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COMMUTATIVE CURVATURE OPERATORS OVER FOUR-DIMENSIONAL GENERALIZED SYMMETRIC SPACES

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Abstract. Commutative properties of four-dimensional generalized symmetric pseudo-Riemannian manifolds were considered. Specially, in this paper, we studied Skew-Tsankov and Jacobi-Tsankov conditions in 4-dimensional pseudo-Riemannian generalized symmetric manifolds.

1. Introduction

Generalized symmetric spaces have been intensively studied from several different points of view. In [13], O. Kowalski studied generalized symmetric spaces in an elementary way, that is, without involving neither topological invariants nor advanced algebra. Homogeneous structures of generalized symmetric Riemannian spaces were studied in [10]. S. Terzić classified generalized symmetric spaces defined as quotients of compact simple Lie groups, describing explicitly their real cohomology algebras [15] and calculating their real Pontryagin characteristic classes [16]. Moreover, D. Kotschick and S. Terzić [12] proved that all generalized symmetric spaces are formal, that is, their rational homotopy type is determined by their rational cohomology algebra alone. Also, some geometric structures over four-dimensional generalized symmetric spaces here considered in [6].

This paper is organized as following: In Section 2, we report some basic materials on the 4-dimensional general symmetric spaces. Section 3 is devoted to present manifolds which are Einstein-like in frame field of

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this paper. Finally, four-dimensional general symmetry spaces which have commutative curvature operator will be presented in last section.

2. Preliminaries

Let (M, g) be a connected pseudo-Riemannian manifold and x a point of M. A symmetry at x is an isometry s_x of M, having x as isolated fixed point. When (M, g) is a symmetric space, each point x admits a symmetry s_x reversing geodesics through the point. Hence, s_x is involutive for all x. This property was generalized by A. J. Ledger, who defined a regular s-structure as a family $\{s_x : x \in M\}$ of symmetries of (M, g) satisfying

$$s_x \circ s_y = s_z \circ s_x, \quad z = s_x(y),$$

for all points x, y of M. The order of an s-structure is the least integer $k \geq 2$, such that $(s_x)^k = id_M$ for all x (it may happen that $k = \infty$). A generalized symmetric space is a connected pseudo-Riemannian manifold (M, g) admitting a regular s-structure. The order of a generalized symmetric space is the infimum of all integers $k \geq 2$ such that M admits a regular s-structure of order k. The classification of four-dimensional generalized symmetric spaces was obtained by J. Černý and O. Kowalski and is resumed in the following.

Theorem 2.1. [7] All proper, simply connected generalized symmetric spaces (M, g) of dimension $n = 4$ are of order 3 or infinity. All these spaces are indecomposable, and belong (up to an isometry) to the following four types:

Type A. The underlying homogeneous space is G/H , where

$$G = \begin{bmatrix} a & b & u \\ c & d & v \\ 0 & 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $ad - bc = 1$. (M, g) is the space $R_4(x, y, u, v)$ with the pseudo-Riemannian metric

$$\begin{aligned} g = \pm[(-x + \sqrt{1 + x^2 + y^2})du^2 + (x + \sqrt{1 + x^2 + y^2})du^2 \\ - 2y^2 dudv] + \lambda[(1 + y^2)dx^2 \\ + (1 + x^2)dy^2 - 2xydxdy]/(1 + x^2 + y^2), \end{aligned}$$

where $\lambda \neq 0$ is a real constant. The order is $k = 3$ and possible signatures are $(4, 0)$,$(0, 4)$,$(2, 2)$. The typical symmetry of order 3 at the initial point $(0, 0, 0, 0)$ is the transformation

$$u' = -(\frac{1}{2})u - (\frac{\sqrt{3}}{2})v, \quad v' = -(\frac{\sqrt{3}}{2})u - (\frac{1}{2})v,$$
$$x' = -(\frac{1}{2})x - (\frac{\sqrt{3}}{2})y, \quad v' = -(\frac{\sqrt{3}}{2})x - (\frac{1}{2}).$$

Type B. The underlying homogeneous space is G/H, where

$$G = \begin{bmatrix} e^{-(x+y)} & 0 & 0 & a \\ 0 & e^x & 0 & b \\ 0 & 0 & e^y & c \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 & -w \\ 0 & 1 & 0 & -2w \\ 0 & 0 & 1 & 2w \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(M, g) is the space $R_4(x, y, u, v)$ with the pseudo-Riemannian metric

$$g = \lambda(dx^2 + dy^2 + dxdy) + e^{-y}(2dx + dy) + e^{-x}(dx + 2dy)du,$$

where λ is a real constant. The order is $k = 3$ and the signature is $(2, 2)$. The typical symmetry of order 3 at the initial point $(0, 0, 0, 0)$ is the transformation

$$u' = -u^{(y-z)} - v, \quad v' = -ue^{-(y+2x)},$$
$$x' = y, \quad y' = -(x + y).$$

Type C. The underlying homogeneous space is the matrix group

$$G = \begin{bmatrix} e^{-t} & 0 & 0 & x \\ 0 & e^t & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

(M, g) is the space $R_4(x, z, u, t)$ with the pseudo-Riemannian metric

$$g = \pm(e^{2t}dx^2 + e^{-2t}dy^2) + dzdt.$$

The possible signatures are $(1, 3)$ and $(3, 1)$. These spaces are indeed symmetric.

Type D. The underlying homogeneous space is G/H, where

$$G = \begin{bmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with $ad - bc = 1$. (M, g) is the space $R_4(x, y, u, v)$ with the pseudo-Riemannian metric

$$\begin{aligned} g = &(\sinh(2u) - \cosh(2u)\sin(2u))dx^2 + (\sinh(2u) \\ &+ \cosh(2u)\sin(2u))dy^2 - 2\cosh(2u)\cos(2u)(2u)dxdy \\ &+ \lambda(du^2 - \cosh^2(2u)dv^2), \end{aligned}$$

where $\lambda \neq$ is a real constant. The order is infinite and the signature is $(2, 2)$. The typical symmetry at the initial point $(0, 0, 0, 0)$ is induced by the automorphism of G of the form:

$$\begin{aligned} a^{'} &= a, \quad b^{'} = (\frac{1}{\alpha^2})b, \quad c^{'} = \alpha^2 c, \\ d^{'} &= d, \quad x^{'} = (\frac{1}{\alpha})x, \quad y^{'} = \alpha y, \end{aligned}$$

where $\alpha \neq 0, \pm 1$.

Any generalized symmetric pseudo-Riemannian space is homogeneous. Moreover, it admits at least one structure of reductive homogeneous space with an invariant metric [7]. With regard to the four-dimensional examples, such a reductive decomposition corresponds to their realizations as coset spaces G/H listed in Theorem 2.1 above. Let now $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ be the Lie algebra of G, and $\{u_i\}$, $\{h_r\}$ a basis of $\mathfrak{m}$ and $\mathfrak{h}$ respectively. The Lie algebra structure of $\mathfrak{g}$ is completely described by the multiplication table listing $[u_i, u_j]$, $[u_i, h_r]$, $[h_r, h_s]$, and the inner product g on $\mathfrak{m}$ by its components $g_{ij} = g(u_i, u_j)$. The invariant metric g on $\mathfrak{m}$ uniquely defines its invariant linear Levi-Civita connection, described in terms of the corresponding homomorphism of $\mathfrak{h}$-modules $\Lambda : \mathfrak{g} \longrightarrow \mathfrak{g}$, where

$$\Lambda(x)(y_{\mathfrak{m}}) = [x, y]_{\mathfrak{m}}, \tag{2.1}$$

for all $x \in \mathfrak{h}$, $y \in \mathfrak{g}$ (see for example [11]). Explicitly, one has

$$\Lambda(x)(y_{\mathfrak{m}}) = \frac{1}{2}[x, y]_{\mathfrak{m}} + v(x, y), \quad \forall x, y \in \mathfrak{g}, \tag{2.2}$$

where $v : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{m}$ is the h-invariant symmetric map uniquely determined by

$$2g(v(x, y), z_{\mathfrak{m}}) = g(x_{\mathfrak{m}}, [z, y]_{\mathfrak{m}}) + g(y_{\mathfrak{m}}, [z, x]_{\mathfrak{m}}), \quad \forall x, y, z \in \mathfrak{g}. \tag{2.3}$$

In this way, we can describe the Levi-Civita connection associated to g for any four-dimensional generalized symmetric space. The curvature tensor is then determined by

$$\begin{aligned} R : \mathfrak{m} \times \mathfrak{m} &\longrightarrow \mathfrak{gl}(\mathfrak{m}), \\ (x, y) &\longmapsto [\Lambda(x), \Lambda(y)] - \Lambda([x, y]). \end{aligned} \tag{2.4}$$

Moreover, $\mathcal{J}$ is Jacobi operator determined by

$$\mathcal{J}_R(x) : y \longrightarrow R(y,x)x. \tag{2.5}$$

Finally, the Ricci tensor ϱ of g, described in terms of its components with respect to $\{u_i\}$, is given by

$$\varrho(u_i,u_j) = \sum_{r=1}^{4} R_{ri}(u_r,u_j), \quad i,j = 1,...,4. \tag{2.6}$$

The scalar curvature τ is the trace of ϱ. Finally, with respect to $\{u_i\}$, Weyl conformal curvature tensor is completely determined by its components

$$W_{ijkl} = R_{ijkl} - \frac{\tau}{6}(g_{il}g_{jk} - g_{ik}g_{jl}) + \frac{1}{2}(g_{il}\varrho_{jk} - \varrho_{ik}g_{jl} + \varrho_{il}g_{jk} - g_{ik}\varrho_{jl}), \tag{2.7}$$

where R_{ijkl} are the components of the $(0,4)$-curvature tensor. We shall now explicitly describe the Levi-Civita connection and curvature of generalized symmetric spaces A, B, C, D .

2.1. **Pseudo-Riemannian case of type A.** Let $(M = G/H, g)$ be a four-dimensional generalized symmetric space of type A, where g is an invariant metric of neutral signature $(2,2)$. Following [7], the Lie algebra $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ admits a basis $\{u_1,u_2,u_3,u_4,h_1\}$, where$\{u_1,u_2,u_3,u_4\}$ and $\{h_1\}$ are bases of $\mathfrak{m}$ and $\mathfrak{h}$ respectively, such that (reversing the metric [14] when needed) the Lie bracket on $\mathfrak{g}$ and the inner product on $\mathfrak{m}$ are completely determined by

(2.8)

[,]	u_1	u_2	u_3	u_4	h_1
u_1	0	0	$-\delta u_1$	δu_2	u_2
u_2	0	0	δu_2	δu_1	$-u_1$
u_3	δu_1	$-\delta u_2$	0	$-2\delta^2 h_1$	$-2u_4$
u_4	$-\delta u_2$	$-\delta u_1$	$2\delta^2 h_1$	0	$2u_3$
h_1	$-u_2$	u_1	$2u_4$	$-2u_3$	0

where $\delta > 0$ is a real constant, and

$$g(u_1,u_1) = g(u_2,u_2) = 1, \; g(u_3,u_3) = g(u_4,u_4) = -2. \tag{2.9}$$

Setting $\Lambda[i] = \Lambda(u_i)$ and applying (2.2) and (2.3), a direct calculation yields that we can describe the Levi-Civita connection as follows:

$$\Lambda[1] = \begin{bmatrix} 0 & 0 & -\delta & 0 \\ 0 & 0 & 0 & \delta \\ -\frac{\delta}{2} & 0 & 0 & 0 \\ 0 & \frac{\delta}{2} & 0 & 0 \end{bmatrix}, \quad \Lambda[2] = \begin{bmatrix} 0 & 0 & 0 & \delta \\ 0 & 0 & \delta & 0 \\ 0 & \frac{\delta}{2} & 0 & 0 \\ \frac{\delta}{2} & 0 & 0 & 0 \end{bmatrix}, \tag{2.10}$$

and $\Lambda[3] = \Lambda[4] = 0$. With respect to $\{u_i\}$, the non-zero components of the curvature tensor, determined according to equation (2.4), are the following:

$$R_{12} = \begin{bmatrix} 0 & -\delta^2 & 0 & 0 \\ \delta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta^2 \\ 0 & 0 & \delta^2 & 0 \end{bmatrix}, \quad R_{34} = \begin{bmatrix} 0 & 2\delta^2 & 0 & 0 \\ -2\delta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4\delta^2 \\ 0 & 0 & 4\delta^2 & 0 \end{bmatrix},$$

$$R_{13} = -R_{24} = \begin{bmatrix} 0 & 0 & -\delta^2 & 0 \\ 0 & 0 & 0 & \delta^2 \\ \frac{-\delta^2}{2} & 0 & 0 & 0 \\ 0 & \frac{\delta^2}{2} & 0 & 0 \end{bmatrix}, \tag{2.11}$$

$$R_{14} = R_{23} = \begin{bmatrix} 0 & 0 & 0 & -\delta^2 \\ 0 & 0 & -\delta^2 & 0 \\ 0 & -\frac{\delta^2}{2} & 0 & 0 \\ -\frac{\delta^2}{2} & 0 & 0 & 0 \end{bmatrix}.$$

Moreover, applying (2.4) and (2.5), some standard calculations give that with respect to $\{u_i\}$, that Jacobi operator $\mathcal{J}_i$ is given by

$$\mathcal{J}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\delta^2 & 0 & 0 & 0 \\ -\delta^2 & 0 & 0 & 0 \\ -\delta^2 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{J}_2 = \begin{bmatrix} 0 & -\delta^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\delta^2 & 0 & 0 \\ 0 & -\delta^2 & 0 & 0 \end{bmatrix}, \tag{2.12}$$

$$\mathcal{J}_3 = \begin{bmatrix} 0 & 0 & \frac{\delta^2}{2} & 0 \\ 0 & 0 & \frac{\delta^2}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{J}_4 = \begin{bmatrix} 0 & 0 & 0 & \frac{\delta^2}{2} \\ 0 & 0 & 0 & \frac{\delta^2}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4\delta^2 \end{bmatrix}.$$

Applying (2.4) and (2.6), some standard calculations give that with respect to $\{u_i\}$, that Ricci tensor ϱ is given by

$$\varrho = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -6\delta^2 & 0 \\ 0 & 0 & 0 & -6\delta^2 \end{bmatrix}. \tag{2.13}$$

With respect to $\{u_i\}$, the non-zero components of the Weyl conformal curvature tensor, determined according to equation (2.7), are the following:
(2.14)

$$\begin{array}{llll}
W_{1234} = 2\delta^2, & W_{2134} = -2\delta^2, & W_{3124} = -\delta^2, & W_{4123} = \delta^2, \\
W_{1243} = -2\delta^2, & W_{2143} = 2\delta^2, & W_{3142} = \delta^2, & W_{4132} = -\delta^2, \\
W_{1324} = \delta^2, & W_{2314} = -\delta^2, & W_{3214} = \delta^2, & W_{4213} = -\delta^2, \\
W_{1342} = -\delta^2, & W_{2341} = \delta^2, & W_{3241} = -\delta^2, & W_{4231} = \delta^2, \\
W_{1423} = -\delta^2, & W_{2413} = \delta^2, & W_{3412} = 2\delta^2, & W_{4312} = -2\delta^2, \\
W_{1432} = \delta^2, & W_{2431} = -\delta^2, & W_{3421} = -2\delta^2, & W_{4321} = 2\delta^2.
\end{array}$$

2.2. **Pseudo-Riemannian case of type B.** Let (M, g) be a four-dimensional generalized symmetric space of type B. Then, $(M = G/H, g)$, $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and $\{u_1, u_2, u_3, u_4\}$, $\{h_1\}$ are respectively a basis of $\mathfrak{m}$ and of $\mathfrak{h}$, such that the Lie bracket on $\mathfrak{g}$ and the scalar product on $\mathfrak{m}$ are respectively given by the following tables:

(2.15)

$[,]$	u_1	u_2	u_3	u_4	h_1
u_1	0	0	$-u_1$	$\varepsilon h_1 + u_2$	0
u_2	0	0	$-\varepsilon h_1 + u_2$	u_1	0
u_3	u_1	$\varepsilon h_1 - u_2$	0	0	$2u_2$
u_4	$-\varepsilon h_1 - u_2$	$-u_1$	0	0	$-2u_1$
h_1	0	0	$-2u_2$	$2u_1$	$0,$

where $\varepsilon = \pm 1$, and

$$g(u_1, u_3) = g(u_2, u_4) = -1, \; g(u_3, u_3) = g(u_4, u_4) = 2\lambda, \tag{2.16}$$

where λ is a real constant (see [7]). Notice that the isotropy representation for h_1, which can be deduced at once from (2.15), easily implies that, a vector field $V \in \mathfrak{m}$ is invariant if and only if $V \in Span\{u_1, u_2\}$. With respect to $\{ui\}$, applying equations (2.2) and (2.3) we get $\Lambda[1] = \Lambda[2] = 0$ and

$$\Lambda[3] = \begin{bmatrix} 1 & 0 & -2\lambda & 0 \\ 0 & -1 & 0 & 2\lambda \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \Lambda[4] = \begin{bmatrix} 0 & -1 & 0 & 2\lambda \\ -1 & 0 & 2\lambda & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{2.17}$$

With respect to $\{u_i\}$, the non-zero components of the curvature tensor are given by

$$R_{14} = -R_{23} = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_{34} = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \tag{2.18}$$

and

$$\mathcal{J}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{J}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{J}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}, \quad \mathcal{J}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.19}$$

and

$$\varrho = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}. \tag{2.20}$$

With respect to $\{u_i\}$, the non-zero components of the Weyl conformal curvature tensor, determined according to equation (2.7), are the following:

$$W_{3434} = 4\lambda, \quad W_{3443} = -4\lambda, \quad W_{4334} = -4\lambda, \quad W_{4343} = 4\lambda. \tag{2.21}$$

2.3. **Pseudo-Riemannian case of type C.** Let $(M = G/H, g)$ be a generalized symmetric space of type C. The Lie algebra $\mathfrak{g}$ admits a basis $\{u_1, u_2, u_3, u_4\}$, such that, reversing the metric when needed,

$$[u_1, u_4] = -u_1, \quad [u_2, u_4] = u_2, \tag{2.22}$$

and

$$g(u_1, u_1) = g(u_2, u_2) = -1, \quad g(u_3, u_4) = \frac{1}{2}, \tag{2.23}$$

(see [7]). Applying equations (2.2) and (2.3), with respect to $\{u_i\}$ we then find

$$\Lambda[1] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Lambda[2] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.24}$$

while $\Lambda[3] = \Lambda[4] = 0$. The non-zero components of the curvature tensor with respect to $\{u_i\}$ are described in the following way:

$$R_{14} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_{24} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \tag{2.25}$$

and

$$\mathcal{J}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{J}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$\mathcal{J}_3 = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{J}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.26}$$

and

$$\varrho = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}. \tag{2.27}$$

With respect to $\{u_i\}$, the components of the Weyl conformal curvature tensor identically zero.

2.4. **Pseudo-Riemannian case of type D.** Let $(M = G/H, g)$ denote a generalized symmetric space of type D. For the Lie algebra $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ of the Lie group G, there exist a basis $\{u_1, u_2, u_3, u_4, h_1\}$, with $\{u_1, u_2, u_3, u_4\}$ and $\{h_1\}$ bases of $\mathfrak{m}$ and $\mathfrak{h}$ respectively, such that

(2.28)

$[,]$	u_1	u_2	u_3	u_4	h_1
u_1	0	0	0	$-u_2$	u_1
u_2	0	0	$-u_1$	0	$-u_2$
u_3	0	u_1	0	$-h_1$	$2u_3$
u_4	u_2	0	h_1	0	$-2u_4$
h_1	$-u_1$	u_2	$-2u_3$	$2u_4$	0

and

$$g(u_1, u_2) = 1, \quad g(u_3, u_4) = \lambda, \tag{2.29}$$

where $\lambda \neq 0$ is a real constant [7]. Calculating the isotropy representation for h_1 from the above multiplication table, we see at once that no invariant vector fields $V \neq 0$ occur in $\mathfrak{m}$. With respect to $\{u_i\}$, from equations (2.2) and (2.3) we deduce

$$\Lambda[1] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \frac{1}{\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Lambda[2] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 & 0 \end{bmatrix}, \tag{2.30}$$

and $\Lambda[3] = \Lambda[4] = 0$. With respect to $\{u_i\}$, the non-zero components of the curvature tensor are given by

$$R_{12} = \begin{bmatrix} \frac{1}{\lambda} & 0 & 0 & 0 \\ 0 & -\frac{1}{\lambda} & 0 & 0 \\ 0 & 0 & -\frac{1}{\lambda} & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda} \end{bmatrix}, \quad R_{14} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\lambda} & 0 & 0 & 0 \end{bmatrix},$$

$$R_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ \frac{1}{\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_{34} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \tag{2.31}$$

and

$$\mathcal{J}_1 = \begin{bmatrix} 0 & -\frac{1}{\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{J}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{J}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{J}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}. \tag{2.32}$$

and

$$\varrho = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & -3 & 0 \end{bmatrix}. \tag{2.33}$$

With respect to $\{u_i\}$, the non-zero components of the Weyl conformal curvature tensor, determined according to equation (2.7), are the following:

(2.34)

$$\begin{array}{llll} W_{1234} = 1, & W_{2134} = 1, & W_{3124} = -\frac{1}{2}, & W_{4132} = -\frac{1}{2}, \\ W_{1243} = -1, & W_{2314} = -\frac{1}{2}, & W_{3142} = \frac{1}{2}, & W_{4213} = -\frac{1}{2}, \\ W_{1324} = \frac{1}{2}, & W_{2314} = \frac{1}{2}, & W_{3214} = \frac{1}{2}, & W_{4213} = \frac{1}{2}, \\ W_{1342} = -\frac{1}{2}, & W_{2413} = \frac{1}{2}, & W_{3241} = -\frac{1}{2}, & W_{4312} = -1, \\ W_{1423} = -\frac{1}{2}, & W_{2431} = -\frac{1}{2}, & W_{3412} = 1, & W_{4312} = 1, \\ W_{1432} = \frac{1}{2}, & W_{3421} = -1. & & \end{array}$$

3. Einstein-like manifolds

A pseudo-Riemannian manifold M is called an Einstein manifold provided that $\varrho = cg$ for some constant c.

It is easily shown that for all manifold in frame filed of this paper we have:

Remark 3.1. A four-dimensional pseudo-Riemannian generalized symmetric space $(G/H, g)$ is not an Einstein manifold.

The Ricci tensor is called cyclic parallel if the following condition is satisfied

$$(\nabla_X \varrho)(Y, Z) + (\nabla_Y \varrho)(Z, X) + (\nabla_Z \varrho)(X, Y) = 0, \tag{3.1}$$

and the Ricci tensor is called a Codazzi tensor if

$$(\nabla_X \varrho)(Y, Z) = (\nabla_Y \varrho)(X, Z), \tag{3.2}$$

for arbitrary vector fields X, Y, Z tangent to M. These two classes of pseudo-Rimannian manifolds are called Einstein-like manifolds. The Einstein-like property on pseudo-Rienannian generalized symmetric spaces according to all information and computation which is presented in subsection (2.1) up to subsection (2.4) and by a long and hard process computation using the Mathematica package according to (3.1) and (3.2) for each type A, B, C and D yields to the following remarks.

Remark 3.2. A four-dimensional pseudo-Riemannian generalized symmetric spaces $(M = G/H, g)$ of type A, B and D is never Ricci cyclic parallel, while type C is always Ricci cyclic parallel.

Remark 3.3. A four-dimensional pseudo-Riemannian generalized symmetric spaces $(M = G/H, g)$ of type A, B and D is never Codazzi Ricci tensor, while type C is always Corazzi Ricci tensor.

Remark 3.4. A four-dimensional pseudo-Riemannian generalized symmetric spaces $(M = G/H, g)$ of type A, B and D is not Ricci parallel, while type C is always Ricci parallel.

Corollary 3.5. *A four-dimensional pseudo-Riemannian generalized symmetric spaces $(M = G/H, g)$ of type A, B and D is not Einstein-like, while type C is Einstein-Like.*

4. Commutatively Property

Commuting properties of curvature operators have been systematically investigated by several authors. Commutativity properties of the skew-symmetric curvature operator and of the Jacobi operator were first studied in the Riemannian setting by Tsankov [17] for hypersurfaces in

$\mathbb{R}^{n+1}$ and subsequently extended to the general pseudo-Riemannian context in [3] (see also [4,5]). Commutativity properties among the Ricci, the Jacobi and the skew-symmetric curvature operators have also been considered in the literature. Refer to [1, 8, 9] for more information.

Definition 4.1. Let $\mathcal{M} := (M, g)$ be a pseudo-Riemannian manifold

a) $\mathcal{M}$ is Jacobi-Tsankov if

$$\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x)$$

where $x, y \in \mathfrak{m}$,

b) $\mathcal{M}$ is mixed-Tsankov if

$$R(x, y)\mathcal{J}(z) = \mathcal{J}(z)R(x, y)$$

where $x, y, z \in \mathfrak{m}$,

c) $\mathcal{M}$ is skew-Tsankov if

$$R(x, y)R(z, w) = R(z, w)R(x, y)$$

where $x, y, z, w \in \mathfrak{m}$,

d) $\mathcal{M}$ is Jacobi-Videv if

$$\mathcal{J}(x)\varrho = \varrho\mathcal{J}(x)$$

for all $x \in \mathfrak{m}$,

f) $\mathcal{M}$ is skew-Videv if

$$R(x, y)\varrho = \varrho R(x, y)$$

where $x, y \in \mathfrak{m}$.

It is worth to emphasize here that skew-Videv commuting and Jacobi-Videv commuting are equivalent conditions [9] as well as skew-Tsankov commuting and mixed-Tsankov commuting [9]. Moreover, mixed-Tsankov commuting Lorentzian manifolds are flat [9]. The geometrical significance of the curvature-curvature commuting condition

(i.e., $R(w, x)R(y, z) = R(y, z)R(w, x)$ for all w, x, y, z)

is not well-understood yet, though some progresses have been made in the Riemannian setting [2] and also in the 3-dimensional Lorentzian manifold [8]. The study facuses on the analysis of condition (b), (c) and (f) of definition (4.1). The study presents,

Theorem 4.2. *A four-dimensional pseudo-Riemannian generalized symmetric space $(M = G/H, g)$ is:*

(i) *skew-Tsankov if and only if that is of type B or C.*
(ii) *Jacobi-Tsankov if ond only if that is of type B.*

Proof. Now we have all argument to handle the proof of Theorem, according to all information and computation which is presented in subsection (2.1) up to subsection (2.4), by a straightforward computation according to case (f) in definition (4.1) we set $R_{ij}R_{kl} - R_{kl}R_{ij} = 0,\ 1 \leq i < j, k < l \leq 4$, and will obtain that types A and D are not satisfy in this condition so the first part of theorem deduced. To check the Jacobi-Tsankov commuting, we set $\mathcal{J}_i\mathcal{J}_j - \mathcal{J}_j\mathcal{J}_i = 0,\ 1 \leq i < j \leq 4$. The same method for this solution shows that only type B satisfy in this commuting condition, this matter finishes the proof. □

Remark 4.3. A four-dimensional pseudo-Riemannian generalized symmetric spaces $(M = G/H, g)$ of type C is 2-step nilpotent.

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References

1. M. Brozos-Vázquez, E. Garcia, P. Gilkey and R. -Vázquez-Lorenzo, *Examples of signature (2, 2)-manifolds with commuting curvature operators*, J. Phys. A: Math. Theor. 40 (2007) 13149–13159.
2. M. Brozos-Vázquez and P. Gilkey, *The global geometry of Riemannian manifolds with commuting curvature operators*, J. Fixed Point Theory Appl. 1 (2007) 87–96.
3. M. Brozos-Vázquez and P. Gilkey, *Manifolds with commuting Jacobi operators*, J. Geom. 86 (2007) 21–30.
4. G. Calvaruso, Harmonicity of vector fields on four-dimensional generalized symmetric spaces, *Cent. Eur. J. Math.* 10 (2012), 411–425.
5. G. Calvaruso and B. De Leo, Curvature Properties of Four-Dimensional Generalized Symmetric Spaces, *J. Geom.* 90 (no. 1-2) (2008), 30–46.
6. G. Calvaruso and A. H. Zaeim, Geometric Structures over Four-Dimensional Generalized Symmetric Spaces, *Mediterr. J. Math.* 10 (2013), 971–987.
7. J. *Černý* and O. Kowalski, Classification of generalized symmetric pseudo-Riemannian spaces of dimension $n \leq 4$, *Tensor (N.S.)* 38 (1982), 256–267.
8. E. Garcia-Rio, A. Haji-Badali, M. E. Vázquez-Abal and R. Vázqes-Lorenzo, Lorentzian 3-manifold with commuting curvature operators, *Int. J. Geom. Meth. Modern Phys.* 5 (4) (2008), 557–572.
9. P. Gilkey, *Geometric Properties of neutral Operators Defined by the Riemannian Curvature Tensor* World Scientific Publishing Co., Inc., River Edge, NJ, 2001.
10. C. Gonzalez and D. Chinea, Estructuras homogeneas sobre espacios simetricos generalizados, Proceedings of the XIIth Portuguese-Spanish Conference on Mathematics, Vol. II, 572–578, Univ. Minho, Braga, 1987.
11. B. Komrakov Jnr., Einstein-Maxwell equation on four-dimensional homogeneous spaces, *Lobachevskii J. Math.* 8 (2001), 33–165.
12. D. Kotschick and S. Terzić, On formality of generalized symmetric spaces, *Math. Proc. Cambridge Philos. Soc.* 134 (2003), 491–505.
13. O. Kowalski, Generalized symmetric spaces, Lectures Notes in Math. 805, Springer-Verlag, Berlin-New York, 1980.

14. B. O'Neill, Semi-Riemannian Geometry, Pure and Applied Mathematics 103, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983.
15. S. Terzić, Real cohomology of generalized symmetric spaces, *Fundam. Prikl. Mat.* 7(2001), 131–157.
16. S. Terzić, Pontryagin classes of generalized symmetric spaces *(Russian), Mat. Zametki* 69 (2001), 613–621. Translation in Math. Notes 69 (no. 3–4) (2001), 559–566.
17. Y. Tsankov, *A characterization of n-dimensional hypersurface in Euclidean space with commuting curvature operators*, Banach Center Publ. 69 (2005) 205–209.

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