

INVERSE STURM-LIOUVILLE PROBLEM WITH DISCONTINUITY CONDITIONS

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ABSTRACT. This paper deals with the boundary value problem involving the differential equation

$$\ell y := -y'' + qy = \lambda y,$$

subject to the standard boundary conditions along with the following discontinuity conditions at a point $a \in (0, \pi)$

$$y(a+0) = a_1 y(a-0), \quad y'(a+0) = a_1^{-1} y'(a-0) + a_2 y(a-0),$$

where $q(x)$, a_1 , a_2 are real, $q \in L^2(0, \pi)$ and λ is a parameter independent of x . We develop the Hochstadt's result based on the transformation operator for inverse Sturm-Liouville problem when there are discontinuous conditions. Furthermore, we establish a formula for $q(x) - \tilde{q}(x)$ in the finite interval where $q(x)$ and $\tilde{q}(x)$ are analogous functions.

1. INTRODUCTION

The method of separation of variables for solving PDEs with discontinuous boundary conditions naturally led to ODE with discontinuities inside of the interval which often appear in mathematics. Inverse spectral problem consists in recovering operators from their spectral characteristics. For example the mathematical formulation of a large variety of technical and physical problem led to inverse problems such as identifying the density of the thing from data collected from the sets of frequencies of oscillations of the string with barrier.

The inverse spectral Sturm-Liouville problem can be regarded as three aspects, e.g., existence, uniqueness and reconstruction of the coefficients

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given specific properties of eigenvalues and eigenfunctions. Our work concerns uniqueness and other property of potential function.

The applications of boundary value problems with discontinuity conditions inside the interval are connected with discontinuous material properties. Inverse problems with a discontinuity condition inside the interval play an important role in mathematics, mechanics, radio electronics, geophysics, and other fields of science and technology. As a rule, such problems are related to discontinuous and non-smooth properties of a medium (e.g., see [1]-[3] and [18]).

We refer to the somewhat complementary surveys in inverse Sturm-Liouville problems in [3]-[12], [19], and [21] for further aspects of this field. In this work, we generalize the Hochstadt's result [13], refining the approach of Levinson [8] to show that precisely how much q has freedom where the μ_n and all but finitely many of the λ_n are specified. Note that the eigenvalues μ_n is obtained with replacing β by γ in (2.2). There are many papers concerning problems with discontinuous conditions. One can find the similar works in [3] and [14]-[17].

2. ASYMPTOTIC FORM OF SOLUTIONS AND EIGENVALUES

We consider the boundary value problem

$$(2.1) \quad \ell y := -y'' + qy = \lambda y,$$

$$(2.2) \quad \begin{aligned} U(y) &:= y(0) \cos \alpha + y'(0) \sin \alpha = 0, \\ V(y) &:= y(\pi) \cos \beta + y'(\pi) \sin \beta = 0, \end{aligned}$$

with the jump conditions

$$(2.3) \quad y(a+0) = a_1 y(a-0), \quad y'(a+0) = a_1^{-1} y'(a-0) + a_2 y(a-0),$$

where $q(x) \in L^2(0, \pi)$ and $\alpha, \beta \in [0, \pi)$, $a \in (0, \pi)$, a_1, a_2 are real. The coefficients a_1 and a_2 are assumed to be known a priori and fixed. For simplicity we use the notation $L = L(q(x); \alpha; \beta; a)$, for the problem (2.1)–(2.3).

Our Hilbert space will be $L_2(0, \pi)$ associated with the inner product

$$(2.4) \quad \langle f, g \rangle = \int_0^{a-0} f \bar{g} dt + \int_{a+0}^{\pi} f \bar{g} dt,$$

where

$$(2.5) \quad \text{dom}(L) = \left\{ y \left| \begin{array}{l} y \text{ and } y' \in AC[0, a) \cup (a, \pi], \ell y \in L^2[0, \pi], \\ y'(a+0) = a_1^{-1} y'(a-0) + a_2 y(a-0), \\ y(a+0) = a_1 y(a-0), U(y) = V(y) = 0 \end{array} \right. \right\}.$$

Lemma 2.1. *The operator L is self-adjoint in $L^2(0, \pi)$.*

From the linear differential equations we obtain that the Wronskian

$$(2.6) \quad W(u, v) = u(x)v'(x) - u'(x)v(x)$$

is constant for on $x \in [0, a) \cup (a, \pi]$ for two solutions $\ell u = \lambda u$, $\ell v = \lambda v$ satisfying the transmission conditions (2.3). Moreover, we set

$$\Delta(\lambda) := W(u, v).$$

Then $\Delta(\lambda)$ is an entire function whose roots λ_n coincide with the eigenvalues of L . Suppose that functions $w(x, \lambda)$ and $v(x, \lambda)$ are solutions of (2.1) under the jump conditions (2.3) and initial conditions:

$$(2.7) \quad \begin{aligned} w(0, \lambda) &= \sin \alpha, & w'(0, \lambda) &= -\cos \alpha, \\ v(\pi, \lambda) &= \sin \beta, & v'(\pi, \lambda) &= -\cos \beta. \end{aligned}$$

By attaching a subscript 1 or 2 to the functions w and v we mean to refer to the first subinterval $[0, a)$ or to the second subinterval $(a, \pi]$. For example by $w_{1n}(x)$ we mean $w(x, \lambda_n)$ in the case where $x \in [0, a)$ and by $w_{2n}(x)$ we mean $w(x, \lambda_n)$ where $x \in (a, \pi]$ and $w_n(x) = w(x, \lambda_n)$.

Theorem 2.2. *Let $\lambda = \rho^2$ and $\tau := \Im \rho$. For equation (2.1) with boundary conditions (2.2) and jump conditions (2.3) as $|\lambda| \rightarrow \infty$, the following asymptotic formulas hold.*

For $\sin \alpha \neq 0$,

$$(2.8) \quad w(x; \lambda) = \begin{cases} \sin \alpha \left(\cos \rho x + \frac{1}{2} \frac{\sin \rho x}{\rho} \int_0^x q(t) dt \right) - \cos \alpha \frac{\sin \rho x}{\rho} \\ \quad + O\left(\frac{\exp(|\tau|x)}{\rho^2}\right), & x < a, \\ \sin \alpha (b_1 \cos \rho x + b_2 \cos \rho(2a - x)) + f_1(x) \frac{\sin \rho x}{\rho} \\ \quad + f_2(x) \frac{\sin \rho(2a - x)}{\rho} + O\left(\frac{\exp(|\tau|x)}{\rho^2}\right), & x > a, \end{cases}$$

$$(2.9) \quad w'(x; \lambda) = \begin{cases} \sin \alpha \left(-\rho \sin \rho x + \frac{1}{2} \cos \rho x \int_0^x q(t) dt \right) - \cos \alpha \cos \rho x \\ \quad + O\left(\frac{\exp(|\tau|x)}{\rho}\right), & x < a, \\ \rho \sin \alpha [(-b_1 \sin \rho x + b_2 \sin \rho(2a - x))] + f_1(x) \cos \rho x \\ \quad - f_2(x) \cos \rho(2a - x) + O\left(\frac{\exp(|\tau|x)}{\rho}\right), & x > a, \end{cases}$$

and for $\sin \alpha = 0$,

$$(2.10) \quad w(x; \lambda) = \begin{cases} \frac{\sin \rho x}{\rho} + O\left(\frac{\exp(|\tau|x)}{\rho^2}\right), & x < a, \\ \frac{1}{\rho} (-b_1 \sin \rho x + b_2 \sin \rho(2a - x)) + O\left(\frac{\exp(|\tau|x)}{\rho^2}\right), & x > a, \end{cases}$$

$$(2.11) \quad w'(x; \lambda) = \begin{cases} \cos \rho x + O\left(\frac{\exp(|\tau|x)}{\rho}\right), & x < a, \\ -b_1 \cos \rho x - b_2 \cos \rho(2a - x) + O\left(\frac{\exp(|\tau|x)}{\rho}\right), & x > a, \end{cases}$$

where

$$(2.12) \quad b_1 = (a_1 + a_1^{-1})/2, \quad b_2 = (a_1 - a_1^{-1})/2,$$

and

$$\begin{aligned} f_1(x) &= b_1(-\cos \alpha + \frac{1}{2} \sin \alpha \int_0^x q(t)dt) + \frac{a_2}{2} \sin \alpha, \\ f_2(x) &= b_2(-\cos \alpha + \sin \alpha(-\frac{1}{2} \int_0^x q(t)dt + \int_0^a q(t)dt)) + \frac{a_2}{2} \sin \alpha. \end{aligned}$$

The characteristic function is

$$(2.13) \quad \Delta(\lambda) = V(w) = w(\pi, \lambda) \cos \beta + w'(\pi, \lambda) \sin \beta.$$

For $\sin \alpha \neq 0$,

$$(2.14) \quad \Delta(\lambda) = \begin{cases} \rho \sin \alpha \sin \beta (-b_1 \sin \rho \pi + b_2 \sin \rho(2a - \pi)) + O(\exp(|\tau|\pi)), & \sin \beta \neq 0 \\ \sin \alpha (b_1 \cos \rho \pi + b_2 \cos \rho(2a - \pi)) + O(\frac{\exp(|\tau|\pi)}{\rho}), & \sin \beta = 0, \end{cases}$$

and for $\sin \alpha = 0$,

$$(2.15) \quad \Delta(\lambda) = \begin{cases} -\sin \beta (b_1 \cos \rho \pi + b_2 \cos \rho(2a - \pi)) + O(\frac{\exp(|\tau|\pi)}{\rho}), & \sin \beta \neq 0 \\ -\frac{1}{\rho} (b_1 \sin \rho \pi + b_2 \sin \rho(2a - \pi)) + O(\frac{\exp(|\tau|\pi)}{\rho^2}), & \sin \beta = 0. \end{cases}$$

Proof. The arguments for obtaining the asymptotic formulas is similar to that of [18]. Note that by changing x to $\pi - x$ one can obtain the asymptotic form of $v(x, \lambda)$ and $v'(x, \lambda)$. \square

As a consequence of Valiron's theorem ([22, Thm. 13.4]) we obtain:

Theorem 2.3. *The corresponding eigenvalues $\{\lambda_n\}$ of the boundary value problem L admit the following asymptotic form as $n \rightarrow \infty$:*

$$(2.16) \quad \rho_n := \sqrt{\lambda_n} = n + O(1).$$

If $w(x, \lambda_n)$ is the eigenfunction corresponding to eigenvalues λ_n , then by using Eqs. (2.8)-(2.11) we get

$$(2.17) \quad \int_0^\pi w_n^2(x)dx = \begin{cases} \sin^2 \alpha [\frac{1}{2}(a + (b_1^2 + b_2^2 + 2b_1b_2 \cos(2a\rho_n))(\pi - a))] & \sin \alpha \neq 0 \neq \sin \beta, \\ \quad (1 + O(\frac{1}{n})), & \\ \sin^2 \alpha [\frac{1}{2}(a + (b_1^2 + b_2^2 + 2b_1b_2 \cos(2a\rho_n))(\pi - a))] & \sin \alpha \neq 0 = \sin \beta, \\ \quad (1 + O(\frac{1}{n})), & \\ \frac{1}{\rho_n^2} [\frac{1}{2}(a + (b_1^2 + b_2^2 + 2b_1b_2 \cos(2a\rho_n))(\pi - a))] & \sin \alpha = 0 \neq \sin \beta, \\ \quad (1 + O(\frac{1}{n})), & \\ \frac{1}{\rho_n^2} [\frac{1}{2}(a + (b_1^2 + b_2^2 + 2b_1b_2 \cos(2a\rho_n))(\pi - a))] & \sin \alpha = 0 = \sin \beta, \\ \quad (1 + O(\frac{1}{n})), & \end{cases}$$

We can write

$$\|w_n\|^2 = \int_0^\pi w_n^2(x)dx = \mu(\rho_n; \alpha; \beta; a)(1 + O(\frac{1}{n})).$$

Lemma 2.4. (see [20] P.241) *The sequence $\{w_n\}$ of the orthonormal eigenfunctions of the self-adjoint system (2.1), (2.2), and (2.3) is complete in $L^2(0, \pi)$.*

3. TRANSFORMATION OPERATOR

In this section we investigate the transformation operator for two operators L and \tilde{L} , where $\tilde{L} = L(\tilde{q}(x); \alpha; \beta; a)$ and $\text{dom}(\tilde{L})$ defined by an analogous manner with ℓ replaced by $\tilde{\ell}$.

Let $v_n(x)$ be an eigenfunction of $L(q(x); \alpha; \beta; a)$ corresponding to the eigenvalue λ_n , then w_n, v_n are linearly dependent for each $n \in \mathbb{N}$. Let k_n be such that

$$(3.1) \quad v_n = k_n w_n.$$

Define \tilde{w}_n, \tilde{v}_n , and \tilde{k}_n in a similar manner. Let $L(q(x); \alpha; \gamma; a)$ be an eigenvalue problem such that by assuming $\sin(\beta - \gamma) \neq 0$. It is clear that

$$\phi(\lambda) := w(\pi, \lambda) \cos \gamma + w'(\pi, \lambda) \sin \gamma$$

is the characteristic function of $L(q(x); \alpha; \gamma; a)$ and the zeros of $\phi(\lambda)$ are eigenvalues of $L(q(x); \alpha; \gamma; a)$, say $\{\mu_n\}_{n=1}^{\infty}$, are real and simple. Define $\tilde{\phi}$ by an analogous manner.

Lemma 3.1. *If $L(q(x); \alpha; \gamma; a)$ and $L(\tilde{q}(x); \alpha; \gamma; a)$ have the same eigenvalues, then $\phi = \tilde{\phi}$.*

Proof. Using the Hadamard's factorization for entire functions $\phi(\lambda)$ and $\tilde{\phi}(\lambda)$ and by applying the Liouville's theorem for $\frac{\phi(\lambda)}{\tilde{\phi}(\lambda)}$, we get $\phi = \tilde{\phi}$. \square

Theorem 3.2. *Let $\Lambda_0 \subseteq \mathbb{N}$ be a finite set and $\Lambda = \mathbb{N} \setminus \Lambda_0$. If $L(q(x); \alpha; \gamma; a)$ and $L(\tilde{q}(x); \alpha; \gamma; a)$ have the same eigenvalues and, as well as, $\lambda_n = \tilde{\lambda}_n$ for all $n \in \Lambda$, where λ_n and $\tilde{\lambda}_n$ are the eigenvalues of $L(q(x); \alpha; \beta; a)$ and $L(\tilde{q}(x); \alpha; \beta; a)$, respectively, then $k_n = \tilde{k}_n$ for all $n \in \Lambda$.*

Proof. From definition of ϕ it follows that

$$(3.2) \quad \begin{cases} \cos \beta w_{2n}(\pi) + \sin \beta w'_{2n}(\pi) = 0, \\ \cos \gamma w_{2n}(\pi) + \sin \gamma w'_{2n}(\pi) = \phi(\lambda_n). \end{cases}$$

The above linear system has unique solution

$$(3.3) \quad w_{2n}(\pi) = -\frac{\sin \beta}{\sin(\gamma - \beta)} \phi(\lambda_n), \quad w'_{2n}(\pi) = -\frac{\cos \beta}{\sin(\gamma - \beta)} \phi(\lambda_n).$$

We obtain similarly

$$(3.4) \quad \tilde{w}_{2n}(\pi) = -\frac{\sin \beta}{\sin(\gamma - \beta)} \tilde{\phi}(\lambda_n), \quad \tilde{w}'_{2n}(\pi) = -\frac{\cos \beta}{\sin(\gamma - \beta)} \tilde{\phi}(\lambda_n).$$

But $\lambda_n = \tilde{\lambda}_n$ for all $n \in \Lambda$ and from Lemma 3.1, $\phi \equiv \tilde{\phi}$, and so $w_{2n}(\pi) = \tilde{w}_{2n}(\pi)$ and $w'_{2n}(\pi) = \tilde{w}'_{2n}(\pi)$ for $n \in \Lambda$. From definition of k_n, \tilde{k}_n , and w_{2n}, \tilde{w}_{2n} it follows that

$$(3.5) \quad v_{2n} = k_n w_{2n}$$

for each $n \in \Lambda$. By using the jump conditions we have $k_n = \tilde{k}_n$ for $x \in [0, a) \cup (a, \pi]$. \square

Suppose

$$(3.6) \quad H := \text{dom}(L) \ominus \{w_n : n \in \Lambda_0\},$$

$$(3.7) \quad \tilde{H} := \text{dom}(\tilde{L}) \ominus \{\tilde{w}_n : n \in \Lambda_0\}.$$

Define the transformation operator $T : H \rightarrow \tilde{H}$ by

$$(3.8) \quad Tw_n = \tilde{w}_n,$$

for all $n \in \Lambda$. Note that by $\text{dom}(L) \ominus \{w_n : n \in \Lambda_0\}$ we mean $\text{dom}(L)$ contains all of $\{w_n\}_{n=1}^\infty$ except $\{w_n\}_{n \in \Lambda_0}$.

Lemma 3.3. *The operator $T : H \rightarrow \tilde{H}$ is bounded.*

Proof. From equation (2.17) we see that

$$(3.9) \quad \|\tilde{w}_n\|^2 = \int_0^{a-0} \tilde{w}_{1n}^2(x) dt + \int_{a+0}^\pi \tilde{w}_{2n}^2(x) dt = \mu(\rho_n; \alpha; \beta; a) [1 + O(\frac{1}{n})]$$

and

$$(3.10) \quad \|w_n\|^2 = \int_0^{a-0} w_{1n}^2(x) dt + \int_{a+0}^\pi w_{2n}^2(x) dt = \mu(\tilde{\rho}_n; \alpha; \beta; a) [1 + O(\frac{1}{n})]$$

for all $n \in \Lambda$. Thus by (3.9) and (3.10) we get

$$(3.11) \quad \frac{\|Tw_n\|^2}{\|w_n\|^2} = \frac{\|\tilde{w}_n\|^2}{\|w_n\|^2} = 1 + O(\frac{1}{n}).$$

\square

Lemma 3.4. *The relation $(\lambda - \tilde{L})T(\lambda - L)^{-1} = T$ holds for the operator T .*

Proof. Assume that $f \in H$. We can extend f in term of the set $\{w_n\}$,

$$(3.12) \quad f(x) = \begin{cases} \sum_{\Lambda} f_n w_{1n}(x), & x < a, \\ \sum_{\Lambda} f_n w_{2n}(x), & x > a, \end{cases}$$

where

$$f_n = \frac{\langle f, w_n \rangle_{\text{dom}(L)}}{\langle w_n, w_n \rangle_{\text{dom}(L)}}.$$

Assume that λ is in the complex plane and is not an eigenvalue of $L(q(x); \alpha; \beta; a)$, then the operator $(\lambda - L)^{-1}$ exists and bounded. So we can write as the following form

$$(\lambda - L)^{-1}f(x) = \begin{cases} \sum_{\Lambda} \frac{f_n w_{1n}(x)}{\lambda - \lambda_n}, & x < a, \\ \sum_{\Lambda} \frac{f_n w_{2n}(x)}{\lambda - \lambda_n}, & x > a. \end{cases}$$

We now get

$$T(\lambda - L)^{-1}f(x) = \begin{cases} \sum_{\Lambda} \frac{f_n \tilde{w}_{1n}(x)}{\lambda - \lambda_n}, & x < a, \\ \sum_{\Lambda} \frac{f_n \tilde{w}_{2n}(x)}{\lambda - \lambda_n}, & x > a, \end{cases}$$

and

$$\begin{aligned} (\lambda - \tilde{L})T(\lambda - L)^{-1}f(x) &= \begin{cases} \sum_{\Lambda} (\lambda - \lambda_n) \frac{f_n \tilde{w}_{1n}(x)}{\lambda - \lambda_n}, & x < a, \\ \sum_{\Lambda} (\lambda - \lambda_n) \frac{f_n \tilde{w}_{2n}(x)}{\lambda - \lambda_n}, & x > a. \end{cases} \\ &= \begin{cases} \sum_{\Lambda} f_n \tilde{w}_{1n}(x), & x < a, \\ \sum_{\Lambda} f_n \tilde{w}_{2n}(x), & x > a. \end{cases} \end{aligned}$$

Then we have

$$(3.13) \quad (\lambda - \tilde{L})T(\lambda - L)^{-1} = T.$$

□

4. HOCHSTADT'S RESULT

In this section we examine a different representation for T . In a general case when there are discontinuous conditions we generalize the well-known result of Hochstadt [13]. We construct Green's function for L by using its solutions $w(x, \lambda)$ and $v(x, \lambda)$. The Green's function $G(x, y; \lambda)$ is defined as following

$$(4.1) \quad G(x, y; \lambda) = \begin{cases} \frac{w(x, \lambda)v(y, \lambda)}{\Delta(\lambda)}, & 0 < x < y < \pi, \\ \frac{w(y, \lambda)v(x, \lambda)}{\Delta(\lambda)}, & 0 > x > y > \pi, \end{cases}$$

where $x, y \neq a$. For simplicity we can write

$$G(x, y; \lambda) = \frac{w(x <)v(x >)}{\Delta(\lambda)},$$

where $x < := \min\{x, y\}$ and $x > := \max\{x, y\}$.

Theorem 4.1. *If $L(q(x); \alpha; \gamma; a)$ and $L(\tilde{q}(x); \alpha; \gamma; a)$ have the same spectrum and $\lambda_n = \tilde{\lambda}_n$ for all $n \in \Lambda$, then*

$$(4.2) \quad q(x) - \tilde{q}(x) = \begin{cases} \sum_{\Lambda_0} (\tilde{y}_{1n} w_{1n})'(x), & x < a, \\ \sum_{\Lambda_0} (\tilde{y}_{2n} w_{2n})'(x), & x > a, \end{cases}$$

a.e. on $[0, a) \cup (a, \pi]$, where \tilde{y}_n and w_n are suitable solutions of $\tilde{L}y = \lambda_n y$ and $Ly = \lambda_n y$, respectively.

Proof. Let C_n be a sequence of circles about the origin intersecting the positive λ -axis between λ_n and λ_{n+1} . By using (4.1), we get

$$(4.3) \quad \lim_{n \rightarrow \infty} \int_{C_n} \frac{G(x, y; \mu)}{\lambda - \mu} d\mu = 0, \quad \lambda \in \text{int } C_n.$$

From residue integration, it follows that

$$(4.4) \quad \frac{1}{2\pi i} \int_{C_n} \frac{G(x, y; \mu)}{\lambda - \mu} d\mu = -G(x, y; \lambda) + \sum_{k=0}^n \frac{w_k(x <) v_k(x >)}{\dot{\Delta}(\lambda_k)(\lambda - \lambda_k)}.$$

By applying Mittag-Leffler expansion for $G(x, y; \lambda)$ and using (4.3) and (4.4) we obtain

$$(4.5) \quad G(x, y; \lambda) = \sum_{k=0}^{\infty} \frac{w_k(x <) v_k(x >)}{\dot{\Delta}(\lambda_k)(\lambda - \lambda_k)},$$

where w_k, v_k are eigenfunctions corresponding to the eigenvalues λ_k . Therefore

$$(4.6) \quad \begin{aligned} (\lambda - L)^{-1} f(x) &= \int_0^\pi G(x, y; \lambda) f(y) dy \\ &= \frac{v_1(x) \int_0^x w_1(y) f(y) dy + w_1(x) (\int_x^a v_1(y) f(y) dy + \int_a^\pi v_2(y) f(y) dy)}{\Delta(\lambda)} \\ &= \sum_{\Lambda} \frac{v_{1n}(x) \int_0^x w_{1n}(y) f(y) dy + w_{1n}(x) (\int_x^a v_{1n}(y) f(y) dy + \int_a^\pi v_{2n}(y) f(y) dy)}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)} \\ &= \sum_{\Lambda} \frac{k_n w_{1n}(x) (\int_0^a w_{1n}(y) f(y) dy + \int_a^\pi w_{2n}(y) f(y) dy)}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)} \end{aligned}$$

for $f \in H$ and for $x < a$. We have

$$(4.7) \quad T(\lambda - L)^{-1} f(x) = \sum_{\Lambda} \frac{k_n \tilde{w}_{1n}(x) (\int_0^a w_{1n}(y) f(y) dy + \int_a^\pi w_{2n}(y) f(y) dy)}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)}.$$

From the formula (4.6), (4.7), and (3.13) we get

$$(4.8) \quad \begin{aligned} &(\lambda - \tilde{L})T(\lambda - L)^{-1} f(x) \\ &= (\lambda - \tilde{L}) \sum_{\Lambda} \frac{\tilde{v}_{1n}(x) \int_0^x w_{1n}(y) f(y) dy + \tilde{w}_{1n}(x) (\int_x^a v_{1n}(y) f(y) dy + \int_a^\pi v_{2n}(y) f(y) dy)}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)} \\ &= Tf(x). \end{aligned}$$

We define

$$(4.9) \quad g(x) := \frac{\tilde{v}_1(x) \int_0^x w_1(y) f(y) dy + \tilde{w}_1(x) (\int_x^a v_1(y) f(y) dy + \int_a^\pi v_2(y) f(y) dy)}{\Delta(\lambda)}.$$

By applying the Mittag-Leffler expansion for $g(x)$, we have

$$\begin{aligned}
 g(x) &= \sum_{\Lambda_0} \frac{\tilde{u}_{1n}(x) \int_0^x w_{1n}(y) f(y) dy + \tilde{z}_{1n}(x) (\int_x^a v_{1n}(y) f(y) dy + \int_a^\pi v_{2n}(y) f(y) dy)}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)} \\
 &+ \sum_{\Lambda} \frac{\tilde{v}_{1n}(x) \int_0^x w_{1n}(y) f(y) dy + \tilde{w}_{1n}(x) (\int_x^a v_{1n}(y) f(y) dy + \int_a^\pi v_{2n}(y) f(y) dy)}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)}.
 \end{aligned}
 \tag{4.10}$$

By using (4.7) the second term of the above expression is equal to $T(\lambda - L)^{-1}f$ and $\tilde{u}_n(x)$ represents $\tilde{v}(x, \lambda_n)$ and $\tilde{z}_n(x)$ represents $\tilde{w}(x, \lambda_n)$. Hence

$$\begin{aligned}
 (\lambda - \tilde{L})^{-1}Tf(x) &= g(x) \\
 &- \sum_{\Lambda_0} \frac{\tilde{u}_{1n}(x) \int_0^x w_{1n}(y) f(y) dy + \tilde{z}_{1n}(x) (\int_x^a v_{1n}(y) f(y) dy + \int_a^\pi v_{2n}(y) f(y) dy)}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)}.
 \end{aligned}
 \tag{4.11}$$

The right and left-hand side of (4.11) are in the domain of $(\lambda - \tilde{L})$. Therefore both sides of (4.11) are continuous. By using (4.6) and differentiation of the right hand side of (4.11), we obtain

$$\begin{aligned}
 &\left[\frac{\tilde{v}'_1(x) \int_0^x w_1(y) f(y) dy + \tilde{w}'_1(x) (\int_x^a v_1(y) f(y) dy + \int_a^\pi v_2(y) f(y) dy)}{\Delta(\lambda)} \right. \\
 &- \sum_{\Lambda_0} \left. \frac{\tilde{u}'_{1n}(x) \int_0^x w_{1n}(y) f(y) dy + \tilde{z}'_{1n}(x) (\int_x^a v_{1n}(y) f(y) dy + \int_a^\pi v_{2n}(y) f(y) dy)}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)} \right] \\
 &+ \left[\frac{\tilde{v}_1(x)w_1(x) - \tilde{w}_1(x)v_1(x)}{\Delta(\lambda)} - \sum_{\Lambda_0} \frac{\tilde{u}_{1n}(x)w_{1n}(x) + \tilde{z}_{1n}(x)v_{1n}(x)}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)} \right] f(x)
 \end{aligned}$$

An inspection of the term in the second set of braces shows that it vanishes identically. To do that, one merely computes the residue at each λ_n and observes that it becomes zero. One can differentiate the expression in the braces in the last expression and then we obtain from (4.11)

$$\begin{aligned}
 Tf(x) &= \left[\frac{\tilde{v}_1(x)w_1(x) - \tilde{w}_1(x)v_1(x)}{\Delta(\lambda)} - \sum_{\Lambda_0} \frac{\tilde{u}_{1n}(x)w_{1n}(x) + \tilde{z}_{1n}(x)v_{1n}(x)}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)} \right] f(x) \\
 &- \sum_{\Lambda_0} \frac{\tilde{u}_{1n}(x) \int_0^x w_{1n}(y) f(y) dy + \tilde{z}_{1n}(x) (\int_x^a v_{1n}(y) f(y) dy + \int_a^\pi v_{2n}(y) f(y) dy)}{\dot{\Delta}(\lambda_n)}.
 \end{aligned}
 \tag{4.12}$$

The operator T is independent of λ . Using the asymptotic formulas, we see that the terms in the braces reduce to unity. To simplify the second term in (4.12) we recall that $v_{1n} = k_n w_{1n}$, $v_{2n} = k_n w_{2n}$ and $\int_0^a w_{1n}(y) f(y) dy + \int_a^\pi w_{2n}(y) f(y) dy = 0$. Then

$$Tf(x) = f(x) - \frac{1}{2} \sum_{\Lambda_0} \tilde{y}_{1n}(x) \int_0^x w_{1n}(y) f(y) dy,
 \tag{4.13}$$

for $x < a$, where

$$\frac{1}{2} \tilde{y}_{1n}(x) = \frac{\tilde{u}_{1n}(x) - k_n \tilde{z}_{1n}(x)}{\dot{\Delta}(\lambda_n)}.$$

By the similar computation we find that

$$\begin{aligned} Tf(x) &= f(x) - \frac{1}{2} \sum_{\Lambda_0} \tilde{y}_{2n}(x) \left(\int_0^a w_{1n}(y) f(y) dy + \int_a^x w_{2n}(y) f(y) dy \right) \\ &= f(x) + \frac{1}{2} \sum_{\Lambda_0} \tilde{y}_{2n}(x) \int_x^\pi w_{2n}(y) f(y) dy, \end{aligned}$$

for $x < a$, where

$$\frac{1}{2} \tilde{y}_{2n}(x) = \frac{\tilde{u}_{2n}(x) - k_n \tilde{z}_{2n}(x)}{\tilde{\Delta}(\lambda_n)}.$$

From (4.11) it follows that $\tilde{L}Tf = TLf$. For $x < a$ we have

$$\begin{aligned} \tilde{L}T w_{1n} &= \tilde{L} \left(w_{1n} - \frac{1}{2} \sum_{m \in \Lambda_0} \tilde{y}_{1m} \int_0^x w_{1m} w_{1n} \right) \\ &= -w_{1n}'' + \tilde{q} w_{1n} - \frac{1}{2} \sum_{m \in \Lambda_0} \tilde{L} \left(\tilde{y}_{1m} \int_0^x w_{1m} w_{1n} \right) \\ &= -w_{1n}'' + \tilde{q} w_{1n} - \frac{1}{2} \sum_{m \in \Lambda_0} \tilde{L} \tilde{y}_{1m} \int_0^x w_{1m} w_{1n} \\ &\quad + \frac{1}{2} \sum_{m \in \Lambda_0} 2\tilde{y}'_{1m} (w_{1m} w_{1n})(x) + \frac{1}{2} \sum_{m \in \Lambda_0} \tilde{y}_{1m} (w'_{1m} w_{1n} + w_{1m} w'_{1n})(x). \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} TL w_{1n} &= T(-w_{1n}'' + q w_{1n}) \\ &= -w_{1n}'' + q w_{1n} - \frac{1}{2} \sum_{m \in \Lambda_0} \tilde{y}_{1m} \int_0^x w_{1m} L w_{1n} \\ &= -w_{1n}'' + q w_{1n} - \frac{1}{2} \sum_{m \in \Lambda_0} \tilde{y}_{1m} \int_0^x w_{1n} L w_{1m} \\ &\quad - \frac{1}{2} \sum_{m \in \Lambda_0} \tilde{y}_{1m} (w_{1n} w'_{1m} - w_{1m} w'_{1n}). \end{aligned} \tag{4.15}$$

Note that

$$\begin{aligned} \sum_{m \in \Lambda_0} \tilde{y}_{1m} \int_0^x w_{1n} L w_{1m} &= \sum_{m \in \Lambda_0} \tilde{y}_{1m} \int_0^x \lambda_m w_{1m} w_{1n} \\ &= \sum_{m \in \Lambda_0} \lambda_m \tilde{y}_{1m} \int_0^x w_{1m} w_{1n} \\ &= \sum_{m \in \Lambda_0} \tilde{L} \tilde{y}_{1m} \int_0^x w_{1m} w_{1n}. \end{aligned}$$

From (4.14) and (4.15) we deduce that

$$q(x) - \tilde{q}(x) = \sum_{\Lambda_0} (\tilde{y}_{1m} w_{1m})'.$$

By the similar calculation for $x > a$, we have

$$q(x) - \tilde{q}(x) = \sum_{\Lambda_0} (\tilde{y}_{2m} w_{2m})'.$$

If Λ_0 is empty, then T is a unitary operator and $L = \tilde{L}$. Hence $q = \tilde{q}$. This completes the proof. \square

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REFERENCES

1. R. J. Krueger, *Inverse problems for nonabsorbing media with discontinuous material properties*, J. Math. Phys. **23** (1982) 396-404.
2. R. S. Anderssen, *The effect of discontinuities in density and shear velocity on the asymptotic overtone structure of torsional eigenfrequencies of the earth*, Geophys. J. R. Astron. Soc. **50** (1997) 303-309.
3. O. H. Hald, *Discontinuous Inverse Eigenvalue Problem*, Commun. Pure. Appl. Math. **37** (1984), 539-577.
4. V. A. Ambartsumyan, *Über eine frage der eigenwerttheorie*, Z. Phys. **53** (1929), 690-695.
5. G. Borg, *Eine umkehrung der Sturm-Liowilleschen eigenwertaufgabe*, Acta Math. **78** (1945), 1-96
6. J. R. McLaughlin, *Analytical methods for recovering coefficients in differential equations from spectral data* SIAM Rev. **28** (1986), 53-72.
7. B. M. Levitan, *Inverse Sturm-Liouville Problems*, VNU Science Press, 1987.
8. N. Levinson, *The inverse Sturm-Liouville problem*, Mat. Tidskr. **3** (1949), 25-30.
9. I. M. Gelfand and B. M. Levitan, *On the determination of of a differential equation from its spectral function*, Amer. Math. Soc. Transl. Ser. **21** (1955), 253-304.
10. V. A. Yurko, *Inverse problem for Sturm-Liouville operators on hedgehog-type graphs*, Math. Notes, vol **89** no **3-4** (2011), 438-449.
11. M. Shahriari, A. J. Akbarfam, G. Teschl, *Uniqueness for Inverse Sturm-Liouville Problems with a Finite Number of Transmission Conditions*, J. Math. Anal. Appl., **395** (2012) 19-29.
12. H. Hochstadt, *On inverse problems associated with Sturm-Liouville operators*, J. Differential Equations, **17** (1975), 220-235.
13. H. Hochstadt, *The inverse Sturm-Liouville problem*, Comm.pure.Math. **26** (1973), 715-729.
14. P. A. Binding, P. J. Browne and B. A. Watson, *Inverse spectral problems for left-definite Sturm-Liouville equations with indefinite weight*, J. Math. Anal. Appl. **271** (2002), 383-408.
15. R. Kh. Amirov, *On Sturm-Liouville operators with discontinuity conditions inside an interval*, J. Math. Anal. Appl. **317** (2006), 163-176.
16. H. Koyunbakan and E. S. Panakhov, *Inverse problem for a singular differential operator*, Mathematical and Computer Modelling, **47** (2008), 178-185.
17. Hikmet Koyunbakan, *Inverse problem for a quadratic pencil of Sturm-Liouville operator*, J. Math. Anal. Appl. **378** (2011) 549-554.
18. V. Yurko, *Integral transforms connected with discontinuous boundary value problems*, Int. Trans. Spec. Functions, **10** (2000), 141-164.
19. J. Pöschel and E. Trowbowitz, *Inverse Spectral Theory*, Academic Press, 1987.

20. A. Friendman, *foundations of modern analysis*, Dover Publications, Inc. New Yurk, 1982.
21. G. Freiling and V. A. Yurko, *Inverse Sturm-Liouville Problems and Their Applications*, NOVA Science Publishers, New Yurk, 2001.
22. B. Ya. Levin, *Lectures on Entire Functions*, Transl. Math. Monographs **150**, Amer. Math. Soc., Providence, RI, 1996.

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