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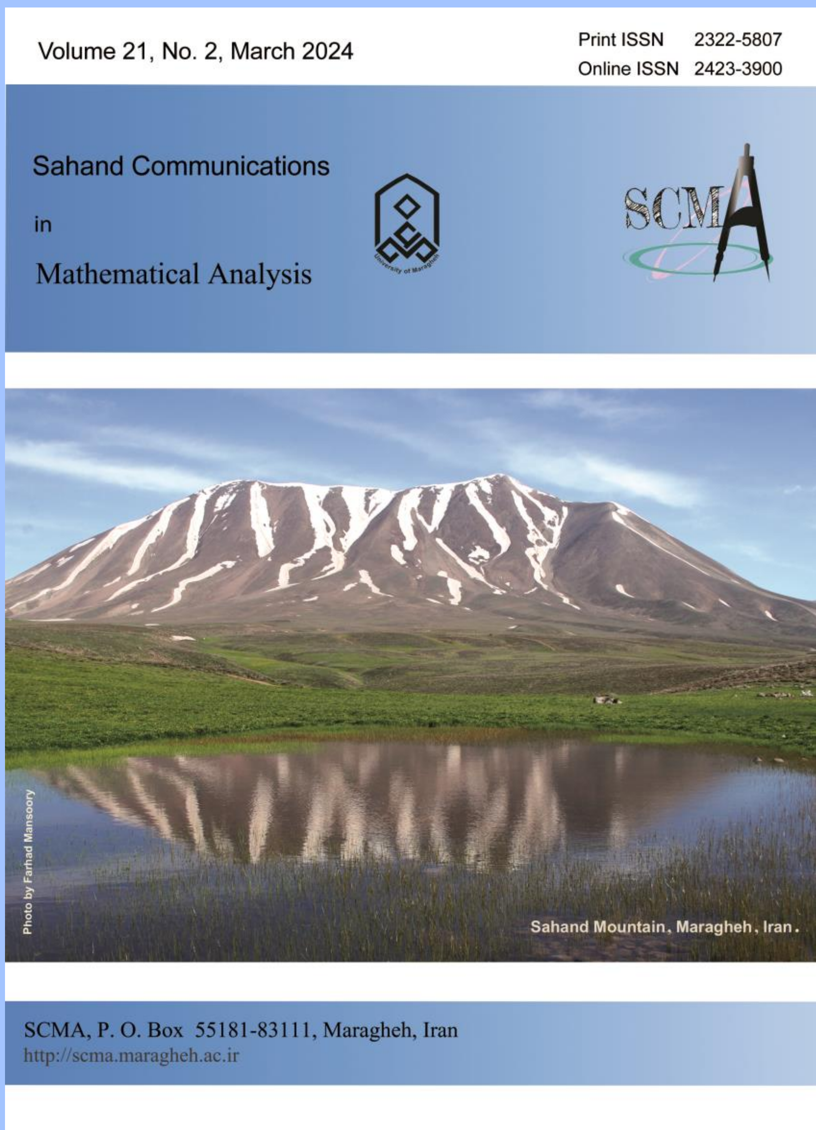

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Notes about Quasi-Mixing Operators

Mansooreh Moosapoor[1*] and Ismail Nikoufar[2]

Abstract. In this article, we introduce quasi-mixing operators and construct various examples. We prove that quasi-mixing operators exist on all finite-dimensional and infinite-dimensional Banach spaces. We also prove that an invertible operator T is quasi-mixing if and only if T^{-1} is quasi-mixing. We state some sufficient conditions under which an operator is quasi-mixing. Moreover, we prove that the direct sum of two operators is quasi-mixing if and only if any of them is quasi-mixing.

1. Introduction

Let X be a Banach space, and $B(X)$ the set of bounded linear operators on X. An operator $T \in B(X)$ is named topologically transitive if, for any two nonempty open sets $U \subseteq X$ and $V \subseteq X$, there exists a non-negative integer n such that $T^n(U) \cap V \neq \phi$. This definition is equivalent to the condition that $\overline{\bigcup_{n=0}^{\infty} T^n(U)} = X$ for any nonempty open set $U \subseteq X$ [7, Proposition 1.10]. Also, topological transitivity is equivalent to this condition that there exists an element $x \in X$ with such property that $orb(T,x) = \{x, Tx, \ldots, T^n x, \ldots\}$ is dense in X which is named hypercyclicity. Hence, hypercyclicity occurs only on separable Banach spaces. So, we consider X a separable Banach space in this paper.

One can see [3, 4, 6, 10] for a history and more information.

If for any two nonempty open sets $U \subseteq X$ and $V \subseteq X$, a natural number N exists such that for any $n \geq N$, $T^n(U) \cap V \neq \phi$, then we call the operator T a mixing operator. Hence, these operators satisfied

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much stronger conditions concerning topologically transitive operators. Any separable infinite-dimensional Banach spaces support mixing operators [8, Theorem 2.6]. However, these operators do not exist in finite-dimensional Banach spaces. Obviously, mixing operators are hypercyclic and hypercyclic operators can not be constructed on finite-dimensional spaces [7, Corollary 2.59]. Authors in [5] characterized mixing weighted backward shift operators. Also, the mixing of composition operators is investigated in [2, 11].

A generalization of the concept of mixing is super-mixing. An operator $T \in B(X)$ is named super-mixing if, for any nonempty open subset V of X, $\bigcup_{i=0}^{\infty}\bigcap_{n=i}^{\infty}T^n(V)$ is dense in X [1]. There are a criterion and some exciting results about this type of operator in [1]. One can also see [9] for more results.

Let us replace V with U in the definition of a mixing operator. That means, consider this property that for any nonempty open set $U \subseteq X$, there is a natural number M such that for any $n \geq M$, $T^n(U) \cap U \neq \phi$ or equivalently $\bigcup_{i=0}^{\infty}\bigcap_{n=i}^{\infty}T^n(U) \cap U \neq \phi$. In this way, we obtain a new class of operators, and we call them quasi-mixing operators.

We organize the article as follows. In Section 2, we present some examples of quasi-mixing operators. We prove that quasi-mixing operators exist on both infinite-dimensional and finite-dimensional Banach spaces. We prove that quasi-mixing preserves under quasi-conjugacy. Also, we state that an invertible operator T is quasi-mixing if and only if its inverse is quasi-mixing. Section 3 states some sufficient conditions for quasi-mixing operators by using a set of points with particular properties, and with using open sets and neighborhoods of zero. In Section 4, we investigate the properties of the direct sum of the quasi-mixing operators. We demonstrate that the direct sum of two operators is quasi-mixing if and only if any of them is quasi-mixing.

2. Definitions and Some Results

We start this section with our main definition.

Definition 2.1. Let $T \in B(X)$. We say the operator T is quasi-mixing if for any nonempty open set U of X, there exists the natural number M such that for any $n \geq M$, $T^n(U) \cap U \neq \phi$.

By the definition of a mixing operator and Definition 2.1, the following lemma can be stated immediately.

Lemma 2.2. *Let $T \in B(X)$.*

(i) *If T is mixing, then T is quasi-mixing.*
(ii) *If T is super-mixing, then T is quasi-mixing.*

By using Lemma 2.2, one can construct various examples of quasi-mixing operators.

Example 2.3. Let D be the derivation operator on $H(\mathbb{C})$, the space of holomorphic functions. Then, λD is a mixing operator on $H(\mathbb{C})$ for any $\lambda \neq 0$. Hence, λD is quasi-mixing for any $\lambda \neq 0$.

But there is no operator T which λT is hypercyclic for any $\lambda \neq 0$ [7, p. 60].

In the following, we make some quasi-mixing operators using weighted backward shifts on l^p spaces. Recall that if $(e_i)_{i\geq 1}$ is the canonical basis for l^p, $1 \leq p \leq \infty$, then the weighted shift B_w with weight $(w_n)_{n\geq 1}$ is defined by $B_w(e_1) = 0$ and $B_w(e_n) = w_n e_{n-1}$, for any $n \geq 2$ [8].

Example 2.4. Assume B_w is a weighted backward shift on l^p with $1 \leq p < \infty$. Suppose that $(w_n)_{n=1}^{\infty}$ is its weight, where $w_n > 0$ for any n. Then $I + B_w$ is mixing [8, Lemma 2.3]. Hence, $I + B_w$ is quasi-mixing by Lemma 2.2.

Example 2.5. It is proved in [5, Theorem 1.2] that a weighted backward shift B_w on $l^2(\mathbb{N})$ with weight $(w_n)_{n=1}^{\infty}$ is mixing if and only if $\lim_n \prod_{i=1}^{n} w_i = \infty$. Hence, any weighted shift with such property is quasi-mixing.

In Example 2.3, Example 2.4, and Example 2.5, we construct operators that are mixing and quasi-mixing. But there are quasi-mixing operators that are not mixing, as we show in the subsequent example.

Example 2.6. Assume X is a Banach space with infinite-dimensional or finite-dimensional. Suppose $I : X \to X$ is the identity operator. Then I is a quasi-mixing operator, since for any open and nonempty U,

$$I^n(U) \cap U = U \cap U = U \neq \phi.$$

But I is not mixing, since for any nonempty open sets U and V, $U \cap V$ may be empty.

Now, this question arises which Banach spaces support quasi-mixing operators? Infinite-dimensional Banach spaces, finite-dimensional Banach spaces, or both of them? We answer this question in the following theorem.

Theorem 2.7. *Quasi-mixing operators exist on any infinite-dimensional and finite-dimensional Banach space.*

Proof. Mixing operators can be found in any infinite-dimensional separable Banach space [8, Theorem 2.6]. Also, mixing operators are quasi-mixing. So, we conclude that quasi-mixing operators exist in any infinite-dimensional Banach space.

Moreover, by Example 2.6, the identity operator on any finite- dimensional space is quasi-mixing. □

According to our discussion in the introduction, we know that mixing operators exist only in infinite-dimensional spaces. So, Theorem 2.7 shows that the set of mixing operators is a proper subset of the set of quasi-mixing operators. Since by Theorem 2.7, quasi-mixing operators exist on any infinite-dimensional and finite-dimensional separable Banach space.

Let $T \in B(X)$ and $S \in B(Y)$, where Y is a Banach space. The operators T and S are quasi-conjugate if a continuous operator $\Phi : X \to Y$ exists with dense range such that $T \circ \Phi = \Phi \circ S$. It is well-known that the mixing property preserves under quasi-conjugacy. We show in the next proposition that quasi-mixing property preserves under quasi-conjugacy, too.

Proposition 2.8. *Quasi-mixing property is preserved under quasi-conjugacy.*

Proof. Assume $T \in B(X)$ is quasi-mixing. Assume $S \in B(Y)$ is quasi-conjugate to T. Suppose $U \subseteq Y$ is a nonempty open set. Since the operator Φ is continuous, $\Phi^{-1}(U)$ is open. So, there is $M \in \mathbb{N}$ such that for any $n \geq M$,

$$T^n \left(\Phi^{-1}(U)\right) \cap \Phi^{-1}(U) \neq \phi.$$

Therefore, for any $n \geq M$, there is $x_n \in \Phi^{-1}(U)$ such that $T^n x_n \in \Phi^{-1}(U)$. So, for any $n \geq M$,

$$\Phi(x_n) \in U, \qquad T^n x_n \in \Phi^{-1}(U).$$

Note that $T^n x_n \in \Phi^{-1}(U)$ indicates that $\Phi \circ T^n x_n \in U$. Hence $S^n \circ \Phi x_n \in U$, because of $S^n \circ \Phi = \Phi \circ T^n$.

Therefore, for any $n \geq M$, $\Phi(x_n) \in U$ and $S^n\left(\Phi(x_n)\right) \in U$. That means $S^n(U) \cap U \neq \phi$ for any $n \geq M$. Hence, S is quasi-mixing either. □

In the following theorem, we show that for an invertible operator T, if the operator T is quasi-mixing, then the operator T^{-1} is quasi-mixing and vice versa.

Theorem 2.9. *Suppose $T \in B(X)$ is an invertible operator. Then the operator T is quasi-mixing if and only if T^{-1} is quasi-mixing.*

Proof. Assume $U \subseteq X$ is nonempty and open. Suppose T is quasi-mixing. Therefore, there is $M \in \mathbb{N}$ which for any $n \geq M$,

$$T^n(U) \cap U \neq \phi. \tag{2.1}$$

It follows from (2.1) that $T^{-n}(U) \cap U \neq \phi$. Hence, for any $n \geq M$,

$$(T^{-1})^n(U) \cap U \neq \phi. \tag{2.2}$$

So, T^{-1} is quasi-mixing. The proof of the converse of the theorem is similar. □

Theorem 2.10. *Suppose $T \in B(X)$ is quasi-mixing. Then for any $q \in \mathbb{N}$, T^q is quasi-mixing.*

Proof. Let $q \in \mathbb{N}$ and $U \subseteq X$ be a nonempty open set. Since the operator T is quasi-mixing , there exists a natural number M such that for any $n \geq M$,

$$T^n(U) \cap U \neq \phi.$$

Especially, for any $m \in \mathbb{N}$ with $mq \geq M$, $T^{mq}(U) \cap U \neq \phi$. Therefore, $(T^q)^m(U) \cap U \neq \phi$ for any $m \geq \left[\frac{M}{q}\right] + 1$. That means T^q is quasi-mixing. □

The following corollary is straightforward, by Theorem 2.9 and Theorem 2.10.

Corollary 2.11. *Assume $T \in B(X)$ is an invertible operator. If T is quasi-mixing, then T^p and T^{-p} are quasi-mixing for any $p \in \mathbb{N}$.*

3. Some Sufficient Conditions

The points that iterates of an operator tending to itself, can play an essential role for quasi-mixing of an operator, as we will see in the first theorem of this section.

Theorem 3.1. *Let $T \in B(X)$. Let $F := \{w \in X : T^n w \to w\}$. If F is dense in X, then T is quasi-mixing.*

Proof. Let $U \subseteq X$ be a nonempty open set. By density of F it is concluded that $U \cap F \neq \phi$. Hence, there exists $a \in U \cap F$. Therefore, $\varepsilon > 0$ can be found such that $B(a, \varepsilon) \subseteq U$. Hence, $T^n a \to a$. So, there is $M \in \mathbb{N}$ such that for any $n \geq M$,

$$\|T^n a - a\| < \varepsilon.$$

Hence, for any $n \geq M$,

$$T^n a \in B(a, \varepsilon) \subseteq U.$$

So, $T^n(U) \cap U \neq \phi$ for any $n \geq M$. □

As well as we can state another sufficient condition as follows.

Theorem 3.2. *Suppose $T \in B(X)$ and Z_0 is a dense subset of X. If there is $S : Z_0 \to Z_0$ such that for any $z \in Z_0$,*

(1) $S^n z \to z$,

(2) $TSz = z$.

Then T is quasi-mixing.

Proof. Similar to the proof of Theorem 3.1, for any open set $U \subseteq X$, there is $z \in Z_0 \cap U$, and $M \in \mathbb{N}$ such that $T^n z \in U$ for any $n \geq M$.

By condition (2), $T^n S^n z = z$ for any $z \in Z_0$. So, for any $n \geq M$, $T^n(S^n z) \in U$. But $S^n z \in U$. Hence, for any $n \geq M$, $T^n(U) \cap U \neq \phi$. $\square$

The idea of the following theorem is given from [7, Proposition 2.37] that states a sufficient condition for quasi-mixing operators.

Theorem 3.3. *Suppose $T \in B(X)$. Assume for any nonempty open set $U \subseteq X$ and any neighborhood W of zero, $M \in \mathbb{N}$ exists such that for any $n \geq M$,*

$$T^n(U) \cap W \neq \phi, \qquad T^n(W) \cap U \neq \phi. \tag{3.1}$$

Then T is quasi-mixing.

Proof. Let U be an open set. So, there is an open set U_1 and a neighborhood W_1 of zero such that $U_1 + W_1 \subseteq U$ [7, Lemma 2.36]. By (3.1), for this U_1 and W_1, $M \in \mathbb{N}$ can be chosen such that for any $n \geq M$

$$T^n(U_1) \cap W_1 \neq \phi, \qquad T^n(W_1) \cap U_1 \neq \phi. \tag{3.2}$$

Hence, for any $n \geq M$, there is $u_n \in U_1$ and $w_n \in W_1$ such that

$$T^n u_n \in W_1, \qquad T^n w_n \in U_1. \tag{3.3}$$

But $u_n + w_n \in U_1 + W_1 \subseteq U$ and $T^n(u_n + w_n) \in U_1 + W_1 \subseteq U$. Hence, for any $n \geq M$, $T^n(U) \cap U \neq \phi$. Therefore, T is quasi-mixing. $\square$

By the idea of [8, Theorem 3.2], we state the following theorem.

Theorem 3.4. *Suppose $T \in B(X)$ and suppose $p \in \mathbb{N}$. Assume for any nonempty open set $U \subseteq X$ and any neighborhood W of zero, $M \in \mathbb{N}$ exists such that*

$$T^n(U) \cap W \neq \phi, \qquad T^{n+p}(W) \cap U \neq \phi, \tag{3.4}$$

for any $n \geq M$. Then T is quasi-mixing.

Proof. Let U be an open set, and W be a neighborhood of zero. Set $W_0 = W \cap T^{-p}(W)$. So, W is an open and nonempty set. By (3.4), there is $M \in \mathbb{N}$ such that for any $n \geq M$,

$$T^n(U) \cap W_0 \neq \phi, \qquad T^{n+p}(W_0) \cap U \neq \phi. \tag{3.5}$$

So, for any $n \geq M$,

$$T^n(U) \cap W \cap T^{-p}(W) \neq \phi, \qquad T^{n+p}(W) \cap T^{n+p}\left(T^{-p}(W)\right) \cap U \neq \phi. \tag{3.6}$$

Hence, for any $n \geq M$,

$$T^n(U) \cap T^{-p}(W) \neq \phi, \qquad T^{n+p}(W) \cap U \neq \phi. \tag{3.7}$$

This implies for any $n \geq M$,

$$T^{n+p}(U) \cap W \neq \phi, \qquad T^{n+p}(W) \cap U \neq \phi. \tag{3.8}$$

Therefore, T is quasi-mixing by Theorem 3.3. □

By setting $p = 1$ in Theorem 3.4, we deduce the following corollary.

Corollary 3.5. *Suppose $T \in B(X)$. Assume for any nonempty open set $U \subseteq X$ and any neighborhood W of zero, $M \in \mathbb{N}$ exists such that*

$$T^n(U) \cap W \neq \phi, \qquad T^{n+1}(W) \cap U \neq \phi \tag{3.9}$$

for any $n \geq M$. Then T is quasi-mixing.

4. Direct Sum of Quasi-mixing Operators

Let X and Y be two Banach spaces. Suppose $T \in B(X)$ and $S \in B(Y)$. By $T \oplus S$ we mean the direct sum of T and S. For any $x \oplus y \in X \oplus Y$, $(T \oplus S)(x \oplus y)$ is defined by $(T \oplus S)(x \oplus y) = Tx \oplus Sy$.

In this section, we investigate the notion of the quasi-mixing operators for the direct sum of the operators. It is investigated that $T \oplus S$ is mixing if and only if T and S are mixing [7, Proposition 2.40]. We want to know whether it is true for quasi-mixing operators or not. First, we show that quasi-mixing of the two operators leads to quasi-mixing of their direct sum as follows.

Theorem 4.1. *If $T \in B(X)$ and $S \in B(Y)$ are quasi-mixing, then $T \oplus S$ is quasi-mixing on $X \oplus Y$. Especially, $T \oplus T$ is quasi-mixing on $X \oplus X$.*

Proof. Let $U \subseteq X$ and $V \subseteq Y$ be nonempty open sets. By quasi-mixing of T, there is $M_1 \in \mathbb{N}$ such that for any $n \geq M_1$,

$$T^n(U) \cap U \neq \phi. \tag{4.1}$$

By quasi-mixing of S, there is $M_2 \in \mathbb{N}$ such that for any $n \geq M_2$,

$$S^n(V) \cap V \neq \phi. \tag{4.2}$$

If we set $M := \max\{M_1, M_2\}$, then for any $n \geq M$,

$$\begin{aligned}(T \oplus S)^n(U \oplus V) \cap (U \oplus V) &= (T^n(U) \oplus S^n(V)) \cap (U \oplus V) \\ &= (T^n(U) \cap U) \oplus (S^n(V) \cap V) \\ &\neq \phi.\end{aligned}$$

Hence, $T \oplus S$ is quasi-mixing. □

Example 4.2. Let B_w be a weighted backward shift on $l^2(\mathbb{N})$ with weight $(w_n)_{n=1}^{\infty}$ such that $\lim_n \prod_{i=1}^{n} w_i = \infty$. Let I be the identity operator on $l^2(\mathbb{N})$. By Example 2.5, B_w is quasi-mixing and by Example 2.6, I is quasi-mixing. So, by Theorem 4.1, $B_w \oplus I$ is quasi-mixing on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$. But it is not hard to see that $B_w \oplus I$ is not mixing.

Now, this question appears when quasi-mixing of $T \oplus S$ implies quasi-mixing of T and S? In the upcoming theorem, we establish that the answer to this question is positive.

Theorem 4.3. *If $T \oplus S$ is a quasi-mixing operator on $X \oplus Y$, then T is quasi-mixing on X, and S is quasi-mixing on Y.*

Proof. Assume $U \subseteq X$ and $V \subseteq Y$ are nonempty open sets. By quasi-mixing of $T \oplus S$, there is $M \in \mathbb{N}$ such that for any $n \geq M$,

$$(T \oplus S)^n (U \oplus V) \cap (U \oplus V) \neq \phi.$$

So, for any $n \geq M$,

$$T^n(U) \cap U \neq \phi, \qquad S^n(V) \cap V \neq \phi.$$

That means T and S are quasi-mixing. □

As a consequence of Theorems 4.1 and 4.3 we reach the following corollary.

Corollary 4.4. *$T \oplus S$ is quasi-mixing on $X \oplus Y$ if and only if T is quasi-mixing on X, and S is quasi-mixing on Y.*

As well as we derive the following corollary if we replace S by T in Corollary 4.4.

Corollary 4.5. *An operator T on X is quasi-mixing if and only if $T \oplus T$ is quasi-mixing on $X \oplus X$.*

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