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CHAOTIC DYNAMICS AND SYNCHRONIZATION OF FRACTIONAL ORDER PMSM SYSTEM

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Abstract. In this paper, we investigate the chaotic behaviors of the fractional-order permanent magnet synchronous motor (PMSM) system. The necessary condition for the existence of chaos in the fractional-order PMSM system is deduced and an active controller is developed based on the stability theory for fractional systems. The presented control scheme is simple and flexible, and it is suitable both for design and for implementation in practice. Simulation is carried out to verify that the obtained scheme is efficient and robust for controlling the fractional-order PMSM system.

1. Introduction

Fractional order calculus can be traced to the work of Leibniz and Hospital in 1695, which has a history of more than 300 years. Fractional order calculus refers to the arbitrarily differential or integral order, including irrational number and complex number. Integer-order calculus only depends on the local characteristics of the function. However, fractional-order calculus has memory characteristic, and accumulates a certain range's overall information of the function. Fractional calculus has applications in numerous seemingly diverse and widespread fields of science and engineering. Applications including modeling of damping behavior of viscoelastic materials, cell diffusion processes, transmission of signals through strong magnetic fields, and finance systems are some examples [3].

On the other hand, synchronization of chaotic systems has attracted much attentions since the seminal paper by Pecora and Carroll [5]. It

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has attracted great attention due to it superior potential applications, for examples, in communication, laser physics and optics, mechanics, and chemistry. Many methods have been developed for synchronizing of chaos such as, linear control, LMI based approach, backstepping design, sliding control, adaptive control, active control, and passivity based control [2, 6].

In this paper, the chaotic behaviors of the fractional-order PMSM system is investigated. The necessary condition for the existence of chaos in the fractional-order PMSM system is deduced and an active control scheme is presented to study the synchronization behavior between two identical PMSM chaotic systems.

This paper is organized as follows. The section 2 provides a brief review of fractional calculus and the dynamics of PMSM system. Stability analysis of equilibrium points follows in this section. Then the active synchronization of PMSM system is studied in section 3. Numerical simulations are given in section 4. We conclude the paper in section 5.

2. PMSM System

PMSM system [8] can be described by

$$\begin{aligned} \dot{x}_1 &= \frac{1}{\tau_1}(x_2x_3 - x_1 + u_d); \\ \dot{x}_2 &= \frac{1}{\tau_2}(-x_2 - x_1x_3 - x_3 + u_q); \\ \dot{x}_3 &= \frac{1}{\tau_3}(ax_1x_2 + bx_2 - cx_3 - T_l), \end{aligned} \tag{2.1}$$

where τ_1, τ_2, τ_3, a, b, c, u_d, u_q and T_l are constant parameters.

When we set $\tau_1 = 6.45$, $\tau_2 = 7.125$, $\tau_3 = 1$, $a = 1.516$, $b = 16$, $c = 1.8$, $u_d = -12.70$, $u_q = 2.34$ and $T_l = 0.525$, the system is chaotic.

PMSM system has five equilibrium points:

$$\begin{aligned} &P_1(-10.4194, 5.9502, 0.3833), \\ &P_2(-12.6901, -0.0486, -0.2043), \\ &P_3(-0.8844, 1.2225, 9.6648), \\ &P_4(-1.3715, -1.1916, -9.5071), \\ &P_5(-10.4428, -3.5927, -0.6283). \end{aligned}$$

Definition 2.1. The Caputo differential operator of order $\alpha > 0$ is defined by

$$ {}_0^cD^\alpha y(t) = J_0^{m-\alpha} y^{(m)}(t), $$

where, $m := \lceil \alpha \rceil = \min\{z \in Z : z \geq \alpha\}$, $y(t) \in L^1[0, b]$, $y^{(m)} \in L^1[0, b]$ and

$$ J_0^\beta y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta-1} y(\tau) dt, $$

is Reimann-Liouville fractional integral operator of order $\beta > 0$ [1].

We can use the three-dimensional autonomous system to describe fractional order PMSM system

$$
\begin{aligned}
{}_0^cD_t^{q_1} x_1 &= \frac{1}{\tau_1}(x_2 x_3 - x_1 + u_d), \\
{}_0^cD_t^{q_2} x_2 &= \frac{1}{\tau_2}(-x_2 - x_1 x_3 - x_3 + u_q), \qquad (2.2) \\
{}_0^cD_t^{q_3} x_3 &= \frac{1}{\tau_3}(a x_1 x_2 + b x_2 - c x_3 - T_l),
\end{aligned}
$$

where $0 < q_i < 1$, $i = 1, 2, 3$. The parameters we used in system (2.2) are same as system (2.1), so they have the same equilibrium points: P_1, P_2, P_3, P_4 and P_5. Equilibrium points have special significance for the system. To analysis the stability of the equilibrium points, we need the help of Jacobi matrix. We calculated the Jacobi matrix of PMSM system as

$$
J = \begin{bmatrix} -\frac{1}{\tau_1} & \frac{1}{\tau_1}x_3 & \frac{1}{\tau_1}x_2 \\ -\frac{1}{\tau_2}x_3 & -\frac{1}{\tau_2} & -\frac{1}{\tau_2}(x_1+1) \\ \frac{a}{\tau_3}x_2 & \frac{1}{\tau_3}(a x_1 + b) & -\frac{c}{\tau_3} \end{bmatrix}.
$$

Calculating of the Jacobi matrix nearby a point, in fact, is an approximate linear process. So we can use fractional order linear systems stability theory to analysis the stability of this point.

Theorem 2.2 ([4]). *Consider the linear fractional order system:*

$$
{}_0^cD^q x(t) = Ax(t), \qquad x(0) = x_0
$$

with $x \in R^n$, $A \in R^{n\times n}$ and $q = (q_1, q_2, \ldots, q_n)^T$, $0 < q_i \le 1$. Also, $q_i = \frac{n_i}{d_i}$ and $gcd(n_i, d_i) = 1$, $i = 1, 2, \ldots, n$. Let M be the lowest common multiple of the d_1, d_2 and d_3. The zero solution of the system is globally asymptotically stable in the Lyapunov's sense, if all roots λ of the equation $\triangle(\lambda) = \det(\mathrm{diag}(\lambda^{Mq_i}) - A) = 0$, $i = 1, 2, \ldots, n$ satisfy $\arg(\lambda) > \frac{\pi}{2M}$.

We calculated the eigenvalues of JP_2. They are -0.1545 and $-0.9705 \pm 2.1506i$. Since all of them have negative real part, so the equilibrium point is stable. As stable equilibrium point has excellent stability, then the system operates around the neighbor domain of P_2, so the system will be automatically stable, and we can priority choose P_2 as the control objective.

For P_1, the eigenvalues of JP_1 are -3.9914, 2.1166, -0.2206 and $|\arg(eig_2)| = 0$. So no matter the value of q, the system could not stable to P_1 without any control.

Similarly, the eigenvalues of JP_5 are -2.8951, 1.1347 and -0.3351, the system could not stable to P_5 without any control either.

For P_3, the eigenvalues of JP_3 are -2.3802 and $0.1424 \pm 1.7693i$. For P_4, the eigenvalues of JP_4 are -2.5531 and $0.2289 \pm 1.6053i$. The equilibrium points P_3 and P_4 are saddle points of index 2. It is known that scrolls in a chaotic attractor, are generated only around the saddle points of index 2 in chaotic systems of Shinikov's type. Moreover, saddle points of index 1 are responsible only for connecting scrolls. Suppose that Ω is the set of equilibrium points of the system surrounded by scrolls. A necessary condition for fractional order system ${}_0^cD_t^q x = f(x)$, to exhibit the chaotic attractor similar to its integer order counterpart, is instability of the equilibrium points in Ω. This necessary condition is mathematically equivalent to condition: $\frac{\pi}{2M} - \min_i\{\arg(\lambda_i)\} \geq 0$, where λ_i's are roots of $\det(\mathrm{diag}(\lambda^{Mq_i}) - J\,|_Q) = 0$, $\forall Q \in \Omega$ [7].

We assume $(q_1, q_2, q_3) = (0.9, 0.8, 0.7)$. The characteristic equation of the system (2.2), evaluated at the equilibrium point P_3, is

$$\begin{aligned}\triangle(\lambda) &= \det(\mathrm{diag}(\lambda^{Mq_1}, \lambda^{Mq_2}, \lambda^{Mq_3}) - J\,|_{P_3})\\ &= \det(\mathrm{diag}(\lambda^9, \lambda^8, \lambda^7) - J\,|_{P_3}) = 0,\end{aligned}$$

or

$$\begin{aligned}&\lambda^{24} + 1.8\lambda^{17} + 1.1404\lambda^{16} + 0.4905\lambda^9 + 0.1551\lambda^{15}\\ &\quad - 0.0722\lambda^8 + 2.0543\lambda^7 + 7.4992 = 0.\end{aligned}$$

Since

$$\frac{\pi}{2M} - \min_i\{\arg(\lambda_i)\} = \frac{\pi}{20} - 0.05309 = 0.10399 \geq 0,$$

thus the fractional order system (2.2) exhibits the chaotic attractor similar to its integer order counterpart.

3. Active Synchronization of Fractional Order System

Consider the drive-response synchronization scheme of two autonomous different fractional order chaotic systems:

$$\begin{aligned}&\text{Drive:} && {}_a^cD_t^q x(t) = f(x);\\ &\text{Response:} && {}_a^cD_t^q y(t) = g(y) + u,\end{aligned}$$

where q is the fractional order, $x, y \in R^n$ represent the states of the drive and response systems, respectively, $f, g : R^n \to R^n$ are the vector fields of the drive and response systems, respectively. The aim is to choose a suitable control function $u = (u_1, u_2, \ldots, u_n)^T$, such that the states of the drive and response systems are synchronized, i.e.

$$\lim_{t\to\infty} \|e\| = \lim_{t\to\infty} \|y(t) - x(t)\| = 0.$$

In what follows we proceed to design a new synchronization scheme for two chaotic systems which are different in order and initial conditions.

Consider the following chaotic incommensurate fractional order system as the drive system:

$$ {}_{t_0}^{c}D_t^{q_i} x_i(t) = f_i(x_1, x_2, x_3), \tag{3.1}$$

with the initial conditions $(x_1(t_0), x_2(t_0), x_3(t_0)) = (x_{10}, x_{20}, x_{30}) \in R^3$, and f_i, $i = 1, 2, 3$ are nonlinear functions. Also suppose that the structure of the response system is as follows:

$$\dot{y}_i = f_i(y_1, y_2, y_3) + u_i, \tag{3.2}$$

with the initial conditions $(y_1(t_0), y_2(t_0), y_3(t_0)) = (y_{10}, y_{20}, y_{30}) \in R^3$, and u_i, $i = 1, 2, 3$ are the control signals. Note that in spite of discrepancy between initial conditions and difference between orders of the systems, we want to synchronize the systems. Now, if we consider the controller structure as follows:

$$u_i = v_i + \dot{y}_i - {}_{t_0}^{c} D_t^{q_i} y_i(t), \quad i = 1, 2, 3, \tag{3.3}$$

then the slave system (3.2) reduces to:

$$ {}_{t_0}^{c}D_t^{q_i} y_i(t) = f_i(y_1, y_2, y_3) + v_i, \quad i = 1, 2, 3. \tag{3.4}$$

Let $e_i = y_i - x_i$, $i = 1, 2, 3$, is the synchronization error vector. By subtracting (3.1) from (3.4) we have:

$$ {}_{t_0}^{c}D_t^{q_i} e_i(t) = f_i(y_1, y_2, y_3) - f_i(x_1, x_2, x_3) + v_i, \quad i = 1, 2, 3. \tag{3.5}$$

Based on the active control methodology, we choose

$$v_i = f_i(x_1, x_2, x_3) - f_i(y_1, y_2, y_3) + a_i e_1 + b_i e_2 + c_i e_3, \quad i = 1, 2, 3.$$

Therefore, (3.5) reduces to:

$$\begin{aligned} {}_{t_0}^{c}D_t^{q_1} e_1(t) &= a_1 e_1 + b_1 e_2 + c_1 e_3, \\ {}_{t_0}^{c}D_t^{q_2} e_2(t) &= a_2 e_1 + b_2 e_2 + c_2 e_3, \\ {}_{t_0}^{c}D_t^{q_3} e_3(t) &= a_3 e_1 + b_3 e_2 + c_3 e_3. \end{aligned} \tag{3.6}$$

Now, by choosing appropriate constants a_i, b_i, c_i, $i = 1, 2, 3$, we can design a stabilizing controller for our synchronization goal. Note that for checking the stability of (3.6), we must use Theorem 2.2. The details of design procedure, can be seen in the numerical simulations in the next section.

4. Numerical simulation

In this section, the numerical simulations are presented to verify the effectiveness of the proposed method. We consider the system (2.2) with order $(q_1, q_2, q_3) = (0.9, 0.8, 0.7)$ and initial conditions

$$(x_1(0), x_2(0), x_3(0)) = (-10, 0.3, -0.2),$$

which exhibits chaos according to the simulations, and the controlled system

$$\begin{aligned} \dot{y}_1 &= \frac{1}{\tau_1}(y_2y_3 - y_1 + u_d) + u_1, \\ \dot{y}_2 &= \frac{1}{\tau_2}(-y_2 - y_1y_3 - y_3 + u_q) + u_2, \\ \dot{y}_3 &= \frac{1}{\tau_3}(ay_1y_2 + by_2 - cy_3 - T_l) + u_3, \end{aligned} \tag{4.1}$$

which is the response system, where u_1, u_2 and u_3 are control functions and initial conditions are $(y_1(0), y_2(0), y_3(0)) = (-8, -0.7, -1.5)$.

We choose u_i, $i = 1, 2, 3$ as (3.3). Thus, the response system (4.1) reduces to

$$\begin{aligned} {}_{t_0}^{c}D_t^{q_1} y_1 &= \frac{1}{\tau_1}(y_2y_3 - y_1 + u_d) + v_1, \\ {}_{t_0}^{c}D_t^{q_2} y_2 &= \frac{1}{\tau_2}(-y_2 - y_1y_3 - y_3 + u_q) + v_2, \\ {}_{t_0}^{c}D_t^{q_3} y_3 &= \frac{1}{\tau_3}(ay_1y_2 + by_2 - cy_3 - T_l) + v_3. \end{aligned} \tag{4.2}$$

Subtracting (2.2) form (4.2) and considering errors $e_i = y_i - x_i$, $i = 1, 2, 3$, the error system can be described as

$$\begin{aligned} {}_{t_0}^{c}D_t^{q_1} e_1 &= \frac{1}{\tau_1}(y_2y_3 - x_2x_3 - e_1) + v_1, \\ {}_{t_0}^{c}D_t^{q_2} e_2 &= \frac{1}{\tau_2}(-e_2 - e_3 - y_1y_3 + x_1x_3) + v_2, \\ {}_{t_0}^{c}D_t^{q_3} e_3 &= \frac{1}{\tau_3}(ay_1y_2 - ax_1x_2 + be_2 - ce_3) + v_3. \end{aligned} \tag{4.3}$$

Selecting

$$\begin{aligned} v_1 &= e_2 + \frac{1}{\tau_1}(-y_2y_3 + x_2x_3 + e_1), \\ v_2 &= e_3 + \frac{1}{\tau_2}(e_2 + e_3 + y_1y_3 - x_1x_3), \\ v_3 &= \frac{1}{\tau_3}(-ay_1y_2 + ax_1x_2) - k_1e_1 - (k_2 + \frac{b}{\tau_3})e_2 - (k_3 - \frac{c}{\tau_3})e_3, \end{aligned}$$

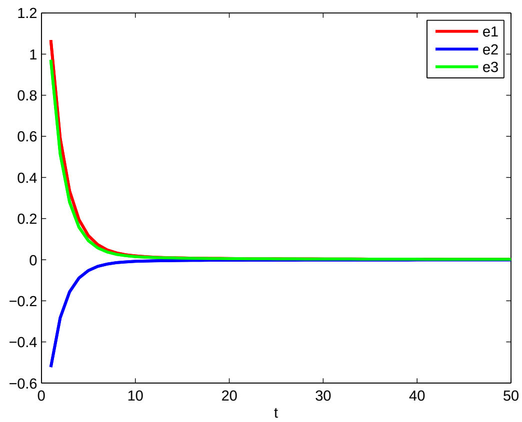

FIGURE 1. The convergence of dynamics of the error system (4.4).

the error system (4.3) reduces to

$$
\begin{aligned}
{}_{t_0}^{c}D_t^{q_1} e_1 &= e_2, \\
{}_{t_0}^{c}D_t^{q_2} e_2 &= e_3, \\
{}_{t_0}^{c}D_t^{q_3} e_3 &= -k_1 e_1 - k_2 e_2 - k_3 e_3.
\end{aligned}
\tag{4.4}
$$

Thus by choosing appropriate k_1, k_2 and k_3, we can stabilize the error vector. If we choose $k_1 = 48$, $k_2 = 44$, and $k_3 = 12$, we see that the eigenvalues of (4.4) are: -2, -4, and -6. Let's determine the stability of (4.4) for these k_i's. According to Theorem 2.2, we constitute $\triangle(\lambda)$ for (4.4) as follows

$$\triangle(\lambda) = \lambda^{24} + 12\lambda^{17} + 44\lambda^{9} + 48 = 0.$$

Solving this equation for λ, we see that $\min(\arg(\lambda_i)) = 0.3196$, which is greater than $\frac{\pi}{2M} = 0.1571$. Therefore based on Theorem 2.2, we conclude the stability of (4.4). Figure 1 shows the numerical results for this synchronization scheme.

5. CONCLUSIONS

In this paper, we proposed a simple active synchronization method that synchronizes two different chaotic systems. The differences are the initial conditions and orders. The drive system was considered a fractional order, and the response system was considered an integer order. We designed an active controller capable to force the trajectories of the response system to track the drive trajectories. Analytical and numerical investigations clarified the effectiveness of the proposed method.

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