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FRAMENESS BOUND FOR FRAME OF SUBSPACES

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Abstract. In this paper, we show that in each finite dimensional Hilbert space, a frame of subspaces is an ultra Bessel sequence of subspaces. We also show that every frame of subspaces in a finite dimensional Hilbert space has frameness bound.

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space. We say that a sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a frame for $\mathcal{H}$, if there exist constants $0 < A, B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}. \tag{1.1}$$

If $A = B$ then $\{f_i\}_{i=1}^{\infty}$ is called a tight frame and if $A = B = 1$, it is called a Parseval frame. If the right hand inequality of (1.1) holds for all $f \in \mathcal{H}$, then we say $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence for $\mathcal{H}$.

In 2008, the concept of ultra Bessel sequences in Hilbert spaces introduced and investigated by Faroughi and Najati [4].

Definition 1.1. Let $\mathcal{H}_0$ be an inner product space. Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of members of $\mathcal{H}_0$. Then $\{f_i\}_{i=1}^{\infty}$ is called an ultra Bessel sequence in $\mathcal{H}_0$, if

$$\sup_{\|f\|=1} \sum_{i=n}^{\infty} |\langle f, f_i\rangle|^2 \to 0,$$

as $n \to \infty$, i.e., the series $\sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2$ converges uniformly in unit sphere of $\mathcal{H}_0$.

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As a generalization of ordinary frame, frame of subspaces introduced by Casazza and Kutyniok in [2].

Definition 1.2. Let $\{v_i\}_{i=1}^{\infty}$ be a family of weights, i.e., $v_i > 0$, for all $i \geq 1$. A family of closed subspaces $\{W_i\}_{i=1}^{\infty}$ of a Hilbert space $\mathcal{H}$ is a frame of subspaces with respect to $\{v_i\}_{i=1}^{\infty}$ for $\mathcal{H}$, if there exist constants $0 < C \leq D < \infty$ such that

$$C\|f\|^2 \leq \sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \leq D\|f\|^2, \quad f \in \mathcal{H}. \tag{1.2}$$

If the right hand inequality in (1.2) holds for all $f \in \mathcal{H}$, we call $\{W_i\}_{i=1}^{\infty}$ a Bessel sequence of subspaces with respect to $\{v_i\}_{i=1}^{\infty}$ with Bessel bound D.

Definition 1.3. For each family of subspaces $\{W_i\}_{i=1}^{\infty}$ of Hilbert space $\mathcal{H}$, we define the set

$$\Big(\sum_{i=1}^{\infty} \oplus W_i\Big)_{\ell^2} = \big\{\{f_i\}_{i=1}^{\infty} | f_i \in W_i, \quad \sum_{i=1}^{\infty} \|f_i\|^2 < \infty\big\}.$$

It is clear that $\Big(\sum_{i=1}^{\infty} \oplus W_i\Big)_{\ell^2}$ is a Hilbert space with the point wise operations and with the inner product given by

$$\langle \{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty} \rangle = \sum_{i=1}^{\infty} \langle f_i, g_i \rangle.$$

It is proved in [2], if $\{W_i\}_{i=1}^{\infty}$ is a frame of subspaces with respect to $\{v_i\}_{i=1}^{\infty}$ for $\mathcal{H}$ then the operator

$$T_{W,v} : \Big(\sum_{i=1}^{\infty} \oplus W_i\Big)_{\ell^2} \to \mathcal{H}, \quad T_{W,v}(\{f_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} v_i f_i$$

is bounded and onto and its adjoint is

$$T_{W,v}^* : \mathcal{H} \to \Big(\sum_{i=1}^{\infty} \oplus W_i\Big)_{\ell^2}, \quad T_{W,v}^*(f) = \{v_i \pi_{W_i}(f)\}_{i=1}^{\infty}.$$

The operators $T_{W,v}$ and $T_{W,v}^*$ are called the synthesis and analysis operators for $\{W_i\}_{i=1}^{\infty}$ and $\{v_i\}_{i=1}^{\infty}$, respectively.

Also, it is proved in [2], if $\{W_i\}_{i=1}^{\infty}$ is a frame of subspaces with respect to $\{v_i\}_{i=1}^{\infty}$, the operator

$$S_{W,v} : \mathcal{H} \to \mathcal{H}, \quad S_{W,v}(f) = TT^*(f)$$

is a positive, self-adjoint and invertible operator on $\mathcal{H}$ and we have the reconstruction formula

$$f = \sum_{i=1}^{\infty} v_i^2 S_{W,v}^{-1} \pi_{W_i}(f), \qquad f \in \mathcal{H}.$$

The operator $S_{W,v}$ is called the frame operator for $\{W_i\}_{i=1}^{\infty}$ and $\{v_i\}_{i=1}^{\infty}$.

The ultra Bessel sequence of subspaces introduced in [1] by the authors of this paper.

Definition 1.4. Let $\mathcal{H}_0$ be an inner product space. Let $\{W_i\}_{i=1}^{\infty}$ be a family of closed subspaces of $\mathcal{H}_0$. Then $\{W_i\}_{i=1}^{\infty}$ is called an ultra Bessel sequence of subspaces in $\mathcal{H}_0$, if

$$\sup_{\|f\|=1} \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \to 0,$$

as $n \to \infty$, i.e., the series $\sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2$ converges uniformly in the unit sphere of $\mathcal{H}_0$.

Following proposition has been proved in [1] and we use it in the rest of this paper.

Proposition 1.5. Let $\{W_i\}_{i=1}^{\infty}$ be a family of closed subspaces in Hilbert space $\mathcal{H}$ and $\{v_i\}_{i=1}^{\infty}$ be a family of weights such that $\sum_{i=1}^{\infty} v_i^2 < \infty$. Then $\{W_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in $\mathcal{H}$.

2. Frameness bound

In this section, we prove that in a finite dimensional Hilbert space, each frame of subspaces is an ultra Bessel sequence of subspaces. Also we can divide a frame of subspaces $\{W_i\}_{i=1}^{\infty}$ in two sets $\{W_i\}_{i=1}^{N-1}$ and $\{W_i\}_{i=N}^{\infty}$, for which $\{W_i\}_{i=1}^{N-1}$ is not a frame of subspaces, but $\{W_i\}_{i=1}^{N}$ is a frame of subspaces.

Theorem 2.1. Let $\{W_i\}_{i=1}^{\infty}$ be a frame of subspaces for Hilbert space $\mathcal{H}$ such that for all $i \geq 1$, $\dim W_i < \infty$. Then

(i) if $\mathcal{H}$ is an infinite dimensional Hilbert space, then $\{W_i\}_{i=1}^{\infty}$ is not an ultra Bessel sequence of subspaces.

(ii) if $\mathcal{H}$ is a finite dimensional Hilbert space, then $\{W_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces, and there exists $N_0 \geqslant 1$ such that for each $1 \leqslant n < N_0$, $\{W_i\}_{i=1}^{n}$ is not a frame of subspaces of $\mathcal{H}$, but $\{W_i\}_{i=1}^{n}$ is a frame of subspaces of $\mathcal{H}$ for each $n \geqslant N_0$.

Proof. (i) Since $\{W_i\}_{i=1}^{\infty}$ is a frame of subspaces of $\mathcal{H}$, there exist constants C, D such that

$$C\|f\|^2 \leq \sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \leq D\|f\|^2, \quad f \in \mathcal{H}.$$

Let $\{W_i\}_{i=1}^{\infty}$ be an ultra Bessel sequence of subspaces. Let $0 < \varepsilon < C$. Then there exists $N > 0$ such that

$$\sum_{i=N+1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 < \varepsilon$$

for each $f \in \mathcal{H}$ with $\|f\| = 1$. Since $\dim(\mathcal{H}) = \infty$, there exists $f_0 \in \mathcal{H}$ such that

$$\|f_0\| = 1, \quad f_0 \perp \mathrm{span}(\cup_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{W_i}).$$

From other hand

(2.1)

$$C\|f\|^2 \leq \sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 = \sum_{i=1}^{N} v_i^2 \|\pi_{W_i}(f)\|^2 + \sum_{i=N+1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2.$$

If we replace f by f_0 in (2.1), then

$$C = C\|f_0\|^2 \leq \sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i}(f_0)\|^2 = \sum_{i=N+1}^{\infty} v_i^2 \|\pi_{W_i}(f_0)\|^2 < \varepsilon$$

and this contradiction proves (i).

(ii) Let $\dim\mathcal{H} = m$ and $\{e_1, ..., e_m\}$ be an orthonormal basis for $\mathcal{H}$.

$$\begin{aligned}
\sum_{i=1}^{\infty} v_i^2 &= \sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i}\|^2 = \sum_{i=1}^{\infty} \|v_i \pi_{W_i}\|^2 \\
&\leq \sum_{i=1}^{\infty} \|v_i \pi_{W_i}\|_2^2 \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{m} \|v_i \pi_{W_i}(e_j)\|^2 \\
&= \sum_{j=1}^{m} \sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i}(e_j)\|^2 \leq mD,
\end{aligned}$$

where $\|v_i \pi_{W_i}\|_2$ denotes the Hilbert-Schmidt norm of the operator $v_i \pi_{W_i}$. Thus by Proposition 1.5, $\{W_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces with respect to $\{v_i\}_{i=1}^{\infty}$.

Let $0 < \varepsilon < C$. Then there exists $N > 0$ such that

$$\sum_{i=N+1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 < \varepsilon \|f\|^2, \quad f \in \mathcal{H}. \tag{2.2}$$

From other hand, we have

$$C\|f\|^2 \leq \sum_{i=1}^{N} v_i^2 \|\pi_{W_i}(f)\|^2 + \sum_{i=N+1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2, \quad f \in \mathcal{H}.$$

So (2.2) implies that

$$(C - \varepsilon)\|f\|^2 \leq \sum_{i=1}^{N} v_i^2 \|\pi_{W_i}(f)\|^2 \leq D\|f\|^2, \quad f \in \mathcal{H}.$$

So for each $n \geq N$, $\{W_i\}_{i=1}^n$ is a frame of subspaces of $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^n$. Now, let N_0 be the minimum of the all such N. Then $\{W_i\}_{i=1}^n$ can not be a frame of subspaces for each $n < N$. □

Definition 2.2. Let $\mathcal{H}$ be a finite dimensional Hilbert space and $\{W_i\}_{i=1}^\infty$ be a frame of subspaces of $\mathcal{H}$. Then we call the number N_0 in the Theorem 2.1, the frameness bound of the frame of subspaces $\{W_i\}_{i=1}^\infty$.

Lemma 2.3. [3] Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces, and suppose that $U : \mathcal{K} \to \mathcal{H}$ is a bounded operator with closed range $\mathcal{R}_U$. Then there exists a bounded operator $U^\dagger : \mathcal{H} \to \mathcal{K}$ for which

$$UU^\dagger f = f, \quad \forall f \in \mathcal{R}_U.$$

If $\{W_i\}_{i=1}^\infty$ is a frame of subspaces for $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^\infty$, then

$$T_{W,v}(T_{W,v}^* S_{W,v}^{-1} f) = f, \quad f \in \mathcal{H},$$

so $T_{W,v}^\dagger = T_{W,v}^* S_{W,v}^{-1}$.

Theorem 2.4. Let $\mathcal{H}$ be a finite dimensional Hilbert space and $\{W_i\}_{i=1}^\infty$ be a frame of subspaces of $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^\infty$ an let N_0 be frameness bound of $\{W_i\}_{i=1}^\infty$. Let $n \geqslant N_0$ and S_n and T_n be the frame operator and synthesis operator of $\{W_i\}_{i=1}^n$, respectively. Then

(i) $S_n \to S_{W,v}$ in $B(\mathcal{H})$,
(ii) $T_n \to T_{W,v}$ in $B\Big((\sum_{i=1}^\infty \oplus W_i)_{\ell^2}, \mathcal{H}\Big)$ and $T_n^\dagger \to T_{W,v}^\dagger$ in $B\Big(\mathcal{H}, (\sum_{i=1}^\infty \oplus W_i)_{\ell^2}\Big)$.

Proof. (i) Let $n \geqslant N_0$ and $f \in \mathcal{H}$ with $\|f\| = 1$. Then

$$\begin{aligned}
\|S_n(f) - S_{W,v}(f)\| &= \Big\| \sum_{i=n+1}^{\infty} v_i^2 \pi_{W_i}(f) \Big\| \\
&= \sup_{\|g\|=1} \Big| \Big\langle \sum_{i=n+1}^{\infty} v_i^2 \pi_{W_i}(f), g \Big\rangle \Big| \\
&\leq \sup_{\|g\|=1} \Big(\sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(g)\|^2 \Big)^{\frac{1}{2}} \Big(\sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \Big)^{\frac{1}{2}} \\
&\leq \sqrt{B} \Big(\sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \Big)^{\frac{1}{2}}.
\end{aligned}$$

Since $\{W_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in $\mathcal{H}$, thus S_n converges to $S_{W,v}$ in $B(\mathcal{H})$.
(ii) Let $n \geqslant N_0$, $\{f_i\}_{i=1}^{\infty} \in \Big(\sum_{i=1}^{\infty} \oplus W_i \Big)_{\ell^2}$ with $\|\{f_i\}_{i=1}^{\infty}\| \leqslant 1$. Then

$$\begin{aligned}
\|T_n(\{f_i\}_{i=1}^{\infty}) - T_{W,v}(\{f_i\}_{i=1}^{\infty})\|^2 &= \Big\| \sum_{i=n+1}^{\infty} v_i f_i \Big\|^2 \\
&= \Big\| \sum_{i=n+1}^{\infty} v_i \pi_{W_i}(f_i) \Big\|^2 \\
&\leq \sum_{i=n+1}^{\infty} \|f_i\|^2 \sup_{\|g\|=1} \sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(g)\|^2 \\
&\leq \sup_{\|g\|=1} \sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(g)\|^2.
\end{aligned}$$

therefore $T_n \to T_{W,v}$ in $B\Big((\sum_{i=1}^{\infty} \oplus W_i)_{\ell^2}, \mathcal{H} \Big)$.
Now let $n \geqslant N_0$ and $f \in \mathcal{H}$ with $\|f\| \leqslant 1$. Then we have

$$\begin{aligned}
\sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(S_{W,v}^{-1}(f))\|^2 &= \|S_{W,v}^{-1}\|^2 \sum_{i=n+1}^{\infty} v_i^2 \Big\| \pi_{W_i} \Big(\frac{S_{W,v}^{-1}(f)}{\|S_{W,v}^{-1}\|} \Big) \Big\|^2 \\
&\leq \|S_{W,v}^{-1}\|^2 \sup_{\|g\| \leq 1} \sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(g)\|^2 \\
&\leq \|S_{W,v}^{-1}\|^2 \sup_{\|g\|=1} \sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(g)\|^2.
\end{aligned}$$

So by (i), for each $\varepsilon > 0$, there exists $M > N_0$ such that for all $f \in \mathcal{H}$ with $\|f\| \leqslant 1$ and all $n \geq M$,

$$\|S_n^{-1}(f) - S_{W,v}^{-1}(f)\| < (\varepsilon/2B)^{\frac{1}{2}}, \tag{2.3}$$

and

$$\sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(S_{W,v}^{-1}(f))\|^2 < \varepsilon/2. \tag{2.4}$$

From other hand we have

$$\begin{aligned}
&\sum_{i=1}^{n} v_i^2 \Big\|\pi_{W_i}\Big(S_n^{-1}(f) - S_{W,v}^{-1}(f)\Big)\Big\|^2 \\
&\qquad = \sum_{i=1}^{n} v_i^2 \Big\langle \pi_{W_i}\Big(S_n^{-1}(f) - S_{W,v}^{-1}(f)\Big), \pi_{W_i}\Big(S_n^{-1}(f) - S_{W,v}^{-1}(f)\Big)\Big\rangle \\
&\qquad = \Big\langle \sum_{i=1}^{n} v_i^2 \pi_{W_i}\Big(S_n^{-1}(f) - S_{W,v}^{-1}(f)\Big), S_n^{-1}(f) - S_{W,v}^{-1}(f)\Big\rangle \\
&\qquad \le \|S_n^{-1}(f) - S_{W,v}^{-1}(f)\| \Big\|\sum_{i=1}^{n} v_i^2 \pi_{W_i}\Big(S_n^{-1}(f) - S_{W,v}^{-1}(f)\Big)\Big\|. \qquad (2.5)
\end{aligned}$$

Also

$$\begin{aligned}
&\Big\|\sum_{i=1}^{n} v_i^2 \pi_{W_i}\Big(S_n^{-1}(f) - S_{W,v}^{-1}(f)\Big)\Big\| \\
&\qquad = \sup_{\|g\|=1} \Big|\Big\langle \sum_{i=1}^{n} v_i^2 \pi_{W_i}\Big(S_n^{-1}(f) - S_{W,v}^{-1}(f)\Big), g\Big\rangle\Big| \\
&\qquad = \sup_{\|g\|=1} \Big|\sum_{i=1}^{n} \Big\langle v_i^2 \pi_{W_i}\Big(S_n^{-1}(f) - S_{W,v}^{-1}(f)\Big), g\Big\rangle\Big| \\
&\qquad = \sup_{\|g\|=1} \Big|\sum_{i=1}^{n} \Big\langle v_i \pi_{W_i}\Big(S_n^{-1}(f) - S_{W,v}^{-1}(f)\Big), v_i \pi_{W_i}(g)\Big\rangle\Big| \\
&\qquad \le \sup_{\|g\|=1} \sum_{i=1}^{n} \Big\|v_i \pi_{W_i}\Big(S_n^{-1}(f) - S_{W,v}^{-1}(f)\Big)\Big\| \|v_i \pi_{W_i}(g)\| \\
&\qquad \le \Big(\sum_{i=1}^{n} v_i^2 \Big\|\pi_{W_i}\Big(S_n^{-1}(f) - S_{W,v}^{-1}(f)\Big)\Big\|^2\Big)^{1/2} \\
&\qquad\quad \sup_{\|g\|=1} \Big(\sum_{i=1}^{n} v_i^2 \|\pi_{W_i}(g)\|^2\Big)^{1/2} \\
&\qquad \le \sqrt{B}\Big(\sum_{i=1}^{n} v_i^2 \Big\|\pi_{W_i}\Big(S_n^{-1}(f) - S_{W,v}^{-1}(f)\Big)\Big\|^2\Big)^{1/2}. \qquad (2.6)
\end{aligned}$$

Now (2.5) and (2.6) imply that

$$\Big(\sum_{i=1}^{n} v_i^2 \Big\|\pi_{W_i}\Big(S_n^{-1}(f) - S_{W,v}^{-1}(f)\Big)\Big\|^2\Big)^{1/2} \leq \sqrt{B}\|S_n^{-1}(f) - S_{W,v}^{-1}(f)\|.$$

Consequently by (2.3) we have

$$\sum_{i=1}^{n} v_i^2 \Big\|\pi_{W_i}\Big(S_n^{-1}(f) - S_{W,v}^{-1}(f)\Big)\Big\|^2 < \varepsilon/2. \tag{2.7}$$

Finally, according to (2.4) and (2.7)

$$\|T_n^{\dagger}f - T_{W,v}^{\dagger}f\|^2 = \sum_{i=1}^{n} v_i^2 \|\pi_{W_i}\Big(S_n^{-1}(f) - S_{W,v}^{-1}(f)\Big)\|^2 + \sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(S_{W,v}^{-1}f)\|^2$$

yields that

$$\|T_n^{\dagger}f - T_{W,v}^{\dagger}f\| < \varepsilon, \quad f \in \mathcal{H}\ , \|f\| = 1,$$

so $T_n^{\dagger} \to T_{W,v}^{\dagger}$ in $B\Big(\mathcal{H}, (\sum_{i=1}^{\infty} \oplus W_i)_{\ell^2}\Big)$. □

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References

1. M. R. Abdollahpour and A. Shekari, *Ultra Bessel sequence of subspaces in Hilbert spaces*, preprint
2. P. G. Casazza and G. Kutynoik, *Frames of subspaces*, Wavelets, Frames and Operator Theory (College Park, MD, 2003), 87-113, Contemp. Math. 345, Amer. Math. Soc., Providence, RI, 2004.
3. O. Christencen, *An introduction to frames and Riesz bases*, Birkhauser, Boston, 2003.
4. M. H. Faroughi and A. Najati, *Ultra Bessel sequences in Hilbert Spaces*, Southeast Asian Bulletin of Mathematics, **32** (2008) 425-436.

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