_{2n}}\right)\right)\right\} \\
&\qquad + \cdots + \bar{\alpha} \max\left\{d\left(f\left(X_\mu^y\right), f\left(X_{\lambda,2n-1}^{x_{2n-1}}\right)\right), d\left(f\left(X_{\lambda,2n-1}^{x_{2n-1}}\right), f\left(X_{\lambda,2n}^{x_{2n}}\right)\right)\right. \\
&\qquad \left., \ldots, d\left(f\left(X_{\lambda,n+k-4}^{x_{n+k-4}}\right), f\left(X_{\lambda,n+k-3}^{x_{n+k-3}}\right)\right)\right\} \\
&\qquad + \bar{\alpha} \max\left\{d\left(X_\mu^y, X_{\lambda,2n-1}^{x_{2n-1}}\right), d\left(f\left(X_{\lambda,2n-1}^{x_{2n-1}}\right), f\left(X_{\lambda,2n}^{x_{2n}}\right)\right)\right. \\
&\qquad \left., \ldots, d\left(f\left(X_{\lambda,2n+2k-3}^{x_{2n+2k-3}}\right), f\left(X_{\lambda,2n+2k-2}^{x_{2n+2k-2}}\right)\right)\right\},
\end{aligned}$$

as $n \to \infty$, we get

$$S\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right) = f\left(X_\mu^y\right). \tag{3.5}$$

Consider

$$\begin{aligned}
d\left(f\left(X_\mu^y\right), T(X_\mu^y, X_\mu^y, \ldots, X_\mu^y)\right) &= d\left(S\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right), T\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right)\right) \\
&\tilde{\leq} \bar{\alpha}(0) \\
&= 0.
\end{aligned}$$

Thus

$$T\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right) = f\left(X_\mu^y\right). \tag{3.6}$$

So that

$$T\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right) = f\left(X_\mu^y\right) = S\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right).$$

Suppose that $f\left(X_\mu^y\right) \neq X_\mu^y$. Then from (3.3), we have

$$\begin{aligned} d\left(f^2\left(X_\mu^y\right), f\left(X_\mu^y\right)\right) &= d\left(S(f\left(X_\mu^y\right), f\left(X_\mu^y\right), \ldots, f\left(X_\mu^y\right)), T\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right)\right) \\ &< d\left(f^2\left(X_\mu^y\right), f\left(X_\mu^y\right)\right). \end{aligned}$$

It is a contradiction. Therefore $f\left(X_\mu^y\right) = X_\mu^y$.
Now from (3.5) and (3.6), we have

$$f\left(X_\mu^y\right) = X_\mu^y = T\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right) = S\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right).$$

For uniqueness suppose that there exists a point $X_\gamma^z \neq X_\mu^y \in \tilde{X}$ such that

$$f(X_\gamma^z) = X_\gamma^z = T(X_\gamma^z, X_\gamma^z, \ldots, X_\gamma^z) = S\left(X_\gamma^z, X_\gamma^z, \ldots, X_\gamma^z\right).$$

Consider

$$\begin{aligned} d\left(f(X_\gamma^z), f\left(X_\mu^y\right)\right) &= d\left(S(X_\gamma^z, X_\gamma^z, \ldots, X_\gamma^z), T\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right)\right) \\ &< d\left(f(X_\gamma^z), f\left(X_\mu^y\right)\right). \end{aligned}$$

It is a contraction. Therefore $X_\gamma^z = X_\mu^y$. □

**Remark 3.4.** If we set $S = T$ and $k$ is taken at place of $2k$ in the above finding, we get the following:

**Corollary 3.5.** *Let $\left(\tilde{X}, d, A\right)$ be a soft metric space with $A$ is a finite set, $k$ is a positive integer and $T : \tilde{X}^k \tilde{\to} \tilde{X}$ and $f : \tilde{X} \tilde{\to} \tilde{X}$ be a soft mapping satisfying*

$$\begin{aligned} (3.7) \qquad & d\left(T\left(X_{\lambda,1}^{x_1}, X_{\lambda,2}^{x_2}, \ldots, X_{\lambda,k}^{x_k}\right), T\left(X_{\lambda,2}^{x_2}, X_{\lambda,3}^{x_3}, \ldots, X_{\lambda,k+1}^{x_{k+1}}\right)\right) \\ & \tilde{\leq} \bar{\alpha} \max\left\{d\left(f\left(X_{\lambda,i}^{x_i}\right), f\left(X_{\lambda,i+1}^{x_{i+1}}\right)\right)\right\}, \end{aligned}$$

$1 \leq i \leq k$ *and for all* $X_{\lambda,1}^{x_1}, X_{\lambda,2}^{x_2}, \ldots, X_{\lambda,k+1}^{x_{k+1}}$ *in* $\tilde{X}$ *where* $0 \tilde{\leq} \bar{\alpha} \tilde{<} 1$.

(3.8)
$$d\left(T\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right), T\left(X_\gamma^z, X_\gamma^z, \ldots, X_\gamma^z\right)\right) \tilde{<} d\left(f\left(X_\mu^y\right), f(X_\gamma^z)\right),$$

*for all $X_\mu^y, X_\gamma^z \in \tilde{X}$ with $X_\mu^y \neq X_\gamma^z$. Suppose that $f\left(\tilde{X}\right)$ is complete and $(f, T)$ is a $k$-weakly soft compatible pair. Then there exists a unique point $X_\lambda^x \in \tilde{X}$ such that*

$$f\left(X_\lambda^x\right) = X_\lambda^x = T\left(X_\lambda^x, X_\lambda^x, \ldots, X_\lambda^x\right).$$

**Remark 3.6.** If we take $f$ as identity function in Corollary 3.5, we find the soft metric version of Prešić's [28].

**Example 3.7.** Let $X$=$[0,1]$ with the finite set $A=\{0,1,2\}$ and mapping $d: SP\left(\tilde{X}\right)\times SP\left(\tilde{X}\right)\to\mathbb{R}(A)^*$ is the usual soft metric and the mapping $S,T$ and $f$ are given as

$$S\left(X_\lambda^x, X_\mu^y\right)=X_0^{\frac{3x+2y}{72}},$$

$$T\left(X_\lambda^x, X_\mu^y\right)=X_0^{\frac{2x+3y}{72}},$$

and

$$f\left(X_\lambda^x\right)=X_0^{\frac{x}{6}}.$$

We find that these mapping satisfies contractive conditions of Theorem 3.3 with $\bar{\alpha}=\bar{\frac{1}{3}}$ and

$$S(X_0^0,X_0^0)=T(X_0^0,X_0^0)=f(X_0^0)=X_0^0.$$

This shows that $X_0^0$ is the unique fixed point.

In next example, we shows that the condition of finiteness on $A$ cannot be omitted in the previous result.

**Example 3.8.** Let $X$=[0,1], $A=\{0,1,2\}$ and the mapping $d: SP\left(\tilde{X}\right)\times SP(\tilde{X})\to\mathbb{R}(A)^*$ is given by

$$d\left(X_\lambda^x, X_\mu^y\right)=|\bar{x}-\bar{y}|+|\bar{\lambda}-\bar{\mu}|,$$

for all $X_\lambda^x, X_\mu^y\in SP\left(\tilde{X}\right)$, where $|.|$ denotes the modulus of soft real number, is a the soft metric on $\tilde{X}$. Furthermore, soft metric space $\left(\tilde{X},d,A\right)$ is complete. Let $T:\tilde{X}^2\to\tilde{X}$ such that $T\left(X_\lambda^x, X_\mu^y\right)=X_0^{\frac{x+y}{4}}$ and $f(X_\lambda^x)=X_0^x$ for all $x,y\in X$ and $\lambda,\mu\in A$. Now for $\eta\in A$, we have

$$\begin{aligned}
d(T\left(X_\lambda^x, X_\mu^y\right),T(X_\mu^y,X_\gamma^x))(\eta) &= d\left(X_0^{\frac{x+y}{4}},X_0^{\frac{y+z}{4}}\right)(\eta)\\
&=\left|\frac{x+y}{4}-\frac{y+z}{4}\right|\\
&=\left|\frac{x-y}{4}+\frac{y-z}{4}\right|\\
&\leq\left|\frac{x-y}{4}\right|+\left|\frac{y-z}{4}\right|\\
&\leq\frac{1}{2}\max\left\{d\left(f\left(X_\lambda^x\right),f\left(X_\mu^y\right)\right),d\left(f(X_\mu^y),f(X_\gamma^z)\right)\right\}(\eta),
\end{aligned}$$

this shows that $T$ and $f$ satisfy the contraction **??** with $\bar{\alpha}=\bar{\frac{1}{2}}$ and $X_0^0$ is the only fixed point.

**Note:** The condition of finiteness of the set $A$ can not be removed. For example, Let $A = \{\frac{1}{n} | \ n \in \mathbb{N}\}$ be an infinite set and the mapping $T$ and $f$ are defined as in above example. Then the mapping satisfies the contraction in the Corollary 3.5 but doesn't have a fixed point.

**Remark 3.9.** The given Inequality (3.7) in the Corollary 3.5 can not be relaxed as illustrated in the following example.

**Example 3.10.** Let $X = [0,1] \cup [2,3]$, $A = \{0,1,2\}$ and mapping $d : SP\left(\tilde{X}\right) \times SP\left(\tilde{X}\right) \to \mathbb{R}(A)^*$ given by

$$d\left(X_\lambda^x, X_\mu^y\right) = |\bar{x} - \bar{y}| + \bar{\lambda} - \bar{\mu}|,$$

for all $X_\lambda^x, X_\mu^y \in SP\left(\tilde{X}\right)$, where $|.|$ denotes the modulus of soft real numbers, is a soft metric on $\tilde{X}$. Let $T : \tilde{X}^2 \tilde{\to} \tilde{X}$ such that

$$\begin{aligned} T\left(X_\lambda^x, X_\mu^y\right) &= X_0^{\frac{x+y}{4}}, && \left(X_\lambda^x, X_\mu^y\right) \in [0,1] \times [0,1], \\ &= X_0^{1+\frac{x+y}{4}}, && \left(X_\lambda^x, X_\mu^y\right) \in [2,3] \times [2,3], \\ &= X_0^{\frac{x+y}{4}-\frac{1}{2}}, && \left(X_\lambda^x, X_\mu^y\right) \in [0,1] \times [2,3], \\ &= X_0^{\frac{x+y}{4}-\frac{1}{2}}, && \left(X_\lambda^x, X_\mu^y\right) \in [2,3] \times [0,1], \end{aligned}$$

and $f\left(X_\lambda^x\right) = X_0^x$. Then for any $X_\lambda^x, X_\mu^y \in [0,1]$ we have $T(X_\lambda^x, X_\mu^y) = X_\gamma^z \in [0,1]$ and for $X_\lambda^x, X_\mu^y \in [2,3]$ we have $T(X_\lambda^x, X_\mu^y) = X_\gamma^z \in [2,3]$, we have

$$\begin{aligned} d\left(T\left(X_\lambda^x, X_\mu^y\right), T(X_\mu^y, X_\gamma^z)\right) &= \left|\frac{x+y}{4} - \frac{y+z}{4}\right| \\ &\tilde{\leq} \left|\frac{x-y}{4}\right| + \left|\frac{y-z}{4}\right| \\ &\tilde{\leq} \frac{1}{2}\max\left\{d\left(f\left(X_\lambda^x\right), f(X_\mu^y)\right), d\left(f\left(X_\mu^y\right), f(X_\gamma^z)\right)\right\}. \end{aligned}$$

For $(X_\lambda^x, X_\mu^y) \in [0,1] \times [2,3]$ or $(X_\lambda^x, X_\mu^y) \in [2,3] \times [0,1]$, we have $T(X_\lambda^x, X_\mu^y) = X_\gamma^z \in [0,1]$.

Therefore, if $X_\mu^y \in$ [2,3], then

$$\begin{aligned} d\left(T\left(X_\lambda^x, X_\mu^y\right), T(X_\mu^y, X_\gamma^z)\right) &= \left|\frac{x+y}{4} - \frac{y+z}{4}\right| \\ &\tilde{\leq} \frac{1}{2}\max\left\{d\left(f\left(X_\lambda^x\right), f\left(X_\mu^y\right)\right), d\left(f\left(X_\mu^y\right), f(X_\gamma^z)\right)\right\}. \end{aligned}$$

If $X_\mu^y \in [0,1]$,then

$$d\left(f\left(X_\lambda^x, X_\mu^y\right), f(X_\mu^y, X_\gamma^z)\right) = \left|\frac{x+y}{4} - \frac{1}{2} - \frac{y+z}{4}\right|$$

$$
\begin{aligned}
&= \left|\frac{x-y}{4} - \frac{1}{2} + \frac{y-z}{4}\right| \\
&\tilde{\leq} \left|\frac{x-y}{4} - \frac{1}{2}\right| + \left|\frac{y-z}{4}\right| \\
&\tilde{<} \left|\frac{x-y}{4}\right| + \left|\frac{y-z}{4}\right| \\
&\tilde{\leq} \frac{1}{2} \max\left\{d\left(f\left(X_\lambda^x\right), f\left(X_\mu^y\right)\right), d\left(f\left(X_\mu^y\right), f(X_\gamma^z)\right)\right\}.
\end{aligned}
$$

Thus, $T$ and $f$ satisfy Inequality (3.7) with $\bar{\alpha} = \bar{\frac{1}{2}}$ but we have $T(X_0^0, X_0^0) = f(X_0^0) = X_0^0$ and $T(X_0^2, X_0^2) = f(X_0^2) = X_0^2$. Therefore, the condition (3.8) in the Corollary 3.5 can't be relaxed.

**Theorem 3.11.** *Let $\left(\tilde{X}, d, A\right)$ be a complete soft metric space with finite set $A$, $k$ is a positive integer and $S, T : \tilde{X}^{2k} \tilde{\to} \tilde{X}$ be soft mappings satisfying the contractive condition*

$$
\begin{aligned}
(3.9) \qquad & d\left(S\left(X_{\lambda,1}^{x_1}, X_{\lambda,2}^{x_2}, \ldots, X_{\lambda,2k}^{x_{2k}}\right), T\left(X_{\lambda,2}^{x_2}, X_{\lambda,3}^{x_3}, \ldots, X_{\lambda,2k+1}^{x_{2k+1}}\right)\right) \\
& \tilde{\leq} \sum_{i=1}^{2k} \bar{\alpha}_i d\left(X_{\lambda,i}^{x_i}, X_{\lambda,i+1}^{x_{i+1}}\right),
\end{aligned}
$$

*for all $X_{\lambda,1}^{x_1}, X_{\lambda,2}^{x_2}, \ldots, X_{\lambda,2k+1}^{x_{2k+1}}$ in $\tilde{X}$,*

$$
\begin{aligned}
(3.10) \qquad & d\left(T\left(X_{\mu,1}^{y_1}, X_{\mu,2}^{y_2}, \ldots, X_{\mu,2k}^{y_{2k}}\right), S\left(X_{\mu,2}^{y_2}, X_{\mu,3}^{y_3}, \ldots, X_{\mu,2k+1}^{y_{2k+1}}\right)\right) \\
& \tilde{\leq} \sum_{i=1}^{2k} \bar{\alpha}_i d\left(X_{\mu,i}^{y_i}, X_{\mu,i+1}^{y_{i+1}}\right),
\end{aligned}
$$

*for all $X_{\mu,1}^{y_1}, X_{\mu,2}^{y_2}, \ldots, X_{\mu,2k+1}^{y_{2k+1}}$ in $\tilde{X}$, where $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{2k}$ are non-negative soft constants such that $\sum_{i=1}^{2k} \bar{\alpha}_i \tilde{<} \bar{1}$ and also suppose that*

$$
(3.11) \qquad d\left(S\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right), T(X_\gamma^z, X_\gamma^z, \ldots, X_\gamma^z\right) \tilde{<} d\left(X_\mu^y, X_\gamma^z\right).
$$

*Then there exits a unique soft point $X_\lambda^x \in \tilde{X}$ such that*

$$
T\left(X_\lambda^x, X_\lambda^x, \ldots, X_\lambda^x\right) = X_\lambda^x = S\left(X_\lambda^x, X_\lambda^x, \ldots, X_\lambda^x\right).
$$

*Proof.* Suppose that $X_{\lambda,1}^{x_1}, X_{\lambda,2}^{x_2}, \ldots, X_{\lambda,2k}^{x_{2k}}$ are arbitrary points in $\tilde{X}$ and for $n \in \mathbb{N}$, define

$$
X_{\lambda,2n+2k-1}^{x_{2n+2k-1}} = S\left(X_{\lambda,2n-1}^{x_{2n-1}}, X_{\lambda,2n}^{x_{2n}}, \ldots, X_{\lambda,2n+2k-2}^{x_{2n+2k-2}}\right),
$$

and

$$
X_{\lambda,2n+2k}^{x_{2n+2k}} = T\left(X_{\lambda,2n}^{x_{2n}}, X_{\lambda,2n+1}^{x_{2n+1}}, \ldots, X_{\lambda,2n+2k-1}^{x_{2n+2k-1}}\right).
$$

Further, we show that the sequence $\{X_{\lambda,n}^{x_n}\}$ is a Cauchy sequence.
For this we set

$$d_n = d\left(X_{\lambda,n}^{x_n}, X_{\lambda,n+1}^{x_{n+1}}\right).$$

We shall prove by induction that inequality

$$d_n \tilde{\leq} \bar{\mu}\bar{\theta}^n, \tag{3.12}$$

is true for $n \in \mathbb{N}$, where

$$\bar{\mu} = \max\left\{\frac{d_1}{\bar{\theta}^1}, \frac{d_2}{\bar{\theta}^2}, \dots, \frac{d_{2k}}{\bar{\theta}^{2k}}\right\}, \qquad \bar{\theta} = \left(\sum_{i=1}^{2k} \bar{\alpha}_i\right)^{\frac{1}{2k}}.$$

By the definition of $\bar{\mu}$ and $\bar{\theta}$ inequality (3.12) is obviously true for $n \in \{1, 2, \dots, 2k\}$. Suppose that the following $(2k-1)$ inequalities

$$\begin{aligned}
&d_{2n} \tilde{\leq} \bar{\mu}\bar{\theta}^{2n},\\
&d_{2n+1} \tilde{\leq} \bar{\mu}\bar{\theta}^{2n+1},\\
&\quad\vdots\\
&d_{2n+2k-2} \tilde{\leq} \bar{\mu}\bar{\theta}^{2n+2k-2},
\end{aligned}$$

holds. Then

$$\begin{aligned}
d_{2n+2k-1} &= d\left(X_{\lambda,2n+2k-1}^{x_{2n+2k-1}}, X_{\lambda,2n+2k}^{x_{2n+2k}}\right)\\
&= d\left(S\left(X_{\lambda,2n-1}^{x_{2n-1}}, X_{\lambda,2n}^{x_{2n}}, \dots, X_{\lambda,2n+2k-2}^{x_{2n+2k-2}}\right)\right.\\
&\qquad \left., T\left(X_{\lambda,2n}^{x_{2n}}, X_{\lambda,2n+1}^{x_{2n+1}}, \dots, X_{\lambda,2n+2k-1}^{x_{2n+2k-1}}\right)\right)\\
&\tilde{\leq} \bar{\alpha}_1 d\left(X_{\lambda,2n-1}^{x_{2n-1}}, X_{\lambda,2n}^{x_{2n}}\right) + \bar{\alpha}_2 d\left(X_{\lambda,2n}^{x_{2n}}, X_{\lambda,2n+1}^{x_{2n+1}}\right)\\
&\qquad + \dots + \bar{\alpha}_{2k} d\left(X_{\lambda,2n+2k-2}^{x_{2n+2k-2}}, X_{\lambda,2n+2k-1}^{x_{2n+2k-1}}\right)\\
&= \bar{\alpha}_1 d_{2n-1} + \bar{\alpha}_2 d_{2n} + \dots + \bar{\alpha}_{2k} d_{2n+2k-2}\\
&\tilde{\leq} \bar{\alpha}_1 \bar{\mu}\bar{\theta}^{2n-1} + \bar{\alpha}_2 \bar{\mu}\bar{\theta}^{2n} + \dots + \bar{\alpha}_{2k} \bar{\mu}\bar{\theta}^{2n+2k-2}\\
&= \{\bar{\alpha}_1 + \bar{\alpha}_2 + \dots + \bar{\alpha}_{2k}\}\bar{\mu}\bar{\theta}^{2n-1}\\
&\tilde{\leq} \bar{\mu}\bar{\theta}^{2n+2k-1}.
\end{aligned}$$

Hence by the induction hypothesis the inequality (3.12) is true for each $n \in \mathbb{N}$
Now, for any $n, p \in \mathbb{N}$, we have

$$\begin{aligned}
d\left(X_{\lambda,n}^{x_n}, X_{\lambda,n+p}^{x_{n+p}}\right) &\tilde{\leq} d\left(X_{\lambda,n}^{x_n}, X_{\lambda,n+1}^{x_{n+1}}\right) + d\left(X_{\lambda,n+1}^{x_{n+1}}, X_{\lambda,n+2}^{x_{n+2}}\right)\\
&\qquad + \dots + d\left(X_{\lambda,n+p-1}^{x_{n+p-1}}, X_{\lambda,n+p}^{x_{n+p}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= d_n + d_{n+1} + \cdots + d_{n+p-1} \\
&\tilde{\leq} \bar{\mu}\bar{\theta}^n + \bar{\mu}\bar{\theta}^{n+1} + \cdots + \bar{\mu}\bar{\theta}^{n+p-1} \\
&= \bar{\mu}\bar{\theta}^n\{1 + \bar{\theta} + \cdots + \bar{\theta}^{p-1}\} \\
&\tilde{<} \frac{\bar{\mu}\bar{\theta}^n}{1-\bar{\theta}},
\end{aligned}
$$

as $\bar{\theta} < \bar{1}$, we conclude that the sequence $\{X_{\lambda,n}^{x_n}\}$ is a Cauchy sequence. Since $\tilde{X}$ is complete, then there exists $X_\lambda^x$ in $\tilde{X}$ such that

$$X_\lambda^x = \lim_{n\to\infty} X_{\lambda,n}^{x_n}.$$

Then for any integer $n \in \mathbb{N}$, we have

$$
\begin{aligned}
&d\left(S\left(X_\lambda^x, X_\lambda^x, \ldots, X_\lambda^x\right), X_{\lambda,2n+2k-1}^{x_{2n+2k-1}}\right) \\
&\quad = d\left(S\left(X_\lambda^x, X_\lambda^x, \ldots, X_\lambda^x\right), S\left(X_{\lambda,2n-1}^{x_{2n-1}}, X_{\lambda,2n}^{x_{2n}}, \ldots, X_{\lambda,2n+2k-2}^{x_{2n+2k-2}}\right)\right) \\
&\quad \tilde{\leq} d\left(S\left(X_\lambda^x, X_\lambda^x, \ldots, X_\lambda^x\right), T\left(X_\lambda^x, X_\lambda^x, \ldots, X_{\lambda,2n-1}^{x_{2n-1}}\right)\right) \\
&\qquad + d\left(T\left(X_\lambda^x, X_\lambda^x, \ldots, X_{\lambda,2n-1}^{x_{2n-1}}\right), S\left(X_\lambda^x, X_\lambda^x, \ldots, X_{\lambda,2n-1}^{x_{2n-1}}, X_{\lambda,2n}^{x_{2n}}\right)\right) \\
&\qquad + d\left(S\left(X_\lambda^x, X_\lambda^x, \ldots, X_{\lambda,2n}^{x_{2n}}\right), T\left(X_\lambda^x, X_\lambda^x, \ldots, X_{\lambda,2n}^{x_{2n}}, X_{\lambda,2n+1}^{x_{2n+1}}\right)\right) \\
&\qquad + \cdots + d\left(S\left(X_\lambda^x, X_\lambda^x, \ldots, X_{\lambda,2n+2k-4}^{x_{2n+2k-4}}\right),\right. \\
&\qquad \left. T\left(X_\lambda^x, X_{\lambda,2n-1}^{x_{2n-1}}, \ldots, X_{\lambda,2n+2k-3}^{x_{2n+2k-3}}\right)\right) \\
&\qquad + d\left(T\left(X_\lambda^x, X_{\lambda,2n-1}^{x_{2n-1}}, \ldots, X_{\lambda,2n+2k-3}^{x_{2n+2k-3}}\right),\right. \\
&\qquad \left. S\left(X_{\lambda,2n-1}^{x_{2n-1}}, X_{\lambda,2n}^{x_{2n}}, \ldots, X_{\lambda,2n+2k-2}^{x_{2n+2k-2}}\right)\right) \\
&\quad \tilde{\leq} \bar{\alpha}_{2k} d\left(X_\lambda^x, X_{\lambda,2n-1}^{x_{2n-1}}\right) \\
&\qquad + \bar{\alpha}_{2k-1} d\left(X_\lambda^x, X_{\lambda,2n-1}^{x_{2n-1}}\right) + \bar{\alpha}_{2k} d\left(X_{\lambda,2n-1}^{x_{2n-1}}, X_{\lambda,2n}^{x_{2n}}\right) \\
&\qquad + \cdots + \bar{\alpha}_2 d\left(X_\lambda^x, X_{\lambda,2n-1}^{x_{2n-1}}\right) + \bar{\alpha}_3 d\left(X_{\lambda,2n-1}^{x_{2n-1}}, X_{\lambda,2n}^{x_{2n}}\right) \\
&\qquad + \cdots + \bar{\alpha}_{2k} d\left(X_{\lambda,n+k-4}^{x_{n+k-4}}, X_{\lambda,n+k-3}^{x_{n+k-3}}\right) \\
&\qquad + \bar{\alpha}_1 d\left(X_\lambda^x, X_{\lambda,2n-1}^{x_{2n-1}}\right) + \bar{\alpha}_2 d\left(X_{\lambda,2n-1}^{x_{2n-1}}, X_{\lambda,2n}^{x_{2n}}\right) \\
&\qquad + \cdots + \bar{\alpha}_{2k} d\left(X_{\lambda,2n+2k-3}^{x_{2n+2k-3}}, X_{\lambda,2n+2k-2}^{x_{2n+2k-2}}\right).
\end{aligned}
$$

As $n \to \infty$, we get

$$S\left(X_\lambda^x, X_\lambda^x, \ldots, X_\lambda^x\right) = X_\lambda^x.$$

Consider

$$\begin{aligned} d\left(X_\lambda^x, T\left(X_\lambda^x, X_\lambda^x, \ldots, X_\lambda^x\right)\right) &= d\left(S(X_\lambda^x, X_\lambda^x, \ldots, X_\lambda^x), T\left(X_\lambda^x, X_\lambda^x, \ldots, X_\lambda^x\right)\right) \\ &\tilde{\leq} \bar{\alpha}(0) \\ &= 0. \end{aligned}$$

Thus

$$T\left(X_\lambda^x, X_\lambda^x, \ldots, X_\lambda^x\right) = X_\lambda^x,$$

this implies that

$$T\left(X_\lambda^x, X_\lambda^x, \ldots, X_\lambda^x\right) = X_\lambda^x = S\left(X_\lambda^x, X_\lambda^x, \ldots, X_\lambda^x\right).$$

Suppose that there exists a soft point $X_\mu^y \neq X_\lambda^x \in \tilde{X}$ such that

$$T\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right) = X_\mu^y = S\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right).$$

Consider

$$\begin{aligned} d\left(X_\mu^y, X_\lambda^x\right) &= d\left(T\left(X_\mu^y, X_\mu^y, \ldots, X_\mu^y\right), S\left(X_\lambda^x, X_\lambda^x, \ldots, X_\lambda^x\right)\right) \\ &< d\left(X_\mu^y, X_\lambda^x\right). \end{aligned}$$

It is a contraction. Therefore $X_\mu^y = X_\lambda^x$. □

**Example 3.12.** Let $X$=[0,1], $A$=$\{0,1,2\}$ and $d$ is the usual soft metric. Define mappings

$$S\left(X_\lambda^x, X_\mu^y\right) = X_0^{\frac{x+y}{4}}, \qquad \left(X_\lambda^x, X_\mu^y\right) = X_0^{\frac{x^2+y^2}{5}}.$$

Note that $S$ and $T$ satisfy the conditions of Theorem 3.11 with $\bar{\alpha}_1 = \frac{\bar{1}}{4}$, $\bar{\alpha}_2 = \frac{\bar{1}}{3}$ and $S(X_0^0, X_0^0) = X_0^0 = T(X_0^0, X_0^0)$ that is $X_0^0$ is the unique fixed point.

**Example 3.13.** Let $X$=[0,1], $A$=$\{0,1,2\}$ and $d$ is the usual soft metric. Define mappings

$$S\left(X_\lambda^x, X_\mu^y\right) = X_0^{\frac{3x^2+2y^2}{48}}, \qquad T(X_\lambda^x, X_\mu^y) = X_0^{\frac{2x^2+3y^2}{48}},$$

satisfy the contractive conditions of Theorem 3.11 with $\bar{\alpha}_1 = \frac{\bar{1}}{8}$, $\bar{\alpha}_2 = \frac{\bar{1}}{7}$.

**Corollary 3.14.** Let $\left(\tilde{X}, d, A\right)$ be a complete soft metric space with finite set $A$, $k$ is a positive integer and $S, T : \tilde{X}^{2k} \tilde{\to} \tilde{X}$ be soft mappings satisfy the contractive condition

$$\begin{aligned} &d\left(S\left(X_{\lambda,1}^{x_1}, X_{\lambda,2}^{x_2}, \ldots, X_{\lambda,2k}^{x_{2k}}\right), T\left(X_{\lambda,2}^{x_2}, X_{\lambda,3}^{x_3}, \ldots, X_{\lambda,2k+1}^{x_{2k+1}}\right)\right) \\ &\quad \tilde{\leq} \bar{\alpha} \max\left\{d\left(X_{\lambda,i}^{x_i}, X_{\lambda,i+1}^{x_{i+1}}\right)\right\}, \end{aligned}$$

$1 \leq i \leq 2k$ for all $X^{x_1}_{\lambda,1}, X^{x_2}_{\lambda,2}, \ldots, X^{x_{2k+1}}_{\lambda,2k+1}$ in $\tilde{X}$,

$$d(T\left(X^{y_1}_{\mu,1}, X^{y_2}_{\mu,2}, \ldots, X^{y_{2k}}_{\mu,2k}\right), S\left(X^{y_2}_{\mu,2}, X^{y_3}_{\mu,3}, \ldots, X^{y_{2k+1}}_{\mu,2k+1}\right)$$
$$\tilde{\leq} \bar{\alpha} \max\left\{d\left(X^{y_i}_{\mu,i}, X^{y_{i+1}}_{\mu,i+1}\right)\right\},$$

$1 \leq i \leq 2k$ for all $X^{y_1}_{\mu,1}, X^{y_2}_{\mu,2}, \ldots, X^{y_{2k+1}}_{\mu,2k+1}$ in $\tilde{X}$ where $0 \leq \bar{\alpha} < 1$.

$$d\left(S\left(X^y_\mu, X^y_\mu, \ldots, X^y_\mu\right), T\left(X^z_\gamma, X^z_\gamma, \ldots, X^z_\gamma\right)\right) \tilde{<} d\left(X^y_\mu, X^z_\gamma\right), \tag{3.13}$$

for all $X^y_\mu, X^z_\gamma \in \tilde{X}$ with $X^y_\mu \neq X^z_\gamma$. Then there exists a unique soft point $X^y_\mu \in \tilde{X}$ such that

$$T\left(X^y_\mu, X^y_\mu, \ldots, X^y_\mu\right) = X^y_\mu = S\left(X^y_\mu, X^y_\mu, \ldots, X^y_\mu\right).$$

**Example 3.15.** Let $X{=}[0,1]$ with finite set $A{=}\{0,1,2\}$ and mapping $d : SP\left(\tilde{X}\right) \times SP\left(\tilde{X}\right) \to \mathbb{R}(A)^*$ and the mapping $S$ and $T$ are given as

$$d\left(X^x_\lambda, X^y_\mu\right) = \begin{cases} 0, & \text{if } x = y = 0, \lambda = \mu = 0, \\ \max\{x, y\}, & \text{otherwise.} \end{cases}$$

And $S$ and $T$ are given by $S(X^x_\lambda, X^y_\mu) = X_0^{\frac{x^2+y}{3}}$ and $T\left(X^x_\lambda, X^y_\mu\right) = X_0^{\frac{2x+y^3}{2}}$. Note that $S$ and $T$ satisfy the contractive condition of Corollary 3.14, where $\bar{\alpha} = \bar{\frac{1}{4}}$ and $S(X^0_0, X^0_0) = X^0_0 = T(X^0_0, X^0_0)$ that is $X^0_0$ is the unique fixed point.

In the next section, we give an application of the soft Banach principle in iterated soft function systems.

## 4. Applications

Let $H\left(\tilde{X}\right) = \{M \subset \tilde{X} : M$ is a non-empty closed and bounded subset of soft points of $\tilde{X}\}$ and $C\left(SP\left(\tilde{X}\right)\right)$ is the set of compact subsets of soft points of $\tilde{X}$. Let $\tilde{X}$ be a complete soft metric space with metric $d$ and suppose that $C\left(SP\left(\tilde{X}\right)\right) \subset H\left(\tilde{X}\right)$ be the soft metric space of non-empty soft compact subsets of soft points of $\tilde{X}$. Metric given by

$$h(M, N) = \sup\left\{d\left(X^x_\lambda, N\right), d\left(X^y_\mu, M\right), X^x_\lambda \in M, X^y_\mu \in N\right\},$$

is Hausdorff soft metric. Let $M \neq \phi$ be the closed subset of soft points of $\tilde{X}$. Then for $\bar{\delta} \tilde{>} \bar{0}$

$$M_{\bar{\delta}} = \left\{X^x_\lambda \in \tilde{X} : d\left(X^x_\lambda, X^y_\mu\right) \tilde{\leq} \bar{\delta}, X^y_\mu \in M\right\}.$$

Let $(\tilde{X}, d)$ be a complete soft metric space and $A$ is a countable finite set of parameters and

$$f_i : \tilde{X} \to \tilde{X},$$

where $i = 1, 2, \ldots, k$ be $k$ soft mappings, which are soft Lipschitz continuous with soft Lipschitz constants $\bar{L_1}, \bar{L_2}, \ldots, \bar{L_k}$ *i.e.*

$$d\left(f_i\left(X_\lambda^x\right), f_i\left(X_\mu^y\right)\right) \tilde{\leq} \bar{L}_i d\left(X_\lambda^x, X_\mu^y\right),$$

for $i = 1, 2, \ldots, k$ and $X_\lambda^x, X_\mu^y \in \tilde{X}$. Define

$$F : C\left(SP\left(\tilde{X}\right)\right) \to C\left(SP\left(\tilde{X}\right)\right),$$

by

$$F(M) = f_1(M) \cup f_2(M) \cup \cdots \cup f_k(M), \quad M \in C\left(SP\left(\tilde{X}\right)\right),$$

then $F$ satisfies soft Lipschitz condition with respect to the Hausdorff soft metric with soft Lipschitz constant

$$\bar{L} = \max\left\{\bar{L}_1, \bar{L}_2, \ldots, \bar{L}_k\right\},$$

*i.e.*

$$h\left(F(M), F(N)\right) \tilde{\leq} \bar{L} h\left(M, N\right).$$

Particularly, if $f_i = 1, 2, \ldots k$ are soft contractions on $\tilde{X}$, then $F$ is a soft contraction on $C\left(SP\left(\tilde{X}\right)\right)$ with respect to the Hausdorff soft metric. $F$ has unique fixed point $M \in C\left(SP\left(\tilde{X}\right)\right)$.
For this, define the Hausdorff soft metric as

$$h(M, N) = \sup\left\{d\left(X_\lambda^x, N\right), d\left(X_\mu^y, M\right), X_\lambda^x \in M, X_\mu^y \in N\right\}.$$

Suppose that $M_1$, $M_2$, $N_1$, $N_2$ are soft compact subsets of $\tilde{X}$ and $M = M_1 \cup M_2$, $N = N_1 \cup N_2$. Now, we want to show that

$$h\left(M, N\right) \tilde{\leq} \max_{i=1,2}\left\{h\left(M_i, N_i\right)\right\} = \bar{m}.$$

Let $X_\lambda^x \in M$, $X_\mu^y \in N$ and prove

$$d\left(X_\lambda^x, N\right), d\left(X_\mu^y, M\right) \tilde{\leq} \bar{m}.$$

Now

$$\begin{aligned} d\left(X_\lambda^x, N\right) &= d\left(X_\lambda^x, N_1 \cup N_2\right) \\ &= \min\left\{d\left(X_\lambda^x, N_i\right), i = 1, 2\right\} \\ &\tilde{\leq} d\left(X_\lambda^x, N_i\right) \\ &\tilde{\leq} h\left(M_i, N_i\right) \\ &\tilde{\leq} \bar{m}, \end{aligned}$$

and also $d\left(X_\mu^y, M\right)\tilde{\leq}\bar{m}$. Let $M, N \in C\left(SP\left(\tilde{X}\right)\right)$ and $h\left(M, N\right) = \bar{\epsilon}$. Then

$$M \subset N_{\bar{\epsilon}}, N \subset M_{\bar{\epsilon}},$$

therefore, we have

$$f_1(N) \subset f_1(M_{\bar{\epsilon}}), f_2(N) \subset f_2(M_{\bar{\epsilon}}),$$
$$f_1(M) \subset f_1(N_{\bar{\epsilon}}), f_2(M) \subset f_2(N_{\bar{\epsilon}}).$$

Also

$$f_i(M_{\bar{\epsilon}}) \subset (f_i(M))_{\bar{L}_i\bar{\epsilon}}, \quad i = 1, 2,$$
$$f_i(N_{\bar{\epsilon}}) \subset (f_i(N))_{\bar{L}_i\bar{\epsilon}}, \quad i = 1, 2,$$

and further

$$f_i(M) \subset (f_1(N) \cup f_2(N))_{\bar{L}\bar{\epsilon}}, \quad i = 1, 2,$$
$$f_i(N) \subset (f_1(M) \cup f_2(M))_{\bar{L}\bar{\epsilon}}, \quad i = 1, 2.$$

By definition of the Hausdorff soft metric, we have

$$h(F(M), F(N))\tilde{\leq}\bar{L}\bar{\epsilon} = \bar{L}h(M, N).$$

$F$ meets soft Lipschitz condition with the Hausdorff soft metric, then $F$ has an unique fixed point, according to Theorem 3.3.

**Remark 4.1.** If $f_1, f_2, \ldots, f_k$ are soft contraction mappings, and the soft function

$$F(M) = f_1(M) \cup f_2(M) \cup \cdots \cup f_k(M), \quad M \in C\left(SP\left(\tilde{X}\right)\right),$$

is called Hutchinson soft operator and the system

$$M_{i+1} = F(M_i),$$

is an iterated soft function system, then this system, by soft Banach contraction principle, has a unique limit.

## 5. Conclusion

In this paper, we generalized Banach contraction principle for Prešić's type contraction mappings in the setting of soft metric spaces. The fact has been substantially by furnishing with example's. We also explained that restriction on a certain parameter set is necessary and an application of soft version of BPC in iterated soft function systems is established. Further some generalization for more mappings or other metric type like b-metric, s-metric, bi-polar metric etc. can be investigated. Also the application of established results are possible in decision making problems. We hope that the results examined in this paper will contribute significantly and scientifically soundly to the field, and will help researchers to further advance their theory work in the soft metric space field and to expand it.

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