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MULTIPLICITY OF POSITIVE SOLUTIONS OF LAPLACIAN SYSTEMS WITH SIGN-CHANGING WEIGHT FUNCTIONS

SEYYED SADEGH KAZEMIPOOR[1*] AND MAHBOOBEH ZAKERI[2]

Abstract. In this paper, we study the multiplicity of positive solutions for the Laplacian systems with sign-changing weight functions. Using the decomposition of the Nehari manifold, we prove that an elliptic system has at least two positive solutions.

1. Introduction

In this paper, we study the multiplicity of positive solutions for the following elliptic system:

$$\begin{cases} -\Delta u = \lambda f(x)|u|^{q-2}u + \frac{r}{r+s}h(x)|u|^{r-2}u|v|^{s} & \text{in } \Omega, \\ -\Delta v = \mu g(x)|v|^{q-2}v + \frac{s}{r+s}h(x)|u|^{r}|v|^{s-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \qquad (E_{\lambda,\mu})$$

Where $r > 1,\ s > 1,\ 1 < q < 2 < r+s < 2^*(2^* = \frac{2N}{N-2}$ if $N > 2,\quad 2^* = \infty$ if $N \leq 2)$and $\Omega \subset \mathbb{R}^N$ is a bounded domain, the pair of parameters $(\lambda, \mu) \in \mathbb{R}^2 - \{(0,0)\}$, and the weight functions $f, g, h \in C(\overline{\Omega})$ are satisfying $f^{\pm} = \max\{\pm f, 0\} \not\equiv 0, \quad g^{\pm} = \max\{\pm g, 0\} \not\equiv 0, \quad h^{\pm} = \max\{\pm h, 0\} \not\equiv 0$.
The fact that number of positive solutions of equation (E_λ) is affected by the nonlinearity terms has been the focus of a great deal of research in recent years.

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* Corresponding author.

the authors Ambrosetti-Brezis-Cerami [1] considered the following system:

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + |u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (E_\lambda)$$

They found that there exists $\lambda_0 > 0$ such that equation (E_λ) admit at least two positive solutions for $\lambda \in (0, \lambda_0)$, has a positive solution for $\lambda = \lambda_0$ and no positive solution exsist for $\lambda > \lambda_0$. Wu [8] proved that equation (E_λ) have at least two positive solutions under the assumptions the weight functions f change sign in $\overline{\Omega}$, $\quad g \equiv 1$ and λ is sufficiently small. For more general results, were done by de Figueiredo-Grossez-Ubilla[5], Wu[9] and Brown-Wu[2].
In this paper, we give a very simple variational proof which is similar to proof of Wu (see [10]) to prove the existence of at least two positive solutions of system $(E_{\lambda,\mu})$ and so the system $(E_{\lambda,\mu})$ is similar to the Wu system [11] (a semilinear elliptic system involving sign-changing weight functions). In fact, we use the decomposition of the Nehari manifold as the pair of parameters (λ, μ) varies to prove the following result.

Theorem 1.1. *There exist $\lambda_0 > 0$ and $\mu_0 > 0$ such that for $0 < |\lambda| < \lambda_0$ and $0 < |\mu| < \mu_0$, system $(E_{\lambda,\mu})$ have at least two positive solutions.*

This paper is organized as follows. In section 2, we give some notations and preliminaries. In section 3, we prove the system $(E_{\lambda,\mu})$ has at least two positive solutions.

2. Notations and Preliminaries

Throughout this paper, we denote by S_l the best Sobolev constant for the operators $W = W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \hookrightarrow L = L^l(\Omega) \times L^l(\Omega)$ is given by

$$S_l = \inf_{(u,v)\in W-\{(0,0)\}} \frac{(\int_\Omega |\nabla u|^2 + \int_\Omega |\nabla v|^2)}{(\int_\Omega |u|^l + \int_\Omega |v|^l)^{\frac{2}{l}}} > 0$$

Where $1 < l \leq 2^*$. In particular, $(\int_\Omega |u|^l + \int_\Omega |v|^l) \leq S_l^{-\frac{l}{2}} ||(u,v)||^l$ for all $(u,v) \in W$ with the standard norm $||(u,v)|| = (\int_\Omega |\nabla u|^2 + \int_\Omega |\nabla v|^2)^{\frac{1}{2}}$.

Definition 2.1. Assume that $\phi \in C^1(X, \mathbb{R})$ is such that $\phi'(0) = 0$. A necessary condition for $u \in X$

$$\langle \phi'(u), u \rangle = 0$$

to be a critical point of ϕ. This condition defines the Nehari manifold [7]

$$N := \{u \in X \ : \ \langle \phi'(u), u \rangle = 0, u \neq 0\}.$$

Definition 2.2. The function $f \in C(\overline{\Omega})$ is called the sign changing [7] if and only if $f^{\pm} = \max\{\pm f, 0\} \not\equiv 0$.

System $(E_{\lambda,\mu})$ is posed in the framework of the Sobolev space W. Moreover, a function $(u, v) \in W$ is said to be a weak solution of system $(E_{\lambda,\mu})$ if

$$\int_\Omega \nabla u \nabla \varphi - \lambda \int_\Omega f|u|^{q-2}u\varphi - \frac{r}{r+s}\int_\Omega h|u|^{r-2}u|v|^s\varphi = 0$$

for all $\varphi \in W_0^{1,2}(\Omega)$ and

$$\int_\Omega \nabla v \nabla \varphi - \mu \int_\Omega g|v|^{q-2}v\varphi - \frac{s}{r+s}\int_\Omega h|u|^{r}|v|^{s-2}v\varphi = 0$$

for all $\varphi \in W_0^{1,2}(\Omega)$. Thus, the corresponding energy functional of system $(E_{\lambda,\mu})$ is defined by

$$\begin{aligned} J_{\lambda,\mu}(u,v) &= \tfrac{1}{2}||(u,v)||^2 - \tfrac{1}{q}(\lambda \textstyle\int_\Omega f|u|^q \\ &+ \mu \textstyle\int_\Omega g|v|^q) - \tfrac{1}{r+s}(\textstyle\int_\Omega h|u|^r|v|^s) \qquad for\ (u,v) \in W. \end{aligned}$$

As the energy functional $J_{\lambda,\mu}$ is not bounded below on W, it is useful to consider the functional on the Nehari manifold

$$M_{\lambda,\mu} = \{(u,v) \in W - \{(0,0)\} \mid \langle J'_{\lambda,\mu}(u,v), (u,v)\rangle = 0\}.$$

Thus, $(u, v) \in M_{\lambda,\mu}$ if and only if

$$||(u,v)||^2 - (\lambda \int_\Omega f|u|^q + \mu \int_\Omega g|v|^q) - (\int_\Omega h|u|^r|v|^s) = 0. \qquad (1)$$

Define

$$\begin{aligned} \psi_{\lambda,\mu}(u,v) = \langle J'_{\lambda,\mu}(u,v), (u,v)\rangle &= ||(u,v)||^2 - (\lambda \textstyle\int_\Omega f|u|^q + \mu \int_\Omega g|v|^q) \\ &- \textstyle\int_\Omega h|u|^r|v|^s. \end{aligned}$$

Then for $(u, v) \in M_{\lambda,\mu}$,

$$\begin{aligned} \langle \psi'_{\lambda,\mu}(u,v), (u,v)\rangle &= 2||(u,v)||^2 - q(\lambda \textstyle\int_\Omega f|u|^q + \mu \int_\Omega g|v|^q) \\ &- (r+s)\textstyle\int_\Omega h|u|^r|v|^s \\ &= (2-q)(\lambda \textstyle\int_\Omega f|u|^q + \mu \int_\Omega g|v|^q) \\ &+ (2-r-s)\textstyle\int_\Omega h|u|^r|v|^s. \qquad (2) \end{aligned}$$

Now, we split $M_{\lambda,\mu}$ into three parts:

$$\begin{aligned} M^+_{\lambda,\mu} &= \{(u,v) \in M_{\lambda,\mu} \mid \langle \psi'_{\lambda,\mu}(u,v), (u,v)\rangle > 0\} \\ M^0_{\lambda,\mu} &= \{(u,v) \in M_{\lambda,\mu} \mid \langle \psi'_{\lambda,\mu}(u,v), (u,v)\rangle = 0\} \\ M^-_{\lambda,\mu} &= \{(u,v) \in M_{\lambda,\mu} \mid \langle \psi'_{\lambda,\mu}(u,v), (u,v)\rangle < 0\} \end{aligned}$$

Then, we have the following results.

Lemma 2.3. *If (u_0, v_0) is a local minimizer for $J_{\lambda,\mu}$ on $M_{\lambda,\mu}$ and $(u_0, v_0) \notin M_\lambda^0$, then $J'_{\lambda,\mu}(u_0, v_0) = 0$ in $W' = W^{-1,2'}(\Omega) \times W^{-1,2'}(\Omega)$.*

Proof. Our proof is almost the same as the proof of the Theorem 2.3. in Brown-Zhang [3]. □

Lemma 2.4. *The energy functional $J_{\lambda,\mu}$ is coercive and bounded below on $M_{\lambda,\mu}$.*

Proof. If $(u, v) \in M_{\lambda,\mu}$, then by the Sobolev trace imbedding theorem

$$\begin{aligned} J_{\lambda,\mu}(u,v) &= \tfrac{1}{2}||(u,v)||^2 - \tfrac{1}{q}(\lambda \textstyle\int_\Omega f|u|^q + \mu \int_\Omega g|v|^q) \\ &- \tfrac{1}{r+s} \textstyle\int_\Omega h|u|^r|v|^s \\ &= \tfrac{r+s-2}{2(r+s)}||(u,v)||^2 + (-\tfrac{1}{q} + \tfrac{1}{r+s})(\lambda \textstyle\int_\Omega f|u|^q + \mu \int_\Omega g|v|^q) \\ &\geq \tfrac{r+s-2}{2(r+s)}||(u,v)||^2 + S_q^{\frac{q}{2}}(\tfrac{r+s-q}{q(r+s)})(|\lambda|||f||_\infty||u||^q + |\mu|||g||_\infty||v||^q) \end{aligned}$$

Thus, J_λ is coercive and bounded below on $M_{\lambda,\mu}$. □

Lemma 2.5. *(i) For any $(u, v) \in M^+_{\lambda,\mu}$, we have $(\lambda \int_\Omega f|u|^q + \mu \int_\Omega g|v|^q) > 0$*

(ii) For any $(u, v) \in M^0_{\lambda,\mu}$, we have $(\lambda \int_\Omega f|u|^q + \mu \int_\Omega g|v|^q) > 0$ and $\int_\Omega h|u|^r|v|^s > 0$

(iii) For any $(u, v) \in M^-_{\lambda,\mu}$, we have $\int_\Omega h|u|^r|v|^s > 0$.

Proof. The proofs are immediate from (1) and (2). □

Lemma 2.6. *There exist $\lambda_0 > 0and\mu_0 > 0$ such that for $0 < |\lambda| < \lambda_0$, $0 < |\mu| < \mu_0$ we have $M^0_{\lambda,\mu} = \emptyset$.*

Proof. Suppose that $M^0_{\lambda,\mu} \neq \emptyset$ for all $(\lambda, \mu) \in \mathbb{R}^2 - \{(0, 0)\}$. Then by lemma 2.5,

$$\begin{aligned} 0 &= \langle J'_{\lambda,\mu}(u,v),(u,v)\rangle = (2-q)||(u,v)||^2 \\ &- (r+s-q)\textstyle\int_\Omega h|u|^r|v|^s \\ &= (2-r-s)||(u,v)||^2 - (q-r-s)(\lambda \textstyle\int_\Omega f|u|^q + \mu \int_\Omega g|v|^q) \end{aligned}$$

for all $(u, v) \in M^0_{\lambda,\mu}$. By the Hõlder inequality, Minkowski inequality and the Sobolev imbedding theorem, we have

$$||(u,v)|| \geq (\frac{2(2-q)}{r+s-q}||h||_\infty S_{r+s}^{\frac{r+s}{2}})^{\frac{1}{(2-r-s)}}$$

and

$$||(u,v)|| \leq (\frac{r+s-q}{r+s-2}(|\lambda|\ ||f||_\infty + |\mu|\ ||g||_\infty))^{\frac{1}{2-q}} S_2^{\frac{q}{2(2-q)}}.$$

If $|\lambda|, |\mu|$ is sufficiently small, this is impossible. Thus, we can conclude that if there exist $\lambda_0 > 0, \mu_0 > 0$ such that $0 < |\lambda| < \lambda_0$ and $0 < |\mu| < \mu_0$, then we have $M^0_{\lambda,\mu} = \emptyset$. □

By Lemma 2.6, for $0 < |\lambda| < \lambda_0$ and $0 < |\mu| < \mu_0$ we write $M_{\lambda,\mu} = M^+_{\lambda,\mu} \cup M^-_{\lambda,\mu}$ and define

$$\alpha^+_{\lambda,\mu} = \inf_{(u,v)\in M^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v)$$

and

$$\alpha^-_{\lambda,\mu} = \inf_{(u,v)\in M^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v).$$

Then we have the following results.

Proposition 2.7. There exist minimizing sequences $\{(u^\pm_n, v^\pm_n)\}$ in $M^\pm_{\lambda,\mu}$ for $J_{\lambda,\mu}$ such that $J_{\lambda,\mu}(u^\pm_n, v^\pm_n) = \alpha^\pm_{\lambda,\mu} + o(1)$ and $J'_{\lambda,\mu}(u^\pm_n, v^\pm_n) = o(1)$ in W'.

Proof. The proof is almost the same as the proof of the Proposition 9 in Wu [8]. □

3. Main results

Throughout this section, we assume that the parameters λ and μ satisfies $0 < |\lambda| < \lambda_0$ and $0 < |\mu| < \mu_0$. Then we have the following results.

Theorem 3.1. *System $(E_{\lambda,\mu})$ has a positive solution $(u^+_0, v^+_0) \in M^+_{\lambda,\mu}$ such that $J_{\lambda,\mu}(u^+_0, v^+_0) = \alpha^+_{\lambda,\mu} < 0$.*

Proof. First, we show $\alpha^+_{\lambda,\mu} < 0$. For $(u,v) \in M^+_{\lambda,\mu}$, we have

$$\begin{aligned} J_{\lambda,\mu}(u,v) &= (\tfrac{1}{2} - \tfrac{1}{q})||(u,v)||^2 \\ &+ (\tfrac{1}{q} - \tfrac{1}{r+s})(\textstyle\int_\Omega h|u|^r|v|^s) \\ &< -\tfrac{(2-q)(r+s-2)}{2q(r+s)}||(u,v)||^2 < 0. \end{aligned}$$

This implies $\alpha^+_{\lambda,\mu} < 0$. By Proposition 2.7, there exists $\{(u^+_n, v^+_n)\} \subset M^+_{\lambda,\mu}$ such that $J_{\lambda,\mu}(u^+_n, v^+_n) = \alpha^+_{\lambda,\mu} + 0(1)$ and $J'_{\lambda,\mu}(u^+_n, v^+_n) = o(1)$ in W'. Then by Lemma 2.4 (ii) and the Rellich-Kondrachov theorem there exist a subsequence $\{(u^+_n, v^+_n)\}$ and $(u^+_0, v^+_0) \in W$ is a solution of system $(E_{\lambda,\mu})$ such that $(u^+_n, v^+_n) \to (u^+_0, v^+_0)$ weakly in W and $(u^+_n, v^+_n) \to (u^+_0, v^+_0)$ strongly in L for all $1 \le l < 2^*$.
Then we have

$$\int_\Omega f|u^+_n|^q + \int_\Omega g|v^+_n|^q = \int_\Omega f|u^+_0|^q + \int_\Omega g|v^+_0|^q + o(1)$$

and $(\lambda \int_\Omega f|u_0^+|^q + \mu \int_\Omega g|v_0^+|^q) \geq 0$.
Now, we prove that $(\lambda \int_\Omega f|u_0^+|^q + \mu \int_\Omega g|v_0^+|^q) > 0$ otherwise, then

$$||(u_n^+, v_n^+)||^2 = \int_\Omega h|u_n^+|^r|v_n^+|^s + o(1)$$

and

$$\begin{aligned}(\tfrac{1}{2} - \tfrac{1}{r+s})||(u_n^+, v_n^+)||_W^2 &= \tfrac{1}{2}||(u_n^+, v_n^+)||^2 \\ &- \tfrac{1}{q}(\lambda \textstyle\int_\Omega f|u_n^+|^q + \mu \int_\Omega g|v_n^+|^q) \\ &- \tfrac{1}{r+s} \textstyle\int_\Omega h|u_n^+|^r|v_n^+|^s + o(1) \\ &= \alpha_{\lambda,\mu}^+ + o(1)\end{aligned}$$

This is contradicts $\alpha_{\lambda,\mu}^+ < 0$. Thus, $(\lambda \int_\Omega f|u_0^+|^q + \mu \int_\Omega g|v_0^+|^q) > 0$. In particular, $(u_0^+, v_0^+) \in M_{\lambda,\mu}^+$ is a nontrivial solution of system $(E_{\lambda,\mu})$ and $J_{\lambda,\mu}(u_0^+, v_0^+) \geq \alpha_{\lambda,\mu}^+$. Moreover,

$$\begin{aligned}\alpha_{\lambda,\mu}^+ &\leq J_{\lambda,\mu}(u_0^+, v_0^+) = (\tfrac{1}{2} - \tfrac{1}{q})(\lambda \textstyle\int_\Omega f|u_0^+|^q + \mu \int_\Omega g|v_0^+|^q) \\ &+ (\tfrac{1}{2} - \tfrac{1}{r+s}) \textstyle\int_\Omega h|u_0^+|^r|v_0^+|^s \\ &= \lim_{n\to\infty} J_{\lambda,\mu}(u_n^+, v_n^+) = \alpha_{\lambda,\mu}^+.\end{aligned}$$

Consequently, $J_{\lambda,\mu}(u_0^+, v_0^+) = \alpha_{\lambda,\mu}^+$. Since $J_{\lambda,\mu}(u_0^+, v_0^+) = J_{\lambda,\mu}(|u_0^+|, |v_0^+|)$ and $(|u_0^+|, |v_0^+|) \in M_{\lambda,\mu}^+$. By Lemma 2.3 we may assume that $u_0^+ \geq 0$, $v_0^+ \geq 0$. Moreover, by the Harnack inequality due to Trudinger [6], we obtain (u_0^+, v_0^+) is a positive solution of system $(E_{\lambda,\mu})$. □

Theorem 3.2. *System $(E_{\lambda,\mu})$ has a positive solution $(u_0^-, v_0^-) \in M_{\lambda,\mu}^-$ such that $J_{\lambda,\mu}(u_0^-, v_0^-) = \alpha_{\lambda,\mu}^-$.*

Proof. By Proposition 2.7, there exist $\{(u_n, v_n)\} \subset M_{\lambda,\mu}^-$ such that $J_{\lambda,\mu}(u_n^-, v_n^-) = \alpha_{\lambda,\mu}^- + o(1)$ and $J'_{\lambda,\mu}(u_n^-, v_n^-) = o(1)$ in W'.
By Lemma 2.4 and the Relich-Kondrachov theorem, there exist a subsequence $\{(u_n^-, v_n^-)\}$ and $(u_0^-, v_0^-) \in M_{\lambda,\mu}^-$, where (u_0^-, v_0^-) is a nonzero solution of system $(E_{\lambda,\mu})$ such that $(u_n^-, v_n^-) \to (u_0^-, v_0^-)$ weakly in W and $(u_n^-, v_n^-) \to (u_0^-, v_0^-)$ strongly in L. Moreover,

$$\begin{aligned}\alpha_{\lambda,\mu}^- &\leq J_{\lambda,\mu}(u_0^-, v_0^-) = (\tfrac{1}{2} - \tfrac{1}{q})(\lambda \textstyle\int_\Omega f|u_0^-|^q + \mu \int_\Omega g|v_0^-|^q) \\ &+ (\tfrac{1}{2} - \tfrac{1}{r+s}) \textstyle\int_\Omega h|u_0^-|^r|v_0^-|^s \\ &= \lim_{n\to\infty} J_{\lambda,\mu}(u_n^-, v_n^-) = \alpha_{\lambda,\mu}^-.\end{aligned}$$

Consequently, $J_{\lambda,\mu}(u_0^-, v_0^-) = \alpha_{\lambda,\mu}^-$. Since $J_{\lambda,\mu}(u_0^-, v_0^-) = J_{\lambda,\mu}(|u_0^-|, |v_0^-|)$ and $(|u_0^-|, |v_0^-|) \in M_{\lambda,\mu}^-$. By Lemma 2.3 we may assume that $u_0^- \geq 0$, $v_0^- \geq 0$.
Moreover, by the Haranack inequality due to Trudinger [6], we obtain

(u_0^-, v_0^-) is a positive solution of system $(E_{\lambda,\mu})$.

□

In the sequel we will prove Theorem 1.1.

Proof of Theorem 1.1. By theorem 3.1, 3.2, system $(E_{\lambda,\mu})$ have two positive solutions (u_0^+, v_0^+) and (u_0^-, v_0^-) such that $(u_0^+, v_0^+) \in M_{\lambda,\mu}^+$ and $(u_0^-, v_0^-) \in M_{\lambda,\mu}^-$. Since $M_{\lambda,\mu}^+ \cap M_{\lambda,\mu}^- = \emptyset$, this implies that (u_0^+, v_0^+) and (u_0^-, v_0^-) are distinct.

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[1]Department of Basic Sciences, Payame Noor University of Karaj, Karaj, Iran.

E-mail address: S.Kazemipoor@umz.ac.ir

[2]Department of Basic Sciences, Payame Noor University of Karaj, Karaj, Iran.

E-mail address: mzakeri.math@gmail.com