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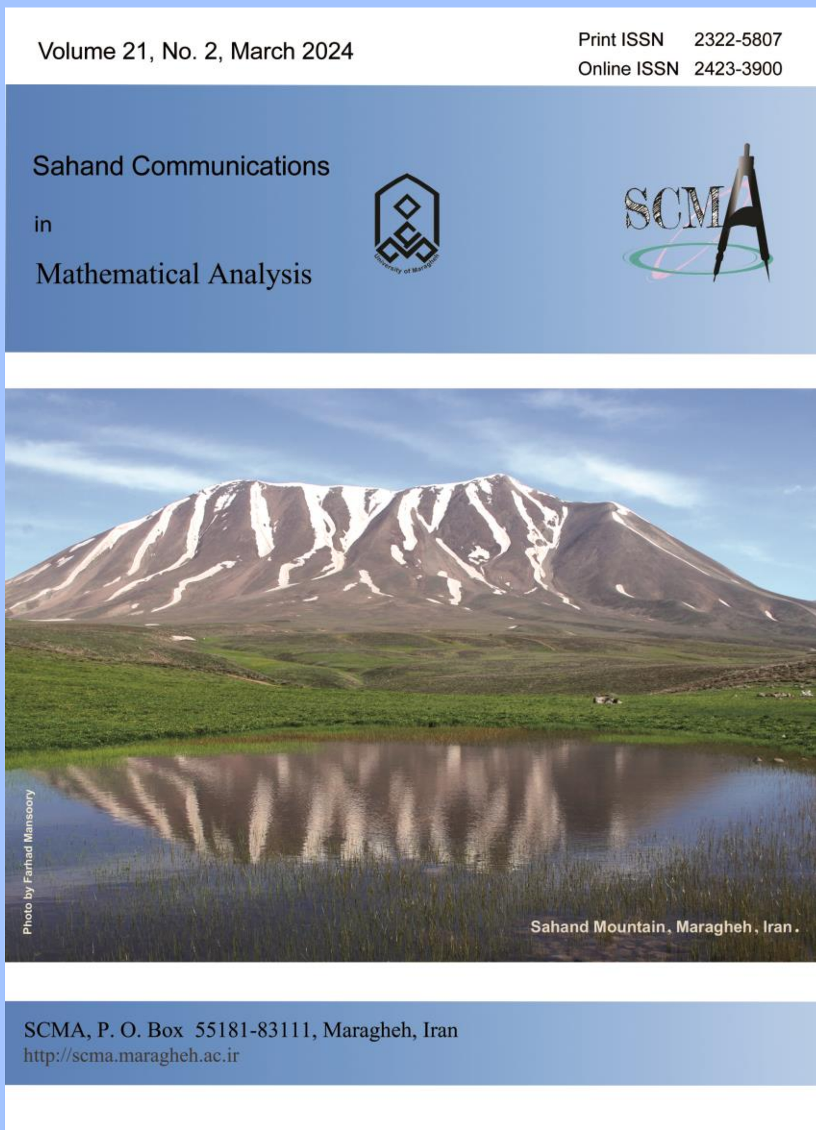

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Exponential Convex Functions with Respect to s

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Abstract. In this paper, we study the concept of exponential convex functions with respect to s and prove Hermite-Hadamard type inequalities for the newly introduced this class of functions. In addition, we get some refinements of the Hermite-Hadamard (H-H) inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is exponential convex with respect to s. Our results coincide with the results obtained previously in special cases.

1. Introduction

The aim of this paper is to give the concept of exponential convex function with respect to s and examine some results connected with new Hermite-hadamard integral inequalities for these type of functions. Especially, it is thought that some important definitions and concepts such as convex function, Hermite-Hadamard integral inequality and h-convex functions are known by the readers. In addition, readers can refer to the articles [1, 4–12, 14–22] and the references therein for more detailed information on both these topics and the different classes of convexity.

In recent years, in [13], Kadakal and İşcan gave and studied the concept of exponential type convex functions and some of their algebraic properties. The authors prove two Hermite-Hadamard type integral inequalities for the newly introduced class of functions and also obtain

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some refinements of the Hermite-Hadamard integral inequality for functions whose first derivative in absolute value at certain power is exponential type convex. In [14], the authors introduced the concept of exponential trigonometric convex functions and obtained Hermite-Hadamard type inequalities for the newly introduced class of functions. The authors show that the result obtained with Hölder-İşcan and improved power-mean integral inequalities give better approximations than that obtained with Hölder and improved power-mean integral inequalities. In [2], the authors investigate the idea and its algebraic properties of n-polynomial exponential type p-convex function. The authors prove new trapezium type inequality for this new class of functions. Some refinements of the trapezium type inequality for functions whose first derivative in absolute value at certain power are n-polynomial exponential type p-convex are also obtained. At the end, some new bounds for special means of different positive real numbers are provided as well. These new results yield us some generalizations of the prior results.

2. Main Results

Definition 2.1. Let $I \subset \mathbb{R}$ be an interval. A non-negative function $\check{Z} : I \subset \mathbb{R} \to \mathbb{R}$ is called exponential convex function with respect to s if for every $\delta_1, \delta_2 \in I$, $\omega \in [0,1]$, and $s > 0$

$$\check{Z}\left(\omega\delta_1 + (1-\omega)\delta_2\right) \leq \omega s^{1-\omega}\check{Z}(\delta_1) + (1-\omega)\, s^{\omega}\check{Z}(\delta_2).$$

Denote by $EXPC\,(I, s)$ the class of all exponential convex with respect to s on interval I.

Proposition 2.2. *Every nonnegative exponential convex with respect to s is a convex for $s \leq 1$.*

Proof. If $s \leq 1$ then $s^{1-\omega} \leq 1$ then $\omega s^{1-\omega} \leq \omega$ and similarly $(1-\omega)\, s^{\omega} \leq 1-\omega$. Thus

$$\begin{aligned}\check{Z}\left(\omega\delta_1 + (1-\omega)\delta_2\right) &\leq \omega s^{1-\omega}\check{Z}(\delta_1) + (1-\omega)\, s^{\omega}\check{Z}(\delta_2) \\ &\leq \omega\check{Z}(\delta_1) + (1-\omega)\,\check{Z}(\delta_2).\end{aligned}$$

Therefore, every nonnegative exponential convex with respect to s is convex. ☐

Example 2.3. Let $c < 0$ be any fixed point. The function $\check{Z} : I \subset \mathbb{R} \to \mathbb{R}$, $\check{Z}(x) = c$ is an exponential convex with respect to s for $s \leq 1$. Since $\omega s^{1-\omega} + (1-\omega)\, s^{\omega} \leq \omega + (1-\omega) = 1$, we have

$$\begin{aligned}\check{Z}\left(\omega\delta_1 + (1-\omega)\delta_2\right) &= c \leq c\left[\omega s^{1-\omega} + (1-\omega)\, s^{\omega}\right] \\ &= \omega s^{1-\omega}\check{Z}(\delta_1) + (1-\omega)\, s^{\omega}\check{Z}(\delta_2)\end{aligned}$$

for all $\delta_1, \delta_2 \in I$, $\omega \in [0,1]$.

Proposition 2.4. *Every nonnegative convex is exponential convex with respect to s for $s \geq 1$.*

Proof. If $s \geq 1$ then $s^{1-\omega} \geq 1$ then $\omega s^{1-\omega} \geq \omega$ and similarly $(1-\omega)\, s^{1-\omega} \geq 1-\omega$, we obtain

$$\begin{aligned}&\check{Z}\left(\omega\delta_1 + (1-\omega)\delta_2\right)\\ &\quad \leq \omega\check{Z}(\delta_1) + (1-\omega)\check{Z}(\delta_2) \leq \omega s^{1-\omega}\check{Z}(\delta_1) + (1-\omega)\, s^{\omega}\check{Z}(\delta_1).\end{aligned}$$ □

Example 2.5. According to the above Proposition, $\check{Z} : [0,\infty) \to \mathbb{R}$, $\check{Z}(x) = x^n, n \geq 1$, is an exponential convex with respect to s for $s \geq 1$.

Definition 2.6 ([22])**.** Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX\,(h, I)$, if f is non-negative and for all $x, y \in I$, $\alpha \in (0, 1)$ we have

$$f\left(\alpha x + (1-\alpha)y\right) \leq h(\alpha)f\left(x\right) + h(1-\alpha)f\left(y\right).$$

If this inequality is reversed, then f is said to be h-concave, i.e. $f \in SV\,(h, I)$.

Remark 2.7. Let $s > 0$ be any fixed point. Then exponential convex with respect to s is a h-convex with $h(z) = zs^{1-z}$.

Theorem 2.8. *Let $\check{Z}_1, \check{Z}_2 : I \subset \mathbb{R} \to \mathbb{R}$. If $\check{Z}_1$ and $\check{Z}_2$ are exponential convex with respect to s, then*

(i) *$\check{Z}_1 + \check{Z}_2$ is exponential convex with respect to s,*
(ii) *For $c \in \mathbb{R}$ ($c \geq 0$) $c\check{Z}$ is exponential convex with respect to s.*

Proof. The proof is clear. □

Theorem 2.9. *If $\check{Z}_1 : I \to J$ is a convex and $\check{Z}_2 : J \to \mathbb{R}$ is an exponential convex function with respect to s and nondecreasing, then $\check{Z}_2 \circ \check{Z}_1 : I \to \mathbb{R}$ is an exponential convex function with respect to s.*

Proof. For $\delta_1, \delta_2 \in I$ and $\omega \in [0,1]$, we get

$$\begin{aligned}\left(\check{Z}_2 \circ \check{Z}_1\right)\left(\omega\delta_1 + (1-\omega)\delta_2\right) &= \check{Z}_2\left(\check{Z}_1\left(\omega\delta_1 + (1-\omega)\delta_2\right)\right)\\ &\leq \check{Z}_2\left(\omega\check{Z}_1(\delta_1) + (1-\omega)\check{Z}_1(\delta_2)\right)\\ &\leq \omega s^{1-\omega}\check{Z}_2\left(\check{Z}_1(\delta_1)\right) + (1-\omega)\, s^{\omega}\check{Z}_2\left(\check{Z}_1(\delta_2)\right).\end{aligned}$$ □

Theorem 2.10. *Let $\check{Z}_1, \check{Z}_2 : I \to \mathbb{R}$ are both nonnegative and monotone increasing. If $\check{Z}_1$ and $\check{Z}_2$ are exponential convex with respect to $s \leq 1$, then $\check{Z}_1\check{Z}_2$ is an exponential convex with respect to s.*

Proof. If $\delta_1 \leq \delta_2$ (the case $\delta_2 \leq \delta_1$is similar) then

$$\left[\check{Z}_1(\delta_1) - \check{Z}_1(\delta_2)\right]\left[\check{Z}_2(\delta_2) - \check{Z}_2(\delta_1)\right] \leq 0$$

which implies

$$\check{Z}_1(\delta_1)\check{Z}_2(\delta_2)+\check{Z}_1(\delta_2)\check{Z}_2(\delta_1)\leq \check{Z}_1(\delta_1)\check{Z}_2(\delta_1)+\check{Z}_1(\delta_2)\check{Z}_2(\delta_2). \tag{2.1}$$

On the other hand for $\delta_1, \delta_2 \in I$ and $\omega \in [0,1]$,

$$\begin{aligned}
&\left(\check{Z}_1\check{Z}_2\right)(\omega\delta_1+(1-\omega)\delta_2)\\
&\quad=\check{Z}_1(\omega\delta_1+(1-\omega)\delta_2)\,\check{Z}_2(\omega\delta_1+(1-\omega)\delta_2)\\
&\quad\leq \left[\omega s^{1-\omega}\check{Z}_1(\delta_1)+(1-\omega)\,s^{\omega}\check{Z}_1(\delta_2)\right]\left[\omega s^{1-\omega}\check{Z}_2(\delta_1)+(1-\omega)\,s^{\omega}\check{Z}_2(\delta_2\right]\\
&\quad=\omega^2 s^{2-2\omega}\check{Z}_1(\delta_1)\check{Z}_2(\delta_1)+\omega(1-\omega)s\left[\check{Z}_1(\delta_1)\check{Z}_2(\delta_2)+\check{Z}_1(\delta_2)\check{Z}_2(\delta_1)\right]\\
&\qquad+(1-\omega)^2 s^{2\omega}\check{Z}_1(\delta_2)\check{Z}_2(\delta_2).
\end{aligned}$$

Using now (2.1), we obtain,

$$\begin{aligned}
&\left(\check{Z}_1\check{Z}_2\right)(\omega\delta_1+(1-\omega)\delta_2)\\
&\quad\leq \omega s^{1-\omega}\left[\omega s^{1-\omega}+(1-\omega)s^{\omega}\right]\check{Z}_1(\delta_1)\check{Z}_2(\delta_2)\\
&\qquad+(1-\omega)\,s^{\omega}\left[\omega s^{1-\omega}+(1-\omega)s^{\omega}\right]\check{Z}_1(\delta_2)\check{Z}_2(\delta_2).
\end{aligned}$$

Since $\omega s^{1-\omega}+(1-\omega)s^{\omega}\leq 1$, we get

$$\begin{aligned}
&\left(\check{Z}_1\check{Z}_2\right)(\omega\delta_1+(1-\omega)\delta_2)\\
&\quad\leq \omega s^{1-\omega}\check{Z}_1(\delta_1)\check{Z}_2(\delta_1)+(1-\omega)\,s^{\omega}\check{Z}_1(\delta_2)\check{Z}_2(\delta_2)\\
&\quad=\omega s^{1-\omega}\left(\check{Z}_1\check{Z}_2\right)(\delta_1)+(1-\omega)\,s^{\omega}\left(\check{Z}_1\check{Z}_2\right)(\delta_2).
\end{aligned}$$

□

Theorem 2.11. *Let $\delta_2>0$ and $\check{Z}_\alpha:[\delta_1,\delta_2]\to\mathbb{R}$ be an arbitrary family of exponential convex functions with respect to s and let $\check{Z}(\delta)=\sup_\alpha \check{Z}_\alpha(\delta)$. If $J=\left\{v\in[\delta_1,\delta_2]:\check{Z}(v)<\infty\right\}$ is nonempty, then J is an interval and $\check{Z}$ is an exponential convex function with respect to s on J.*

Proof. Let $\omega\in[0,1]$ and $\delta_1,\delta_2\in J$ be arbitrary. Then

$$\begin{aligned}
\check{Z}(\omega\delta_1+(1-\omega)\delta_2) &\leq \sup_\alpha\left[\omega s^{1-\omega}\check{Z}_\alpha(\delta_1)+(1-\omega)\,s^{\omega}\check{Z}_\alpha(\delta_2)\right]\\
&\leq \omega s^{1-\omega}\sup_\alpha\check{Z}_\alpha(\delta_1)+(1-\omega)\,s^{\omega}\sup_\alpha\check{Z}_\alpha(\delta_2)\\
&=\omega s^{1-\omega}\check{Z}(\delta_1)+(1-\omega)\,s^{\omega}\check{Z}(\delta_2)\\
&<\infty.
\end{aligned}$$

This shows simultaneously that J is an interval, because it contains every point between any two of its points and that the function $\check{Z}$ is an exponential convex with respect to s on J. □

3. Hermite-Hadamard Inequality For Exponential Convex with Respect to s

The goal of this paper is to establish some inequalities of Hermite-Hadamard integral inequalities for exponential convex functions with respect to s. In this section, we will denote by $L[\delta_1, \delta_2]$ the space of (Lebesgue) integrable functions on $[\delta_1, \delta_2]$.

Theorem 3.1. *Let $f : [\delta_1, \delta_2] \to \mathbb{R}$ be an exponential convex with respect to $s > 0$. If $\delta_1 \leq \delta_2$ and $\check{Z} \in L[\delta_1, \delta_2]$, then the following H-H type integral inequalities hold:*

$$\begin{aligned} \frac{1}{\sqrt{s}} \check{Z}\left(\frac{\delta_1+\delta_2}{2}\right) &\leq \frac{1}{\delta_2-\delta_1} \int_{\delta_1}^{\delta_2} \check{Z}(z)dz \\ &\leq -\frac{\ln s - s + 1}{\ln^2 s}\left[\check{Z}(\delta_1) + \check{Z}(\delta_2)\right] \end{aligned} \tag{3.1}$$

for all $s > 1$, and

$$\begin{aligned} \check{Z}\left(\frac{\delta_1+\delta_2}{2}\right) &\leq \frac{1}{\delta_2-\delta_1} \int_{\delta_1}^{\delta_2} \check{Z}(z)dz \\ &\leq \frac{\check{Z}(\delta_1) + \check{Z}(\delta_2)}{2} \end{aligned} \tag{3.2}$$

for $s = 1$.

Proof. Firstly, let $s > 1$. If we use the property of the exponential convex with respect to s of the function $\check{Z}$, we get

$$\begin{aligned} \check{Z}\left(\frac{\delta_1+\delta_2}{2}\right) &= \check{Z}\left(\frac{1}{2}[\omega\delta_1 + (1-\omega)\delta_2] + \frac{1}{2}[(1-\omega)\delta_1 + \omega\delta_2]\right) \\ &\leq \frac{\sqrt{s}}{2}\check{Z}(\omega\delta_1 + (1-\omega)\delta_2) + \frac{\sqrt{s}}{2}\check{Z}((1-\omega)\delta_1 + \omega\delta_2). \end{aligned}$$

Now, by taking integral in the last inequality with respect to $\omega \in [0,1]$, we deduce that

$$\begin{aligned} &\check{Z}\left(\frac{\delta_1+\delta_2}{2}\right) \\ &\leq \left[\frac{\sqrt{s}}{2}\int_0^1 \check{Z}(\omega\delta_1 + (1-\omega)\delta_2)\,d\omega + \frac{\sqrt{s}}{2}\int_0^1 \check{Z}((1-\omega)\delta_1 + \omega\delta_2)\,d\omega\right] \\ &= \frac{\sqrt{s}}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2} \check{Z}(z)dz. \end{aligned}$$

Then, by using the property of the exponential convexity with respect to s of the function $\check{Z}$, if the variable is changed as $z = \omega\delta_1 + (1-\omega)\delta_2$, then

$$\begin{aligned}\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\check{Z}(z)dz &= \int_0^1 \check{Z}\left(\omega\delta_1+(1-\omega)\delta_2\right)d\omega \\ &\leq \int_0^1 \left\{\omega s^{1-\omega}\check{Z}(\delta_1)+(1-\omega)\,s^{\omega}\check{Z}(\delta_2)\right\}d\omega \\ &= -\frac{\ln s-s+1}{\ln^2 s}\left[\check{Z}\left(\delta_1\right)+\check{Z}\left(\delta_2\right)\right].\end{aligned}$$

Similarly, it is easily seen that the inequalities (3.2) hold for $s=1$. □

4. Some New Inequalities For Exponential Convex Functions with Respect to s

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard integral inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is exponential convex function with respect to s. Dragomir and Agarwal [3] used the following lemma:

Lemma 4.1 ([3]). *Let $\check{Z}: I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I°, $\delta_1, \delta_2 \in I^\circ$ with $\delta_1 \leq \delta_2$. If $f' \in L\left[\delta_1,\delta_2\right]$, then the following equality holds:*

$$\begin{aligned}&\frac{\check{Z}\left(\delta_1\right)+\check{Z}\left(\delta_2\right)}{2}-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\check{Z}(z)dz \\ &= \frac{\delta_2-\delta_1}{2}\int_0^1(1-2\omega)\check{Z}'\left(\omega\delta_1+(1-\omega)\delta_2\right)d\omega.\end{aligned}$$

Definition 4.2. The arithmetic mean of two non-negative numbers δ_2, δ_2 $(\delta_2 > \delta_2)$ is defined as follows:

$$A := A(\delta_1,\delta_2) = \frac{\delta_1+\delta_2}{2}, \quad \delta_1,\delta_2 \geq 0.$$

Theorem 4.3. *Let $s \in (0,\infty)\setminus\{1\}$ and the function $\check{Z}: I \to \mathbb{R}$ be a continuously differentiable function, let $\delta_1 \leq \delta_2$ in I and assume that $\check{Z}' \in L\left[\delta_1,\delta_2\right]$. If $\left|\check{Z}'\right|$ is an exponential convex with respect to s on the interval $\left[\delta_1,\delta_2\right]$, then the following integral inequalities*

$$\begin{aligned}&\left|\frac{\check{Z}\left(\delta_1\right)+\check{Z}\left(\delta_2\right)}{2}-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\check{Z}(z)dz\right| \\ &\leq \left(\delta_2-\delta_1\right)\left(-\frac{\ln^2 s+\left(3-s-2\sqrt{s}\right)\ln s+4\left(s-2\sqrt{s}+1\right)}{\ln^3 s}\right) \\ &\quad \times A\left(\left|\check{Z}'\left(\delta_1\right)\right|,\left|\check{Z}'\left(\delta_2\right)\right|\right)\end{aligned}$$

for all $s>1$, *and*

(4.1)
$$\left|\frac{\check{Z}(\delta_1)+\check{Z}(\delta_2)}{2}-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\check{Z}(z)dz\right|\leq\frac{\delta_2-\delta_1}{4}A\left(\left|\check{Z}'(\delta_1)\right|,\left|\check{Z}'(\delta_2)\right|\right),$$

for $s=1$, *where* A *is the arithmetic mean.*

Proof. Firstly, let $s>1$. By using Lemma 4.1 and the inequality

$$\left|\check{Z}'(\omega\delta_1+(1-\omega)\delta_2)\right|\leq\omega s^{1-\omega}\left|\check{Z}'(\delta_1)\right|+(1-\omega)s^{\omega}\left|\check{Z}'(\delta_2)\right|,$$

we get

$$\begin{aligned}
&\left|\frac{\check{Z}(\delta_1)+\check{Z}(\delta_2)}{2}-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\check{Z}(z)dz\right|\\
&\quad\leq\frac{\delta_2-\delta_1}{2}\int_0^1|1-2\omega|\left|\check{Z}'(\omega\delta_1+(1-\omega)\delta_2)\right|d\omega\\
&\quad\leq\frac{\delta_2-\delta_1}{2}\int_0^1|1-2\omega|\left[\omega s^{1-\omega}\left|\check{Z}'(\delta_1)\right|+(1-\omega)s^{\omega}\left|\check{Z}'(\delta_2)\right|\right]d\omega\\
&\quad=\frac{\delta_2-\delta_1}{2}\left[\left|\check{Z}'(\delta_1)\right|\int_0^1|1-2\omega|\,\omega s^{1-\omega}d\omega\right.\\
&\qquad\left.+\left|\check{Z}'(\delta_2)\right|\int_0^1|1-2\omega|\,(1-\omega)s^{\omega}d\omega\right]\\
&\quad=(\delta_2-\delta_1)\left(-\frac{\ln^2 s+(3-s-2\sqrt{s})\ln s+4(s-2\sqrt{s}+1)}{\ln^3 s}\right)\\
&\qquad\times A\left(\left|\check{Z}'(\delta_1)\right|,\left|\check{Z}'(\delta_2)\right|\right),
\end{aligned}$$

because

(4.2)
$$\begin{aligned}
\int_0^1|1-2\omega|\,\omega s^{1-\omega}d\omega&=\int_0^1|1-2\omega|\,(1-\omega)s^{\omega}d\omega\\
&=-\frac{\ln^2 s+(3-s-2\sqrt{s})\ln s+4(s-2\sqrt{s}+1)}{\ln^3 s}
\end{aligned}$$

and similarly, since

$$\begin{aligned}
\int_0^1|1-2\omega|\,\omega d\omega&=\int_0^1|1-2\omega|\,(1-\omega)d\omega\\
&=\frac{1}{4},
\end{aligned}$$

it is easily seen that the inequality (4.1) hold for $s=1$. □

Remark 4.4. If we take $s=1$ in Theorem 4.3, then (4.1) reduces to the inequality of Theorem 2.2 in [3].

Theorem 4.5. *Let $s \in (0,\infty)\setminus\{1\}$ and the function $\check{Z} : I \to \mathbb{R}$ be a continuously differentiable function, let $\delta_1 \le \delta_2$ in I and assume that $q > 1$. If $\left|\check{Z}'\right|^q$ is an exponential convex function with respect to s on the interval $[\delta_1, \delta_2]$, then the following inequalities hold*

(4.3)

$$\left|\frac{\check{Z}(\delta_1)+\check{Z}(\delta_2)}{2}-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\check{Z}(z)dz\right|$$
$$\le \begin{cases} \frac{\delta_2-\delta_1}{2}\left[-2\frac{\ln s-s+1}{\ln^2 s}\right]^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(\left|\check{Z}'(\delta_1)\right|^q,\left|\check{Z}'(\delta_2)\right|^q\right), & s>1, \\ \frac{\delta_2-\delta_1}{2(p+1)^{\frac{1}{p}}} A^{\frac{1}{q}}\left(\left|\check{Z}'(\delta_1)\right|^q,\left|\check{Z}'(\delta_2)\right|^q\right), & s=1, \end{cases}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and A is the arithmetic mean.

Proof. Firstly, let $s > 1$. By using Lemma 4.1, Hölder's integral inequality and

$$\left|\check{Z}'(\omega\delta_1+(1-\omega)\delta_2)\right|^q \le \omega s^{1-\omega}\left|\check{Z}'(\delta_1)\right|^q+(1-\omega)s^{\omega}\left|\check{Z}'(\delta_2)\right|^q,$$

we get

$$\begin{aligned}
&\left|\frac{\check{Z}(\delta_1)+\check{Z}(\delta_2)}{2}-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\check{Z}(z)dz\right| \\
&\le \frac{\delta_2-\delta_1}{2}\int_0^1 |1-2\omega|\left|\check{Z}'(\omega\delta_1+(1-\omega)\delta_2)\right|d\omega \\
&\le \frac{\delta_2-\delta_1}{2}\left(\int_0^1|1-2\omega|^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1\left|\check{Z}'(\omega\delta_1+(1-\omega)\delta_2)\right|^q d\omega\right)^{\frac{1}{q}} \\
&\le \frac{\delta_2-\delta_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[\omega s^{1-\omega}\left|\check{Z}'(\delta_1)\right|^q+(1-\omega)s^{\omega}\left|\check{Z}'(\delta_2)\right|^q\right]d\omega\right)^{\frac{1}{q}} \\
&= \frac{\delta_2-\delta_1}{2}\left[-2\frac{\ln s-s+1}{\ln^2 s}\right]^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(\left|\check{Z}'(\delta_1)\right|^q,\left|\check{Z}'(\delta_2)\right|^q\right),
\end{aligned}$$

because

$$\int_0^1 |1-2\omega|^p d\omega = \frac{1}{p+1}$$

and

$$\int_0^1 \omega s^{1-\omega} d\omega = \int_0^1 (1-\omega)s^{\omega}dt$$

$$= \begin{cases} -\frac{\ln s - s + 1}{\ln^2 s}, & s > 1, \\ \frac{1}{2}, & s = 1, \end{cases}$$

and similarly, it is easily seen that the inequality (4.3) holds for $s = 1$. □

Remark 4.6. If we take $s = 1$ in Theorem 4.5, then (4.3) reduces to the inequality of Theorem 2.3 in [3].

Theorem 4.7. *Let $s \in (0, \infty) \setminus \{1\}$ and the function $\check{Z} : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function, let $\delta_1 \leq \delta_2$ in I and assume that $q \geq 1$. If $\left|\check{Z}'\right|^q$ is an exponential convex function with respect to s on the interval $[\delta_1, \delta_2]$, then the following inequalities hold*

$$\begin{aligned}
&\left| \frac{\check{Z}(\delta_1) + \check{Z}(\delta_2)}{2} - \frac{1}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} \check{Z}(z) dz \right| \\
&\leq \frac{\delta_2 - \delta_1}{2^{2-\frac{1}{q}}} A^{\frac{1}{q}} \left(\left|\check{Z}'(\delta_1)\right|^q, \left|\check{Z}'(\delta_2)\right|^q \right) \\
&\times \left[2 \left(- \frac{\ln^2 s + (3 - s - 2\sqrt{s}) \ln s + 4 (s - 2\sqrt{s} + 1)}{\ln^3 s} \right) \right]^{\frac{1}{q}},
\end{aligned} \tag{4.4}$$

for all $s > 1$, and

$$\left| \frac{\check{Z}(\delta_1) + \check{Z}(\delta_2)}{2} - \frac{1}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} \check{Z}(z) dz \right| \leq \frac{\delta_2 - \delta_1}{4} A^{\frac{1}{q}} \left(\left|\check{Z}'(\delta_1)\right|^q, \left|\check{Z}'(\delta_2)\right|^q \right), \tag{4.5}$$

for $s = 1$ where A is the arithmetic mean.

Proof. Assume first that $q > 1$. By considering the Lemma 4.1, Hölder's inequality and the property of the exponential convex function with respect to s of $\left|\check{Z}'\right|^q$, by using (4.2), we obtain

$$\begin{aligned}
&\left| \frac{\check{Z}(\delta_1) + \check{Z}(\delta_2)}{2} - \frac{1}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} \check{Z}(z) dz \right| \\
&\leq \frac{\delta_2 - \delta_1}{2} \int_0^1 |1 - 2\omega| \, \check{Z}'(\omega\delta_1 + (1 - \omega)\delta_2) \, d\omega \\
&\leq \frac{\delta_2 - \delta_1}{2} \left(\int_0^1 |1 - 2\omega| \, d\omega \right)^{1 - \frac{1}{q}} \\
&\times \left(\int_0^1 |1 - 2\omega| \left|\check{Z}'(\omega\delta_1 + (1 - \omega)\delta_2)\right|^q d\omega \right)^{\frac{1}{q}}
\end{aligned} \tag{4.6}$$

$$= \frac{\delta_2 - \delta_1}{2} \left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times \left(\int_0^1 |1 - 2\omega| \left[\omega s^{1-\omega} \left| \check{Z}'(\delta_1) \right|^q + (1 - \omega) s^{\omega} \left| \check{Z}'(\delta_2) \right|^q \right] d\omega \right)^{\frac{1}{q}}$$
$$= \frac{\delta_2 - \delta_1}{2^{2-\frac{1}{q}}} \left[2 \left(-\frac{\ln^2 s + (3 - s - 2\sqrt{s}) \ln s + 4 (s - 2\sqrt{s} + 1)}{\ln^3 s} \right) \right]^{\frac{1}{q}}$$
$$\times A^{\frac{1}{q}} \left(\left| \check{Z}'(\delta_1) \right|^q, \left| \check{Z}'(\delta_2) \right|^q \right).$$

For $q = 1$ we use the estimates from the proof of Theorem 4.3, which also follow step by step the above estimates. Similarly, it is easily seen that the inequality (4.5) hold for $s = 1$. □

Corollary 4.8. *If we take $q = 1$ in the Theorem 4.7 with , we get the conclusion of Theorem 4.3.*

Remark 4.9. If we take $s = 1$ and $q = 1$ in Theorem 4.7, then the inequality (4.5) reduces to the inequality of Theorem 2.2 in [3].

5. Applications For Special Means

Throughout this section, for shortness, the following notations will be used for special means of two nonnegative numbers δ_1, δ_2 with $\delta_2 > \delta_1$:

(1). The arithmetic mean

$$A := A(\delta_1, \delta_2) = \frac{\delta_1 + \delta_2}{2}, \quad \delta_1, \delta_2 \geq 0.$$

(2). The geometric mean

$$G := G(\delta_1, \delta_2) = \sqrt{\delta_1 \delta_2}, \quad \delta_1, \delta_2 \geq 0.$$

(3). The harmonic mean

$$H := H(\delta_1, \delta_2) = \frac{2\delta_1 \delta_2}{\delta_1 + \delta_2}, \quad \delta_1, \delta_2 > 0.$$

(4). The logarithmic mean

$$L := L(\delta_1, \delta_2) = \begin{cases} \frac{\delta_2 - \delta_1}{\ln \delta_2 - \ln \delta_1}, & \delta_1 \neq \delta_2 \\ \delta_1, & \delta_1 = \delta_2 \end{cases}, \quad \delta_1, \delta_2 > 0.$$

(5). The p-logarithmic mean

$L_p := L_p(\delta_1, \delta_2)$

$$= \begin{cases} \left(\frac{\delta_2^{p+1}-\delta_1^{p+1}}{(p+1)(\delta_2-\delta_1)}\right)^{\frac{1}{p}}, & \delta_1 \neq \delta_2, p \in \mathbb{R}\setminus\{-1,0\} \\ \delta_1, & \delta_1 = \delta_2 \end{cases}, \quad \delta_1, \delta_2 > 0.$$

(6). The identric mean

$$\begin{aligned} I &:= I(\delta_1, \delta_2) \\ &= \frac{1}{e}\left(\frac{\delta_2^{\delta_2}}{\delta_1^{\delta_1}}\right)^{\frac{1}{\delta_2-\delta_1}}, \quad \delta_1, \delta_2 > 0. \end{aligned}$$

Also, L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 5.1. *Let $\delta_1, \delta_2 \in [0, \infty)$ with $\delta_1 < \delta_2$ and $n \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$. Then, the following inequalities are obtained:*

$$\begin{aligned} \frac{A^n(\delta_1, \delta_2)}{\sqrt{s}} &\leq L_n^n(\delta_1, \delta_2) \\ &\leq -2\frac{\ln s - s + 1}{\ln^2 s} A(\delta_1^n, \delta_2^n), \end{aligned}$$

for all $s > 1$.

Proof. The assertion follows from the inequalities (3.1) applied to the function

$$\check{Z}(x) = x^n, \quad x \in [0, \infty). \qquad \square$$

Proposition 5.2. *Let $\delta_1, \delta_2 \in (0, \infty)$ with $\delta_1 < \delta_2$. Then, the following inequalities are obtained:*

$$\begin{aligned} \frac{A^{-1}(\delta_1, \delta_2)}{\sqrt{s}} &\leq L^{-1}(\delta_1, \delta_2) \\ &\leq -2\frac{\ln s - s + 1}{\ln^2 s} H^{-1}(\delta_1, \delta_2), \end{aligned}$$

for all $s > 1$.

Proof. The assertion follows from the inequalities (3.1) applied to the function

$$\check{Z}(x) = x^{-1}, \quad x \in (0, \infty). \qquad \square$$

Proposition 5.3. *Let $\delta_1, \delta_2 \in (0, 1]$ with $\delta_1 < \delta_2$. Then, the following inequalities are obtained:*

$$-2\frac{\ln s - s + 1}{\ln^2 s} \ln G(\delta_1, \delta_2) \leq \ln I(\delta_1, \delta_2) \leq \frac{\ln A(\delta_1, \delta_2)}{\sqrt{s}},$$

for all $s > 1$.

Proof. The assertion follows from the inequalities (3.1) applied to the function

$$\check{Z}(x) = -\ln x, \quad x \in (0,1]. \qquad \square$$

6. CONCLUSION

In this paper, we give the concept of exponential convex functions with respect to s and prove Hermite-Hadamard type inequalities for the newly introduced this class of functions. Then, we obtain some new Hermite-Hadamard type integral inequalities for the exponential convex functions with respect to s using an identity together with Hölder's integral inequality. Different types of integral inequalities can be obtained using this new definition.

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