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GENERAL MINKOWSKI TYPE AND RELATED INEQUALITIES FOR SEMINORMED FUZZY INTEGRALS

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Abstract. Minkowski type inequalities for the seminormed fuzzy integrals on abstract spaces are studied in a rather general form. Also related inequalities to Minkowski type inequality for the seminormed fuzzy integrals on abstract spaces are studied. Several examples are given to illustrate the validity of theorems. Some results on Chebyshev and Minkowski type inequalities are obtained.

1. Introduction

The theory of fuzzy measures and fuzzy integrals was introduced by Sugeno [28]. The properties and applications of the Sugeno integral have been studied by many authors, including Ralescu and Adams [18], Román-Flores et al. [19-26] and Wang and Klir [32]. Many authors generalized the Sugeno integral by using some other operators to replace the special operators $\wedge$ and/or $\vee$ (see, e.g., [10, 14, 16, 29, 31]). Suárez and Gill presented two families of fuzzy integrals, the so-called seminormed fuzzy integrals and semiconormed fuzzy integrals [27].
The study of inequalities for Sugeno integral was initiated by Román-flores et al. [21-26], and then followed by the authors (see [3-5, 11, 13]). In [6], a fuzzy Chebyshev inequality for a special case was obtained which has been generalized by Y. Ouyang and R. Mesiar. [14].
More precisely they proved the following Chebyshev type inequality for seminormed fuzzy integrals and related inequality for semiconormed fuzzy integrals.

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Theorem 1.1. Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : X \to [0,1]$ two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments. If the seminorm T satisfies

$$T(a \star b, c) \geq (T(a,c) \star b) \vee (a \star T(b,c)), \tag{1.1}$$

then

$$\int_{T,A} f \star g d\mu \geq \Big(\int_{T,A} f d\mu \Big) \star \Big(\int_{T,A} g d\mu \Big) \tag{1.2}$$

holds for any A $\in \mathcal{F}$.

Theorem 1.2. Let$(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : X \to [0,1]$ two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments. If the semiconorm S satisfies

$$S(a \star b, c) \leq (S(a,c) \star b) \wedge (a \star S(b,c)), \tag{1.3}$$

then

$$\int_{S,A} f \star g d\mu \leq \Big(\int_{S,A} f d\mu \Big) \star \Big(\int_{S,A} g d\mu \Big) \tag{1.4}$$

holds for any A $\in \mathcal{F}$.

Later on Agahi, Mesiar and Ouyang [2] proved a general Minkowski type inequality for comonotone functions and arbitrary fuzzy measure-based Sugeno integrals, and then they provided the inverse of this inequality for the same condition [15].

This paper is organized as follows: in Section 2, some preliminaries and summarization of some previous known results are given. Section 3 include general Minkowski type inequalities and the revers of those for seminormed fuzzy integrals. Finally, Section 4 contains a short conclusion.

2. Preliminaries

In this section, we recall some basic definition and previous results that will be used in the next section.

Let X be a non-empty set, $\mathcal{F}$ be a σ-algebra of subsets of X. Let $\mathbb{N}$ denote the set of all positive integers. Throughout this paper, all considered subsets are supposed to belong to $\mathcal{F}$.

Definition 2.1. (Sugeno [28]). A set function $\mu : \mathcal{F} \to [0,1]$ is called a fuzzy measure if the following properties are satisfied:

(FM1) $\mu(\emptyset) = 0$ and $\mu(X) = 1$;

(FM2) $A \subset B$ implies $\mu(A) \leq \mu(B)$;

(FM3) $A_n \to A$ implies $\mu(A_n) \to \mu(A)$.

When μ is a fuzzy measure, the triple $(X, \mathcal{F}, \mu)$ is called a fuzzy measure space.

Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, and $\mathcal{F}_+(X) = \{f | f : X \to [0,1]$ is measurable with respect to $\mathcal{F}\}$. In what follows, all considered functions belong to $\mathcal{F}_+(X)$. For any $\alpha \in [0,1]$, we will denote the set $\{x \in X | f(x) \geq \alpha\}$ by F_α and $\{x \in X | f(x) > \alpha\}$ by $F_{\bar{\alpha}}$. Clearly, both F_α and $F_{\bar{\alpha}}$ are nonincreasing with respect to α, i.e., $\alpha \leq \beta$ implies $F_\alpha \supseteq F_\beta$ and $F_{\bar{\alpha}} \supseteq F_{\bar{\beta}}$.

Definition 2.2. (Pap [17], Sugeno [29]). Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, and A $\in \mathcal{F}$, the Sugeno integral of f over A, with respect to the fuzzy measure μ, is defined by

$$\fint_A f d\mu = \bigvee_{\alpha \in [0,1]} (\alpha \wedge \mu(A \cap F_\alpha)).$$

When A=X, then

$$\fint_X f d\mu = \fint f d\mu = \bigvee_{\alpha \in [0,1]} (\alpha \wedge \mu(F_\alpha)).$$

Notice that Ralescu and Adams (see [18]) extended the range of fuzzy measures and the Sugeno integrals from [0,1] to [0,∞]. But we only deal with the original fuzzy measure and the Sugeno integrals which was introduced by Sugeno in 1974.

Note in the above definition, $\wedge$ is just the prototypical t-norm minimum and $\vee$ the prototypical t-conorm maximum.

A t-norm [8] is a function T : [0,1]×[0,1] → [0,1] satisfying the following conditions:

(A) T(x, 1) = T(1, x) = x $\quad \forall x \in [0,1]$;

(B) $\forall x_1, x_2, y_1, y_2$ in [0,1], if $x_1 \leq x_2, y_1 \leq y_2$, then $T(x_1, y_1) \leq T(x_2, y_2)$;

(C) T(x, y) = T(y, x);

(D) T(T(x, y), z) = T(x, T(y, z)).

A function $S : [0,1] \times [0,1] \to [0,1]$ is called a t-conorm [**?**], if there is a t-norm T such that $S(x, y) = 1 - T(1-x, 1-y)$.

Evidently, a t-conorm S satisfies:

(A') $S(x, 0) = S(0, x) = x, \quad \forall x \in [0,1]$ as well as conditions (B), (C) and (D).

A binary operator T (S) on [0,1] is called a t-seminorm (t-semiconorm) [27] if it satisfies the above condition (A) and(B) ((A') and (B)). By using the concepts of t-seminorm and t-semiconorm, Suárez and Gil proposed two families of fuzzy integrals:

Definition 2.3. Let T be a t-seminorm, then the seminormed fuzzy integral of f over A with respect to T and the fuzzy measure μ is defined by

$$\int_{T,\ A} f d\mu = \bigvee_{\alpha \in [0,1]} T(\alpha, \mu(A \cap F_\alpha)). \tag{2.1}$$

Definition 2.4. Let S be a t-semiconorm, then the semiconormed fuzzy integral of f over A with respect to S and the fuzzy measure μ is defined by

$$\int_{S,\ A} f d\mu = \bigwedge_{\alpha\in[0,1]} S(\alpha, \mu(A\cap F_{\bar{\alpha}})).$$

It is easy to see that the Sugeno integral is a special seminormed fuzzy integral. Moreover, Kandel and Byatt in (see [7]) showed another expression of the Sugeno integral as follows:

$$⨍_A f d\mu = \bigwedge_{\alpha\in[0,1]} (\alpha \vee \mu(A \cap F_{\bar{\alpha}})).$$

So the semiconormed fuzzy integrals also generalized the concept of the Sugeno integral. Notice that the seminormed fuzzy integral is just the family of the weakest universal fuzzy integrals . Note that if $\int_{T,A} f d\mu = a$, then $T(\alpha, \mu(A \cap F_\alpha)) \leq a$ for all $\alpha \in [0, 1]$ and, for $\varepsilon > o$ there exists α_ε such that $T(\alpha_\varepsilon, \mu(A \cap F_{\alpha_\varepsilon})) \geq a - \varepsilon$. Also, if $\int_{S,A} f d\mu = a$, then $S(\alpha, \mu(A \cap F_{\bar{\alpha}})) \geq a$ for all $\alpha \in [0, 1]$ and, for $\varepsilon > o$ there exists α_ε such that $S(\alpha_\varepsilon, \mu(A \cap F_{\bar{\alpha}_\varepsilon})) \geq a + \varepsilon$.

The following theorem is a general version of the Minkowski type inequality for Sugeno integral (See [2]).

Theorem 2.5. Let $f, g \in \mathcal{F}_+(X)$ and μ be an arbitrary fuzzy measure such that $⨍_A f \star g d\mu$ are finite. And let $\star : [0, \infty)^2 \to [0, \infty)$ be continuous and nondecreasing in both arguments and bounded from below by maximum. If f, g are comonotone, then the inequality

$$\Big(⨍_A (f\star g)^s d\mu\Big)^{\frac{1}{s}} \leq \Big(⨍_A f^s d\mu\Big)^{\frac{1}{s}} \star \Big(⨍_A g^s d\mu\Big)^{\frac{1}{s}} \tag{2.2}$$

holds for all $0 < s < \infty$.

In [15] Ouyang, et al. proved an inequality related to Minkowski type for Sugeno integrals. (with respect to a fuzzy measure in the sense of Ralescu and Adams [18]):

Theorem 2.6. Let $f, g \in \mathcal{F}_+(X)$ and μ be an arbitrary fuzzy measure such that both $⨍_A f d\mu$ and $⨍_A g d\mu$ are finite. And let $\star : [0, \infty)^2 \to [0, \infty)$ be continuous and nondecreasing in both arguments and bounded from above by minimum. If f, g are comonotone, then the inequality

$$\Big(⨍_A (f\star g)^s d\mu\Big)^{\frac{1}{s}} \geq \Big(⨍_A f^s d\mu\Big)^{\frac{1}{s}} \star \Big(⨍_A g^s d\mu\Big)^{\frac{1}{s}} \tag{2.3}$$

holds for all $0 < s < \infty$.

3. Main results

The classical Minkowski inequality was published by Minkowski [12, pp. 115-117] in his famous book 'Geometrie der Zahlen'.

In this section we prove general versions of Minkowski type inequalities and those related inequalities for seminormed fuzzy integrals.

Theorem 3.1. Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : X \to [0,1]$ two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments. If the seminorm T satisfies

$$T(a \star b, c) \leq (T(a,c) \star b) \wedge (a \star T(b,c)), \tag{3.1}$$

then the inequality

$$\Big(\int_{T,A} (f \star g)^s d\mu\Big)^{\frac{1}{s}} \leq \Big(\int_{T,A} f^s d\mu\Big)^{\frac{1}{s}} \star \Big(\int_{T,A} g^s d\mu\Big)^{\frac{1}{s}} \tag{3.2}$$

holds for any A $\in \mathcal{F}$ and for all $0 < s < \infty$.

Proof. Let $\int_{T,A} f^s d\mu = a$, $\int_{T,A} g^s d\mu = b$ and $\int_{T,A} (f \star g)^s d\mu = c$. Then for all $\alpha \in [0,1]$ we have $T(\alpha, \mu(A \cap F_{\alpha^{\frac{1}{s}}})) \leq a$, $T(\alpha, \mu(A \cap G_{\alpha^{\frac{1}{s}}})) \leq b$ and $T(\alpha, \mu(A \cap H_{\alpha^{\frac{1}{s}}})) \leq b$, where $H_\alpha = \{x \mid f(x) \star g(x) \geq \alpha\}$. Hence for any $\varepsilon > 0$, there exist a_ε, b_ε and $c_\varepsilon = a_\varepsilon \star b_\varepsilon$ such that $\mu(A \cap F_{(a_\varepsilon)^{\frac{1}{s}}}) = a_1$, $\mu(A \cap G_{(b_\varepsilon)^{\frac{1}{s}}}) = b_1$ and $\mu(A \cap H_{(c_\varepsilon)^{\frac{1}{s}}}) = c_1$, where $T(a_\varepsilon, a_1) \geq a - \varepsilon$, $T(b_\varepsilon, b_1) \geq b - \varepsilon$ and $T(c_\varepsilon, c_1) \geq c - \varepsilon$. (Thus $a_\varepsilon \leq a$, $b_\varepsilon \leq b$, $T(a_\varepsilon, a_1) \leq a$, and $T(b_\varepsilon, b_1) \leq b$). The fact of $F_{(a_\varepsilon)^{\frac{1}{s}}} \cup G_{(b_\varepsilon)^{\frac{1}{s}}} \supset H_{(a_\varepsilon)^{\frac{1}{s}} \star (b_\varepsilon)^{\frac{1}{s}}}$ and the comonotonicity of f, g imply that $\mu(A \cap H_{(a_\varepsilon)^{\frac{1}{s}} \star (b_\varepsilon)^{\frac{1}{s}}}) \leq a_1 \vee b_1$. Now we have

$$\begin{aligned}
c - \varepsilon &\leq T(c_\varepsilon, c_1) \\
&= T(a_\varepsilon \star b_\varepsilon, \mu(A \cap H_{(a_\varepsilon)^{\frac{1}{s}} \star (b_\varepsilon)^{\frac{1}{s}}}) \\
&\leq T(a_\varepsilon \star b_\varepsilon, a_1 \vee b_1) \\
&= T(a_\varepsilon \star b_\varepsilon, a_1) \vee T(a_\varepsilon \star b_\varepsilon, b_1) \\
&\leq [T(a_\varepsilon, a_1) \star b_\varepsilon] \vee [a_\varepsilon \star T(b_\varepsilon, b_1) \\
&\leq (a \star b_\varepsilon) \vee (a_\varepsilon \star b) \\
&\leq (a \star b) \vee (a \star b) \\
&= (a \star b).
\end{aligned}$$

Hence $c \leq a \star b$ follows from the arbitrariness of ε.
Consequently from the continuity of $\star$ we have
$c \leq (a^{\frac{1}{s}} \star b^{\frac{1}{s}})^s$.

This implies

$$\Big(\int_{T,A}(f\star g)^s d\mu\Big)^{\frac{1}{s}} \leq \Big(\int_{T,A} f^s d\mu\Big)^{\frac{1}{s}} \star \Big(\int_{T,A} g^s d\mu\Big)^{\frac{1}{s}}.$$

This completes the proof. □

Example 3.2. Let X = [0, 1] and the fuzzy measure μ be the Lebesgue measure and $\star$ be the usual product and T be defined as T(x,y) = max {x+y-1,0}. Let $f, g : X \to [0, 1]$ be tow comonotone functions defined by $f(x) = \frac{1}{4}$ and $g(x) = \sqrt{x}$. For s = 1 we have:

$$\int_{T,\ X} fdm = \sup_{\alpha\in[0,1]} T(\alpha, m([0,1]\cap\{x \mid \frac{1}{4} \geq \alpha\})) \ = \ \frac{1}{4}\ ,$$

$$\int_{T,\ X} gdm = \sup_{\alpha\in[0,1]} T(\alpha, m([0,1]\cap\{x \mid \sqrt{x} \geq \alpha\})) \ = \ \frac{1}{4}\ ,$$

and

$$\int_{T,\ X} f.gdm = \sup_{\alpha\in[0,1]} T(\alpha, m([0,1]\cap\{x \mid \frac{1}{4}\sqrt{x} \geq \alpha\})) \ = \ \frac{1}{64}\ ,$$

Therefore $\int_{T,\ X} f.gdm \ = \ \frac{1}{64} \ \leq \ (\int_{T,\ X} fdm)(\int_{T,\ X} gdm) \ = \ (\frac{1}{4})(\frac{1}{4})$.

If $(x \star 1) \wedge (1 \star x) \leq x$ for any $x \in [0, 1]$ and if T dominates $\star$, then (3.1) holds readily. Indeed, $T(a \star b, c) \leq T(a \star b, c \star 1) \leq T(a, 1) \star T(b, c) = a \star T(b, c)$ and $T(a \star b, c) \leq T(a, c) \star b$ follows similarly. It is well known that if $\star$ is bounded from blow by maximum, then $\star$ is dominated by maximum. Thus Corollary 3.3 gives us a general version of the Minkowski type inequality for Sugeno integral which appears in [2, Theorem 3.1].

Corollary 3.3. Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : X \to [0, 1]$ be two comonotone measurable functions. Let $\star : [0, 1]^2 \to [0, 1]$ be continuous and nondecreasing in both arguments and bounded from below by maximum. Then the inequality

$$\Big(⨍_A (f\star g)^s d\mu\Big)^{\frac{1}{s}} \leq \Big(⨍_A f^s d\mu\Big)^{\frac{1}{s}}\star\Big(⨍_A g^s d\mu\Big)^{\frac{1}{s}},$$

holds for any $A \in \mathcal{F}$ and for all $0 < s < \infty$.

Corollary 3.4. Let $f_1, f_2, ..., f_n$ be such that any two of them are comonotone and $\star$ be as in Theorem 3.1. Then the inequality

$$\Big(\int_{T,A}((...((f_1 \star f_2) \star f_3) \star ...) \star f_n)^s d\mu\Big)^{\frac{1}{s}} \leq$$

$$\Bigg(\Bigg(...\Big((\int_{T,A} f_1^s d\mu) \star (\int_{T,A} f_2^s d\mu)\Big)^{\frac{1}{s}}\Bigg) \star ...\Bigg) \star \Big(\int_{T,A} f_n^s d\mu\Big)^{\frac{1}{s}},$$

holds for any $A \in \mathcal{F}$ and for all $0 < s < \infty$.

Proof. Sincef_1, f_2 are comonotone, then by Theorem 3.1, we have

$$\Big(\int_{T,\ A}(f_1 \star f_2)^s d\mu\Big)^{\frac{1}{s}} \leq \Big(\int_{T,\ A} f_1^s d\mu\Big)^{\frac{1}{s}} \star \Big(\int_{T,\ A} f_2^s d\mu\Big)^{\frac{1}{s}}. \tag{3.3}$$

Moreover, using (3.3) and comonotonicity of $f_1 \star f_2$ and f_3 we obtain

$$\begin{aligned}\Big(\int_{T,A}((f_1{\star}f_2) \star f_3)^s d\mu\Big)^{\frac{1}{s}} &\leq \Big(\int_{T,A}(f_1 \star f_2)^s d\mu\Big)^{\frac{1}{s}} \star \big(\int_{T,A} f_3^s d\mu\big)^{\frac{1}{s}} \\ &\leq \Big(\big(\int_{T,A} f_1^s d\mu\big)^{\frac{1}{s}} \star \big(\int_{T,A} f_2^s d\mu\big)^{\frac{1}{s}}\Big) \star \big(\int_{T,A} f_3^s d\mu\big)^{\frac{1}{s}}\Big).\end{aligned}$$

Thus we can prove the conclusion by induction. □

Theorem 3.5. Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : X \to [0,1]$ two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments. If the seminorm T satisfies

$$T(a \star b, c) \geq (T(a, c) \star b) \vee (a \star T(b, c)), \tag{3.4}$$

then the inequality

$$\Big(\int_{T,A}(f{\star}g)^s d\mu\Big)^{\frac{1}{s}} \geq \Big(\int_{T,A} f^s d\mu\Big)^{\frac{1}{s}}{\star}\Big(\int_{T,A} g^s d\mu\Big)^{\frac{1}{s}} \tag{3.5}$$

holds for any A $\in \mathcal{F}$ and for all $0 < s < \infty$.

Proof. Let $\int_{T,A} f^s d\mu = a$ and $\int_{T,A} g^s d\mu = b$. Then $T(\alpha, \mu(A \cap F_{\alpha^{\frac{1}{s}}})) \leq a$ and $T(\alpha, \mu(A \cap G_{\alpha^{\frac{1}{s}}})) \leq b$ for all $\alpha \in [0,1]$. Hence for any $\varepsilon > 0$, there exist a_ε and b_ε such that $T(a_\varepsilon, \mu(A \cap F_{(a_\varepsilon)^{\frac{1}{s}}})) \geq a - \varepsilon$ and $T(b_\varepsilon, \mu(A \cap G_{(b_\varepsilon)^{\frac{1}{s}}})) \geq b - \varepsilon$. If we let $\mu(A \cap F_{(a_\varepsilon)^{\frac{1}{s}}}) = a_1$ and $\mu(A \cap G_{(b_\varepsilon)^{\frac{1}{s}}}) = b_1$, then we have $T(a_\varepsilon, a_1) \geq a - \varepsilon$ and $T(b_\varepsilon, b_1) \geq b - \varepsilon$. The fact of $F_{(a_\varepsilon)^{\frac{1}{s}}} \cap G_{(b_\varepsilon)^{\frac{1}{s}}} \subset H_{(a_\varepsilon)^{\frac{1}{s}} \star (b_\varepsilon)^{\frac{1}{s}}}$ and the comonotonicity of f, g imply that $\mu(A \cap H_{(a_\varepsilon)^{\frac{1}{s}} \star (b_\varepsilon)^{\frac{1}{s}}}) \geq a_1 \wedge b_1$, where $H_\alpha = \{x \mid f(x) \star g(x) \geq \alpha\}$.

Hence

$$\begin{aligned}\int_{T,\ A}(f\star g)^s d\mu &= \sup_{\alpha\in[0,1]} T(\alpha,\mu(A\cap H_{\alpha^{\frac{1}{s}}}))\\ &\geq T(a_\varepsilon\star b_\varepsilon, a_1\wedge b_1)\\ &= T(a_\varepsilon\star b_\varepsilon, a_1)\wedge T(a_\varepsilon\star b_\varepsilon, b_1)\\ &\geq [T(a_\varepsilon, a_1)\star b_\varepsilon]\wedge[a_\varepsilon\star T(b_\varepsilon, b_1)]\\ &\geq [(a-\varepsilon)\star b_\varepsilon]\wedge[a_\varepsilon\star(b-\varepsilon)]\\ &\geq (a-\varepsilon)\star(b-\varepsilon).\end{aligned}$$

From the continuity of $\star$ and the arbitrariness of ε, we obtain that $\int_{T,A}(f\star g)^s d\mu \geq a\star b = (a^{\frac{1}{s}}\star b^{\frac{1}{s}})^s$.
Consequently

$$\Big(\int_{T,\ A}(f\star g)^s d\mu\Big)^{\frac{1}{s}} \geq \Big(\int_{T,\ A} f^s d\mu\Big)^{\frac{1}{s}}\star\Big(\int_{T,\ A} g^s d\mu\Big)^{\frac{1}{s}}.$$

This finishes the proof. □

Example 3.6. Let X = [0, 1] and μ be the Lebesgue measure. And let

$$f(x)=\begin{cases} x & x\in[0,\frac{1}{4}],\\ \frac{1}{2} & x\in(\frac{1}{4},\frac{1}{2}),\\ x & x\in[\frac{1}{2},1],\end{cases}$$

and

$$g(x)=\begin{cases} x^2 & x\in[0,\frac{1}{2}),\\ x & x\in[\frac{1}{2},1],\end{cases}$$

if $\star$ and T be the usual product on [0, 1]. Then for s = 2 we have

$$\int_{T,\ X} f^2 dm = \bigvee_{\alpha\in[0,1]}(\alpha.m([0,1]\cap F_{\alpha^{\frac{1}{2}}})) = \frac{4}{27},$$

$$\int_{T,\ X} g^2 dm = \bigvee_{\alpha\in[0,1]}(\alpha.m([0,1]\cap G_{\alpha^{\frac{1}{2}}})) = \frac{4}{27},$$

and

$$\int_{T,\ X}(f.g)^2 dm = \bigvee_{\alpha\in[0,1]}(\alpha.m([0,1]\cap H_{\alpha^{\frac{1}{2}}})) = \frac{4^4}{5^5}$$

where $H_\alpha = \{x \mid f(x).g(x)\geq\alpha\}$. Therefore

$$(\frac{4^4}{5^5})^{\frac{1}{2}} = \Big(\int_{T,\ X}(f.g)^2 dm\Big)^{\frac{1}{2}} \geq \Big(\int_{T,\ X} f^2 dm\Big)^{\frac{1}{2}}.\Big(\int_{T,\ X} g^2 dm\Big)^{\frac{1}{2}} = (\frac{4}{27})^{\frac{1}{2}}.(\frac{4}{27})^{\frac{1}{2}}.$$

Example 3.7. Let $f, g : X \to [0,1]$ be two comonotone functions. If $\star = \vee|_{[0,1]^2}$, then the inequality

$$\Big(\int_{T,\ A}(f\vee g)^s d\mu\Big)^{\frac{1}{s}} = \Big(\int_{T,\ A} f^s d\mu\Big)^{\frac{1}{s}} \vee \Big(\int_{T,\ A} g^s d\mu\Big)^{\frac{1}{s}}$$

holds for any A $\in \mathcal{F}$ and for all $0 < s < \infty$.

If $(x\star 1)\vee(1\star x) \leq x$ for any $x \in [0,1]$ and if T dominates $\star$, then (3.4) holds readily. Indeed, $T(a\star b, c) \geq T(a\star b, c\star 1) \geq T(a,1)\star T(b,c) = a \ \star \ T(b,c)$ and $T(a\star b, c) \geq T(a,c)\star b$ follows similarly. It is well known that if $\star$ is bounded from above by minimum, then $\star$ is dominated by minimum. Thus the following result holds. (Compare with Theorem 2.6)

Corollary 3.8. Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : X \to [0,1]$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from above by minimum. Then, the inequality

$$\Big(\fint_{T,A}(f\star g)^s d\mu\Big)^{\frac{1}{s}} \geq \Big(\fint_{T,A} f^s d\mu\Big)^{\frac{1}{s}} \star \Big(\fint_{T,A} g^s d\mu\Big)^{\frac{1}{s}},$$

holds for any $A \in \mathcal{F}$ and for all $0 < s < \infty$.

In the previous corollary, let us consider s = 1, so we get the Chebyshev type inequality for seminormed fuzzy integrals, that is proved by Ouyang and Mesiar in [14].

Corollary 3.9. Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : X \to [0,1]$ two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments. If the seminorm T satisfies
$T(a\star b, c) \geq (T(a,c)\star b)\vee(a\star T(b,c))$,
then

$$\int_{T,A} f\star g d\mu \geq \Big(\int_{T,A} f d\mu\Big)\star\Big(\int_{T,A} g d\mu\Big)$$

holds for any A $\in \mathcal{F}$.

In the following theorem we have proved a new general extension of Chebyshev type inequality for the seminormed fuzzy integrals. Agahi et al. have obtained similarly inequality for Sugeno integrals [1].

Theorem 3.10. Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : X \to [0,1]$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments, and $\varphi : [0,1] \to [0,1]$ be a continuous and strictly increasing function such that φ commutes whit $\star$. If the seminorm T satisfies

$$T(a\star b, c) \geq (T(a,c)\star b)\vee(a\star T(b,c)),$$

then the inequality

$$\varphi^{-1}\Big(\int_{T,A}\varphi(f\star g)d\mu\Big) \geq \varphi^{-1}\Big(\int_{T,A}\varphi(f)d\mu\Big)\star\ \varphi^{-1}\Big(\int_{T,A}\varphi(g)d\mu\Big) \tag{3.6}$$

holds for any A $\in \mathcal{F}$.

Proof. Since φ commutes whit $\star$, then we have

$$\int_{T,A}\varphi(f\star g)d\mu = \int_{T,A}\big(\varphi(f)\star\varphi(g)\big)d\mu. \tag{3.7}$$

If f, g are comonotone functions and φ is a continuous and strictly increasing function, then $\varphi(f)$ and $\varphi(g)$ are also comonotone. From (3.7) and using the Corollary 3.9, we have

$$\begin{aligned}\int_{T,A}\big(\varphi(f)\star\varphi(g)\big)d\mu &\geq \big(\int_{T,A}\varphi(f)d\mu\big)\star\big(\int_{T,A}\varphi(g)d\mu\big)\\ &= \varphi\Big[\,\varphi^{-1}\big(\int_{T,A}\varphi(f)d\mu\big)\star\;\varphi^{-1}\big(\int_{T,A}\varphi(g)d\mu\big)\Big]\end{aligned}$$

where φ commutes with $\star$. Hence (3.6) is valid.

Corollary 3.11. Let $f_1, f_2, ..., f_n$ be such that any two of them are comonotone and be as in Theorem 3.10. Then

$$\varphi^{-1}\Big(\int_{T,A}\varphi((...((f_1\star f_2)\star f_3)...)\star f_n)d\mu\Big)$$

$$\geq \Big(\Big(...\Big(\varphi^{-1}\Big(\int_{T,A}\varphi(f_1)d\mu\Big)\star\varphi^{-1}\Big(\int_{T,A}\varphi(f_2)d\mu\Big)\Big)\star...\Big)\star\varphi^{-1}\Big(\int_{T,A}\varphi(f_n)d\mu\Big)\Big)$$

holds for any A $\in\mathcal{F}$.

Proof. The proof is easy by using of the Theorem 3.10 and induction. □

4. Conclusion

We have proved general Minkowski type inequalities for seminormed fuzzy integrals on an abstract fuzzy measure space $(X, \mathcal{F}, \mu)$ based on a product like operator $\star$, and a related inequalities for those. As we have seen, this inequality is related to Minkowski type one and chebyshev type one. For further investigation we propose to consider the Minkowski inequality for the semiconormed fuzzy integral. Also other Minkowski-like inequalities can be generalize for seminormed fuzzy integral on an abstract space.

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