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EXISTENCE/UNIQUENESS OF SOLUTIONS TO HEAT EQUATION IN EXTENDED COLOMBEAU ALGEBRA

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Abstract. This work concerns the study of existence and uniqueness to heat equation with fractional Laplacian differentiation in extended Colombeau algebra.

1. Introduction

The fractional Laplacian and the fractional derivative are two different mathematical concepts (Samko et al, 1987). Both are defined through a singular convolution integral, but the former is guaranteed to be the positive definition via the Riesz potential as the standard Laplace operator, while the latter via the Riemann-Liouville integral is not. It is noted that the fractional Laplacian can not be interpreted by the fractional derivative in the sense of either Riemann-Liouville or Caputo. Both the fractional Laplacian and the fractional derivative have found applications in many complicated engineering problems. In particular, the fractional Laplacian attracts new attentions in recent years owing to its unique capability describing anomalous diffusion problems (Hanyga, 2001). Recently the fractional Laplacians attract much interest in nonlinear analysis. Caffarelli and Silvestre have given a new formulation of the fractional Laplacians through Dirichlet-Neumann maps. The reason for introducing fractional derivatives into algebra of generalized functions was the possibility of solving nonlinear problems with singularities and derivatives of arbitrary real order in it ([6]). We use an algebra of generalized functions which will be an extension of the Colombeau algebra in a sense of extension of fractional derivatives. Fractional calculus

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is a generalization of ordinary differentiation and integration to arbitrary non-integer order. Moreover fractional processes have witnessed an increasing development in the last decade. For instance, they are suitable for describing the long memory properties of many time series. Colombeau algebras (usually denoted by the letter $\mathcal{G}$) are differential (quotient) algebras with unit, and were introduced by J. F. Colombeau (cf.[1],[2],[3]) as a nonlinear extension of distribution theory to deal with nonlinearities and singularities in PDE theory. These algebras contain the space of distributions $\mathcal{D}'$ as a subspace with an embedding realized through convolution with a suitable mollifier. Elements of these algebras are classes of nets of smooth functions.
The fractional calculus by application of distributed order PDEs in Colombeau algebra was started by [6].
Fractional derivatives were considered in [4] where the use of Caputo and Riemann-Liouville derivatives of a Colombeau generalized process is proved. The paper is organized as follows. After the introduction some basic preliminaries such as notation and definitions of the used objects are given. Also the spaces of Colombeau generalized functions are introduced. In addition, imbedding the Laplacian fractional derivative into extended Colombeau algebra of generalized functions is shown. Finally, the existence-uniqueness result for a nonlinear heat equation is proven. Furthermore an equation driven by the fractional derivative of delta distribution is certified. This means the equation illustrates the application of the theory in a framework of the extended algebra of generalized functions. Besides moderateness and the negligibility for entire and fractional derivatives are clarified.

2. Colombeau algebra

First the definitions of some generalized function algebras of Colombeau type are mentioned which are as follows.
The elements of Colombeau algebras $\mathcal{G}$ are equivalent classes of regularizations, i.e., sequences of smooth functions satisfying asymptotic conditions in the regularization parameter ϵ.
Therefore, for any set X, the family of sequences $(u_\epsilon)_\epsilon \in (0,1]$ of elements of a set X will be denoted by $X^{(0,1]}$; such sequences will also be called nets and simply written as u_ϵ.
The algebra of generalized functions on equals $\mathcal{G}(\Omega)$, where Ω is an open set, is defined $\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$ where

$$\mathcal{E}_M(\Omega) = \left\{ (u_\epsilon)_\epsilon \in (C^\infty(\Omega))^{(0,1]} \;\middle|\; \begin{array}{c} \forall K \subset\subset \Omega, \forall \alpha \in \mathbb{N}_0^n \; \exists N \in \mathbb{N}, s.t. \\ \sup_{x\in K} |\partial^\alpha u_\epsilon(x)| = O(\epsilon^{-N}),\; \epsilon \to 0 \end{array} \right\}$$

and

$$N(\Omega)=\left\{(u_\epsilon)_\epsilon\in(C^\infty(\Omega))^{(0,1]}\;\middle|\;\begin{array}{c}\forall K\subset\subset\Omega,\forall\alpha\in\mathbb{N}_0^n\;\forall s\in\mathbb{N},\;s.t.\\ \sup_{x\in K}|\partial^\alpha u_\epsilon(x)|=O(\epsilon^s),\;\epsilon\to 0\end{array}\right\}.$$

Element of $\mathcal{E}_M(\Omega)$ and $\mathcal{N}(\Omega)$ are called moderate, negligible functions, respectively. Families $(r_\epsilon)_\epsilon$ of complex numbers such as $|r_\epsilon|=O(\epsilon^{-p})$ as $\epsilon\to 0$ for some $p\geq 0$ are called moderate, in which $|r_\epsilon|=O(\epsilon^q)$ for every $q\geq 0$ are termed negligible. The ring $\tilde{\mathbb{R}}$ of Colombeau generalized numbers is obtained by factoring moderate families of complex numbers with respect to negligible families.
The definition of extended Colombeau algebras of generalized functions on open subset of Ω is in a sense of extension of the entire derivatives to the fractional ones. Let $\mathcal{E}^e(\Omega)$ be an algebra of all sequences $(u_\epsilon)_{\epsilon>0}$ of real valued smooth functions $u_\epsilon\in C^\infty(\Omega)$. Suppose that

$$E^e_M(\Omega)=\left\{(u_\epsilon)_\epsilon\in\mathcal{E}^e(\Omega)\;\middle|\;\begin{array}{c}\forall K\subset\subset\Omega,\forall\alpha\in\mathbb{R}_+\,\cup\,\{0\},\exists N\geq 0,\;s.t.\\ \sup_{x\in K}|D^\alpha u_\epsilon(x)|=O(\epsilon^{-N})\;\;\epsilon\to 0\end{array}\right\},$$

and

$$N^e(\Omega)=\left\{(u_\epsilon)_\epsilon\in\mathcal{E}^e(\Omega)\;\middle|\;\begin{array}{c}\forall K\subset\subset\Omega,\forall\alpha\in\mathbb{R}_+\,\cup\,\{0\},\forall s\geq 0,\;s.t.\\ \sup_{x\in K}|D^\alpha u_\epsilon(x)|=O(\epsilon^{s})\;\;\epsilon\to 0\end{array}\right\}.$$

Where $D^\alpha u_\epsilon(x)$ is the Caputo fractional derivative. The extended Colombeau algebra of generalized functions is the set $\mathcal{G}^e(\Omega)=\mathcal{E}^e_M(\Omega)/\mathcal{N}^e(\Omega)$.

The new definition of extended Colombeau algebra is based on the ratio of spatial variable x. Moreover for a fractional derivative in the Laplacian sense is used. An interval $\Omega=(-\infty,\infty)$, and for PDEs the derivative (w.r.) to spatial variable x in the domain $\Omega=((0,T]\times\mathbb{R})$ is considered. The Colombeau algebra generalized functions is the set $\mathcal{G}^e_{L^\infty}(\Omega)=\mathcal{E}^e_{M,L^\infty}(\Omega)/\mathcal{N}^e_{L^\infty}(\Omega)$ where

$$\mathcal{E}^e_{M,L^\infty}(\Omega)=\left\{(u_\epsilon)_\epsilon\in\mathcal{E}^e(\Omega)\;\middle|\;\begin{array}{c}\forall\gamma\in(0,2),\exists N\geq 0,\;s.t.\\ \|(-\Delta)^{\frac{\gamma}{2}}u_\epsilon(x)\|_{L^\infty(\Omega)}=O(\epsilon^{-N})\;as\;\epsilon\to 0\end{array}\right\},$$

and

$$N^e_{L^\infty}(\Omega)=\left\{(u_\epsilon)_\epsilon\in\mathcal{E}^e(\Omega)\;\middle|\;\begin{array}{c}\forall\gamma\in(0,2),\forall s\geq 0,\;s.t.\\ \|(-\Delta)^{\frac{\gamma}{2}}u_\epsilon(x)\|_{L^\infty(\Omega)}=O(\epsilon^{s})\;as\;\epsilon\to 0\end{array}\right\}.$$

Imbedding the fractional derivatives (w.r.) to the spatial variable is given by the convolution of the fractional Laplacian is shown with the map:
$i_{frac}:w\to[\widetilde{(-\Delta)}^{\frac{\gamma}{2}}(w_\epsilon)_{\epsilon>0}]=[(-\Delta)^{\frac{\gamma}{2}}(w_\epsilon*\varphi_\epsilon(x))_{\epsilon>0}]$, where w_ϵ is a representative for the entire derivative,
where $\varphi(x)\in C_0^\infty(\mathbb{R}),\varphi(x)\geq 0,\int\varphi(x)dx=1,\int x^\alpha\varphi(x)=0,\forall\alpha\in$

$\mathbb{N}, |\alpha| > 0$.

3. Imbedding of The fractional Laplacian into extended Colombeau algebra of generalized functions

The fractional Laplacian $(-\Delta)^{\frac{\gamma}{2}}$ commutes with the primary coordination transformations in the Euclidean space $\mathbb{R}^d$, and has tight link to splines, fractals and stable Levy processes.

The Riesz derivative is a complementary operator to the well known Riesz derivative given explicitly by Samko [5], in the d-dimensional case, as follows:

$$(-\Delta)_d^{\frac{\gamma}{2}} f(x) = \frac{-\Gamma[(d-2+\gamma)/2)]}{\pi^{(2-\gamma)/2} 2^{2-\gamma} \Gamma[(2-\gamma)/2]} \int \frac{\Delta f(\xi) d\xi}{|x-\xi|^{d-2+\gamma}},$$

where $0 < \gamma < 2$.

Consider fractional Laplacian , defined for the Colombeau representative $f_\epsilon(x)$. Fractional Laplacian of order $0 < \gamma < 2$,

$$(-\Delta)_d^{\frac{\gamma}{2}} f(x) = \frac{-\Gamma[(d-2+\gamma)/2)]}{\pi^{(2-\gamma)/2} 2^{2-\gamma} \Gamma[(2-\gamma)/2]} \int \frac{\Delta f(\xi) d\xi}{|x-\xi|^{d-2+\gamma}}.$$

Use the following regularization for $0 < \gamma < 2$,

$$\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} f_\epsilon(x) = \frac{-\Gamma[(d-2+\gamma)/2)]}{\pi^{(2-\gamma)/2} 2^{2-\gamma} \Gamma[(2-\gamma)/2]} \int \frac{\Delta f_\epsilon(\xi)}{|x-\xi|^{d-2+\gamma}} \varphi_\epsilon(x-\xi-t) d\xi dt.$$

Where $f_\epsilon(x)$ is a representative for $f(x)$ in extended Colombeau algebra $\mathcal{G}^e([0,T] \times \mathbb{R})$. The convolution form is given by

$$\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} f_\epsilon(x) = \frac{-\Gamma[(d-2+\gamma)/2)]}{\pi^{(2-\gamma)/2} 2^{2-\gamma} \Gamma[(2-\gamma)/2]} (\Delta f_\epsilon(x) * (|x|^{-d+2-\gamma} * \varphi_\epsilon(x))).$$

We indicate $|\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} f_\epsilon(x) - (-\Delta)_d^{\frac{\gamma}{2}} f_\epsilon(x)| \approx 0$.

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} |\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} f_\epsilon(x) - (-\Delta)_d^{\frac{\gamma}{2}} f_\epsilon(x)| \\
&\quad \frac{-\Gamma[(d-2+\gamma)/2)]}{\pi^{(2-\gamma)/2} 2^{2-\gamma} \Gamma[(2-\gamma)/2]} \sup_{x\in\mathbb{R}} |\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} f_\epsilon(x) - (-\Delta)_d^{\frac{\gamma}{2}} f_\epsilon(x)| \\
&\quad = \frac{-\Gamma[(d-2+\gamma)/2)]}{\pi^{(2-\gamma)/2} 2^{2-\gamma} \Gamma[(2-\gamma)/2]} \sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} \Delta f_\epsilon(x) t^{2-\gamma-d} |\varphi_\epsilon(t) - \delta(t)| \longrightarrow 0,
\end{aligned}$$

as $\epsilon \longrightarrow 0$. Since $\lim_{\epsilon\longrightarrow 0} |\varphi_\epsilon(t) - \delta(t)| \longrightarrow 0$, Then $\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} f_\epsilon(x) \approx (-\Delta)_d^{\frac{\gamma}{2}} f_\epsilon(x)$.

Using the fact that $\varphi_\epsilon(t)$ has the compact support on $[-x,x]$, so by Holder inequalities, have the following calculations:

$$
\begin{aligned}
\sup_{x\in\mathbb{R}}|\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} f_\epsilon(x)| &\leq \frac{-\Gamma[(d-2+\gamma)/2]}{\pi^{(2-\gamma)/2}2^{2-\gamma}\Gamma[(2-\gamma)/2]}\int_{-\infty}^{\infty}\Delta f_\epsilon(t)(|t|^{2-\gamma-d}*\varphi_\epsilon(t))dt \\
&= C_{\gamma,d}\int_{-\infty}^{\infty}\Delta f_\epsilon(t)\int_{-\infty}^{\infty}|t-h|^{2-\gamma-d}\varphi_\epsilon(h)dhdt \\
&= C_{\gamma,d}\int_{-\infty}^{\infty}\Delta f_\epsilon(t)\int_{-\infty}^{\infty}|t-\epsilon p|^{2-\gamma-d}\phi(p)dpdt \\
&\leq C_{\gamma,d}\int_{-\infty}^{\infty}\Delta f_\epsilon(t)\sup_{p\in[-x,x]}\phi(p)\int_{-x}^{x}|t-\epsilon p|^{2-\gamma-d}dpdt \\
&\leq C_{\gamma,d}\int_{-\infty}^{\infty}\Delta f_\epsilon(t)\sup_{p\in[-x,x]}\phi(p)\frac{1}{\epsilon}\int_{t-\epsilon x}^{t+\epsilon x}|k|^{2-\gamma-d}dkdt \\
&\leq C_{\gamma,d}\sup_{t\in\mathbb{R}}\Delta f_\epsilon(t)\sup_{p\in[-x,x]}\phi(p)\int_{-x}^{x}\frac{1}{\epsilon}\int_{t-\epsilon x}^{t+\epsilon x}|k|^{2-\gamma-d}dkdt \\
&\leq C_{\gamma,d}\sup_{t\in\mathbb{R}}\Delta f_\epsilon(t)\sup_{p\in[-x,x]}\phi(p)\int_{-x}^{x}\frac{1}{\epsilon}\frac{1}{3-\gamma-d}(|t+\epsilon x|^{3-\gamma-d}-|t-\epsilon x|^{3-\gamma-d})dt \\
&\leq C_{\gamma,d}\sup_{t\in\mathbb{R}}\Delta f_\epsilon(t)\sup_{p\in[-x,x]}\phi(p)\frac{1}{\epsilon^2}\frac{1}{(4-\gamma-d)(3-\gamma-d)} \\
&\quad\times(|t+\epsilon x|^{3-\gamma-d}-|t-\epsilon x|^{3-\gamma-d})|_{-x}^{x} \\
&\leq C_{\gamma,d}\sup_{t\in\mathbb{R}}\Delta f_\epsilon(t)\sup_{p\in[-x,x]}\phi(p)\frac{1}{\epsilon^2}\frac{1}{(4-\gamma-d)(3-\gamma-d)}C'_{\gamma,d}\epsilon^{4-\gamma-d}X^{4-\gamma-d} \\
&\leq C_{\gamma,d,\phi}\frac{1}{(4-\gamma-d)(3-\gamma-d)}\sup_{t\in\mathbb{R}}\Delta f_\epsilon(t)\epsilon^{-N}X^{4-\gamma-d} \\
&\leq C_{\gamma,d,\phi}\epsilon^{-N}X^{4-\gamma-d},\ 0<\gamma<2.
\end{aligned}
$$

Since $x<X,\ X>0$ and $\Delta f_\epsilon(x)$ is of the moderate class.

$$\sup_{x\in\mathbb{R}}|\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} f_\epsilon(x)| \leq C_{\gamma,d,\phi}\epsilon^{-N}X^{4-\gamma-d},\ \ 0<\gamma<2.$$

3.1.Imbedding of the heat equation into extended Colombeau algebra of generalized functions

We consider the existence and uniqueness result for a nonlinear parabolic heat equation driven by the fractional derivative of the delta distribution in the extended algebra of generalized functions:

$$\partial_t u(t,x)=\Delta u(t,x)+g(u(t,x)),\quad u(0,x)=u_0(x)=V(x)=\delta(x),$$

where $g(u)\in L^\infty_{loc}([0,T),\mathbb{R}^n)$ and the following regularization for delta distribution will be used:

$$u_{0\epsilon}(x)=|\ln\epsilon|^{an}\phi(x.|\ln\epsilon|),\quad \|\nabla g_\epsilon(u_\epsilon)\|_{L^\infty}\leq(|\ln\epsilon|)^b,\ 0<a,b<1,$$

where $\phi(x) \in C_0^\infty(\mathbb{R}^n)$, $\phi(x) \geq 0$, $\int \phi(x)dx = 1$.

Theorem 3.1. Regularized equation to heat equation

$$\partial_t u_\epsilon(t,x) = \Delta u_\epsilon(t,x) + g_\epsilon(u_\epsilon(t,x)), \quad u_{0\epsilon} = \delta_\epsilon(x), \tag{3.1}$$

has a unique solution in the space $\mathcal{G}^e_{L^\infty}([0,T) \times \mathbb{R}^n)$.

Proof. The integral form of the equation (3.1)

$$\begin{aligned} u_\epsilon(t,x) &= E_{n\epsilon}(t,x) * u_{0\epsilon}(x) \\ &+ \int_0^t \int_{\mathbb{R}^n} E_{n\epsilon}(t-\tau, x-y) g_\epsilon(u_\epsilon(\tau,x)) dy d\tau, \quad t \in [0,T],\, x \in \mathbb{R}^n, \end{aligned}$$

where E_n is the heat kernel. By the Holder's inequality,

$$\begin{aligned} \|u_\epsilon(t,\cdot)\|_{L^\infty} &\leq \|E_{n\epsilon}(t, x-\cdot)\|_{L^1} \|u_{0\epsilon}\|_{L^\infty} \\ &\quad + \int_0^t \|E_{n\epsilon}(t-\tau, x-\cdot)\|_{L^1} \|\nabla g_\epsilon(\theta u_\epsilon)\|_{L^\infty} \|u_\epsilon(\tau,\cdot)\|_{L^\infty} d\tau. \end{aligned}$$

$$\|u_\epsilon(t,\cdot)\|_{L^\infty} \leq C|\ln \epsilon|^{an} + \int_0^t C(|\ln \epsilon|)^b \|u_\epsilon(\tau,\cdot)\|_{L^\infty} d\tau.$$

By Gronwall inequality

$$\begin{aligned} \|u_\epsilon(t,\cdot)\|_{L^\infty} &\leq C|\ln \epsilon|^{an} \exp(CT(|\ln \epsilon|)^b) \\ &\leq C\epsilon^{-N},\ \exists N > 0,\ x \in \mathbb{R}^n,\ t \in [0,T], \epsilon \in (0,1]. \end{aligned}$$

For the first derivative we have

$$\begin{aligned} \partial_x u_\epsilon(t,x) &= \int E_{n\epsilon}(t, x-y) \partial_x u_{0\epsilon}(y) dy \\ &\quad + \int_0^t \int \partial_x E_{n\epsilon}(t-\tau, x-y) \nabla g_\epsilon(\theta u_\epsilon) u_\epsilon(\tau, y) dy d\tau. \end{aligned}$$

So,

$$\|\partial_x u_\epsilon(t.\cdot)\|_{L^\infty} \leq C|\ln \epsilon|^{an-1} + C\int_0^t (|\ln \epsilon|)^b \|u_\epsilon(\tau,\cdot)\|_{L^\infty} d\tau.$$

Using the moderateness of $u_\epsilon(t,x)$,

$$\begin{aligned} \|\partial_x u_\epsilon(t.\cdot)\|_{L^\infty} &\leq C|\ln \epsilon|^{an-1}(CT(|\ln \epsilon|)^b) \\ &\leq C\epsilon^{-N},\ \exists N > 0,\ x \in \mathbb{R}^n,\ t \in [0,T],\ \epsilon \in (0,1]. \end{aligned}$$

Take the fractional Laplacian for $0<\gamma<2$

$$|\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} u_\epsilon(x,t))| = \int E_{n\epsilon}(t,x-y)\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} u_{0\epsilon}(y)dy$$
$$+\int\int_0^t E_{n\epsilon}(t-\tau,x-y)\nabla g_\epsilon(\theta u_\epsilon)\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} u_\epsilon(\tau,y)dyd\tau$$

Using the moderateness of $u_\epsilon(t,x)$,

$$\|\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} u_\epsilon(t,\cdot))\|_{L^\infty} \le \|E_{n\epsilon}(t,x-y)\|_{L^1}\|\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} u_{0\epsilon}(y)\|_{L^\infty}$$
$$+\int_0^t \|E_{n\epsilon}(t-\tau,x-\cdot)\|_{L^1}\|\nabla g_\epsilon(\theta u_\epsilon)\|_{L^\infty}\|\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} u_\epsilon(\tau,\cdot)\|_{L^\infty}d\tau.$$

$$\|\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} u_\epsilon(x,t))\|_{L^\infty} \le CX^{4-\gamma-d}\epsilon^{-N}+CTX^{4-\gamma-d}(|\ln\epsilon|)^b\epsilon^{-N}.$$

$$\|\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} u_\epsilon(x,t))\|_{L^\infty} \le \epsilon^{-N}, \exists N>0, x\in\mathbb{R}^n, t\in[0,T],\ \epsilon\in(0,1].$$

It follows moderateness for the fractional Laplacian in the space $\mathcal{G}^e_{L^\infty}([0,T)\times\mathbb{R}^n)$.
For uniqueness suppose that $L_\epsilon(x,t) = u_{1\epsilon}(x,t) - u_{2\epsilon}(x,t)$ be two different solutions whose difference for equation

$$\partial_t L_\epsilon(t,x) = \Delta L_\epsilon(t,x) + k_\epsilon(t,x)L_\epsilon(t,x) + N_\epsilon(t,x),$$
$$(L_\epsilon(x,0))_\epsilon = (N_{0\epsilon}(x)) \in (\mathcal{N}_{L^\infty}(\mathbb{R}^n)),$$

where

$$N_\epsilon(x,t)\in\mathcal{N}_{L^\infty}(\mathbb{R}^n\times[0,T)),\quad \|k_\epsilon(\tau,x)\|_{L^\infty}\le C(|\ln\epsilon|)^b,\ 0<b<1.$$

$$\|\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} L_\epsilon(t,\cdot))\|_{L^\infty} \le \|E_{n\epsilon}(t,x-\cdot)\|_{L^1}\|\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} N_{0\epsilon}(\cdot)\|_{L^\infty}$$
$$+\int_0^t \|E_{n\epsilon}(t-\tau,x-\cdot)\|_{L^1}\|k_\epsilon(\tau,\cdot)\|_{L^\infty}\|\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} L_\epsilon(\tau,\cdot)\|_{L^\infty}d\tau$$
$$+\int_0^t \|E_{n\epsilon}(t-\tau,x-\cdot)\|_{L^1}\|\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} N_\epsilon(\tau,\cdot)\|_{L^\infty}d\tau.$$

$$\|\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} L_\epsilon(t,\cdot)\|_{L^\infty} \le C\epsilon^r+CT(|\ln\epsilon|)^b\epsilon^{-N}+\epsilon^r \le C\epsilon^r.$$

Then $\|\widetilde{(-\Delta)}_d^{\frac{\gamma}{2}} L_\epsilon(t,\cdot)\|_{L^\infty} \le C\epsilon^r$, $\exists N>0$, $x\in\mathbb{R}^n$, $t\in[0,T]$, $\epsilon\in(0,1]$. □

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