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PARABOLIC STARLIKE MAPPINGS OF THE UNIT BALL B^n

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Abstract. Let f be a locally univalent function on the unit disk U. We consider the normalized extensions of f to the Euclidean unit ball $B^n \subseteq \mathbb{C}^n$ given by

$$\Phi_{n,\gamma}(f)(z) = \left(f(z_1), \left(f'(z_1)\right)^{\gamma} \hat{z}\right),$$

where $\gamma \in [0, 1/2]$, $z = (z_1, \hat{z}) \in B^n$ and

$$\Psi_{n,\beta}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\beta} \hat{z}\right),$$

in which $\beta \in [0, 1]$, $f(z_1) \neq 0$ and $z = (z_1, \hat{z}) \in B^n$. In the case $\gamma = 1/2$, the function $\Phi_{n,\gamma}(f)$ reduces to the well known Roper-Suffridge extension operator. By using different methods, we prove that if f is parabolic starlike mapping on U then $\Phi_{n,\gamma}(f)$ and $\Psi_{n,\beta}(f)$ are parabolic starlike mappings on B^n.

1. Introduction

Let $\mathbb{C}^n$ be the vector space of n-complex variables $z = (z_1, \dots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{k=1}^{n} z_k \bar{w}_k$ and Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. The open ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by B_r^n and the unit ball B_1^n by B^n. In the case of one complex variable, B^1 is denoted by U. It is convenient, if $n \geq 2$ to write a vector $z \in \mathbb{C}^n$ as $z = (z_1, \hat{z})$, where $z_1 \in \mathbb{C}$ and $\hat{z} = (z_2, \dots, z_n) \in \mathbb{C}^{n-1}$.

Let $H(B^n, \mathbb{C}^n)$ denotes the topological vector space of all holomorphic mappings $F : B^n \to \mathbb{C}^n$. Let $F \in H(B^n)$, we say that F is normalized if $F(0) = 0$ and $DF(0) = I$, where DF is the Fréchet differential of F and I is the identity operator on $\mathbb{C}^n$. Let $S(B^n)$ be the set of normalized

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biholomorphic mappings on B^n, and $S_1 = S$ is the classical family of univalent mappings of U.

A map $f \in S(B^n)$ is said to be a convex if its image is convex domain in $\mathbb{C}^n$, and starlike if its image is a starlike domain with respect to 0. We denote the classes of normalized convex and starlike mappings on B^n respectively by $K(B^n)$ and $S^*(B^n)$.

In 1995, Roper and Suffridge [8] introduced an extension operator which gives a way of extending a (locally) univalent function on the unit disk U to a (locally) univalent mapping of B^n into $\mathbb{C}^n$. This operator is defined for a normalized locally biholomorphic function f in the unit disk U in $\mathbb{C}$ by (see [3] and [8])

$$[\Phi_n(f)](z) = \left(f(z_1), \sqrt{f'(z_1)}\hat{z}\right),$$

where $z = (z_1, \hat{z}) \in B^n$ and we choose the branch of the square root such that $\sqrt{f'(z_1)}\Big|_{z_1=0} = 1$.

The following results illustrate the importance and usefulness of the Roper-Suffridge extension operator

$$\Phi_n(K) \subseteq K(B^n), \qquad \Phi_n(S^*) \subseteq S^*(B^n).$$

The first was proved by Roper and Suffridge when they introduced their operator [8], while the second result was given by Graham and Kohr [1]. Till now, it has been difficult to make constant the concrete convex mappings and starlike mappings on B^n. By making use of the Roper-Suffridge extension operator, we may easily give many concrete examples about these mappings. This is one important reason why people are interested in this extension operator. A good treatment of further applications of the Roper-Suffridge extension operator can be found in the recent book by Graham and Kohr [3].

The authors [4] considered the following operator

$$\Phi_{n,\gamma}(f)(z) = \left(f(z_1), \left(f'(z_1)\right)^{\gamma}\hat{z}\right), \qquad z = (z_1, \hat{z}) \in B^n,$$

where $\gamma \in [0, 1/2]$ and f is a locally univalent function in U, normalized by $f(0) = f'(0) - 1 = 0$. We choose the branch of the power function such that $\left(f'(z_1)\right)^{\gamma}|_{z_1=0} = 1$. Of course when $\gamma = 1/2$, we obtain the Roper-Suffridge extension operator. In [4], a number of extension results were obtained related to the operator $\Phi_{n,\gamma}$, $\gamma \in [0, 1/2]$: if $f \in S$, then $\Phi_{n,\gamma}(f)$ can be embedded in a Loewner chain and moreover $\Phi_{n,\gamma}(f) \in S^0(B^n)$. In particular, if $f \in S^*$, then $\Phi_{n,\gamma}(f) \in S^*(B^n)$. It was also proved that convexity is preserved only if $\gamma = 1/2$.

In [2], Graham and Kohr introduced another extension operator for the locally biholomorphic function f on U by

$$\Psi_{n,\beta}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1} \right)^{\beta} \hat{z} \right), \qquad z = (z_1, \hat{z}) \in B^n,$$

where $\beta \in [0,1]$ and $f(z_1) \neq 0$, when $z_1 \in U \backslash \{0\}$, and we choose the branch of the power function such that $\left(\frac{f(z_1)}{z_1} \right)^{\beta} \Big|_{z_1=0} = 1$. They proved that the operator $\Psi_{n,\beta}(f)$ maps the normalized starlike function on U to a normalized starlike mapping on B^n. When $\beta = 1$, it was proved and discussed by Pfaltzgraff and Suffridge [7].

Remark 1.1. Let $g : U \to \mathbb{C}$ be a holomorphic univalent function such that $g(0) = 1$, $g(\bar{\eta}) = \overline{g(\eta)}$ for $\eta \in U$ (so, g has real coefficients in its power series expansion), $\mathrm{Re} g(\eta) > 0$ on U and assume g satisfies the following

$$\begin{cases} \min\limits_{|\eta|=r} \mathrm{Re} g(\eta) = \min \{ g(r), g(-r) \}, \\ \\ \max\limits_{|\eta|=r} \mathrm{Re} g(\eta) = \max \{ g(r), g(-r) \}, \end{cases} \tag{1.1}$$

for $r \in (0,1)$.

For example, the condition (1.1) is satisfied by all functions which are convex in the direction of the imaginary axis and symmetric about the real axis (see [6]).

Let

$$\mathcal{M}_g = \left\{ \begin{array}{c} h \in H(B^n) : h(0) = 0, \quad Dh(0) = I_n, \quad \left\langle h(z), \frac{z}{\|z\|^2} \right\rangle \in g(U), \\ z \in B^n \backslash \{0\} \end{array} \right\}.$$

For $g(\xi) = \frac{1+\xi}{1-\xi}$, $\xi \in U$, we obtain the well known set $\mathcal{M}_g = \mathcal{M}$ of mapping with "positive real part on B^n", i.e.

$$\mathcal{M}_g = \left\{ \begin{array}{c} h \in H(B^n) : h(0) = 0, \quad Dh(0) = I_n, \quad \mathrm{Re} \left\langle h(z), \frac{z}{\|z\|^2} \right\rangle > 0, \\ z \in B^n \backslash \{0\} \end{array} \right\}.$$

Now, we give the definition of parabolic starlike mappings on B^n (see [5]). Let

$$q(\eta) = 1 + \frac{4}{\pi^2} \left(\log \frac{1 + \sqrt{\eta}}{1 - \sqrt{\eta}} \right)^2.$$

Then q is a biholomorphic mapping from U onto domain Ω, where

$$\Omega = \{w = u + iv : v^2 < 4u\} \\ = \{w : |w - 1| < 1 + \mathrm{Re} w\}.$$

We note that Ω is a parabolic region in the right half- plane.

Definition 1.2. Let f be a normalized locally biholomorphic mapping on B^n, we say that f is a parabolic starlike mapping if

$$\left\langle [Df(z)]^{-1} f(z), \frac{z}{\|z\|^2} \right\rangle \in g(U), \qquad z \in B^n \backslash \{0\},$$

where $g = \frac{1}{q}$.

Let f be a parabolic starlike mapping on B^n. Since

$$\mathrm{Re} \left\langle [Df(z)]^{-1} f(z), z \right\rangle > 0,$$

parabolic starlike mappings are starlike mappings by Suffridge [9].

In order to prove the main results, we need the following lemma:

Lemma 1.3. [5]. *Let $g = \frac{1}{q}$. Then, $g(U)$ is starlike with respect to* 1.

2. Main Results

Theorem 2.1. *Let $f : U \to C$ be a normalized locally univalent function, which satisfies the condition*

$$\left| \frac{z_1 f'(z_1)}{f(z_1)} - 1 \right| < 1, \qquad z_1 \in U. \tag{2.1}$$

Also, let $F = \Phi_{n,\gamma}(f)$, then

$$\left| \frac{\|z\|^2}{\langle DF^{-1}(z) F(z), z \rangle} - 1 \right| < 1, \qquad z \in B^n \backslash \{0\},$$

and hence F is a parabolic starlike mapping on B^n.

Proof. Without loss of generality, we may assume that f is holomorphic on the closed unit disk $\overline{U}$, since otherwise we use the function $f_r(z_1) = f(rz_1)/r$ for $r \in (0,1)$, which is holomorphic on $\overline{U}$. Taking into account the minimum principle for harmonic functions, we have to prove that

$$\left| \frac{1}{\langle DF^{-1}(z) F(z), z \rangle} - 1 \right| < 1, \qquad z = (z_1, \hat{z}) \in B^n, \qquad \|z\| = 1,$$

i.e.

$$\mathrm{Re} \left\langle DF^{-1}(z) F(z), z \right\rangle > \frac{1}{2}, \qquad z \in \partial B^n.$$

We know that the inequality (2.1) is equivalent to

$$\mathrm{Re} \left\{ \frac{f(z_1)}{z_1 f'(z_1)} \right\} > \frac{1}{2}, \qquad z_1 \in U.$$

Since f is parabolic starlike, $F = \Phi_{n,\gamma}(f)$ is starlike on B^n, and hence biholomorphic. A short computation yields that

$$DF(z) = \begin{bmatrix} f'(z_1) & 0 \\ \gamma\left(f'(z_1)\right)^{\gamma-1} f''(z_1)\hat{z} & \left(f'(z_1)\right)^{\gamma} \end{bmatrix},$$

therefore,

$$DF^{-1}(z)F(z) = \begin{bmatrix} \frac{f(z_1)}{f'(z_1)} \\ -\gamma\frac{f''(z_1)f(z_1)}{(f'(z_1))^2}\hat{z} + \hat{z} \end{bmatrix}.$$

Then we have

$$\begin{aligned}\left\langle DF(z)^{-1}F(z), z\right\rangle &= \frac{f(z_1)}{f'(z_1)}\bar{z}_1 + \|\hat{z}\|^2\left(1 - \gamma\frac{f(z_1)f''(z_1)}{(f'(z_1))^2}\right) \\ &= |z_1|^2\frac{f(z_1)}{z_1 f'(z_1)} + \|\hat{z}\|^2 - \|\hat{z}\|^2\gamma\frac{f(z_1)f''(z_1)}{(f'(z_1))^2},\end{aligned}$$

for $z = (z_1, \hat{z}) \in B^n$. We must show that

$$\text{(2.2)} \qquad \mathrm{Re}\langle DF(z)^{-1}F(z), z\rangle > \frac{1}{2}, \qquad z \in \partial B^n.$$

Therefore, by making use of the equality $|z_1|^2 + \sum_{j=2}^{n} |z_j|^2 = 1$, we must show that

$$|z_1|^2\mathrm{Re}\left\{\frac{f(z_1)}{z_1 f'(z_1)}\right\} + \left(1 - |z_1|^2\right) - \gamma\left(1 - |z_1|^2\right)\mathrm{Re}\left\{\frac{f(z_1)f''(z_1)}{(f'(z_1))^2}\right\} > \frac{1}{2}.$$

If $z = (z_1, 0) \in B^n$, then

$$\begin{aligned}\mathrm{Re}\langle DF(z)^{-1}F(z), z\rangle &> |z_1|^2\mathrm{Re}\left\{\frac{f(z_1)}{z_1 f'(z_1)}\right\} \\ &> \frac{\|z\|^2}{2},\end{aligned}$$

and hence we may assume $\hat{z} \neq 0$. Then F is holomorphic in a neighborhood of each $z \in \bar{B}^n$ for each $\hat{z} \neq 0$. In view of the minimum principle for harmonic functions, it suffices to prove that

$$\mathrm{Re}\langle DF(z)^{-1}F(z), z\rangle \geq \frac{1}{2}, \qquad \|z\| = 1.$$

Let $p(z_1) = \frac{f(z_1)}{z_1 f'(z_1)}$ for $|z_1| < 1$. Since f is parabolic starlike on U, we obtain that $\mathrm{Re}p(z_1) > \frac{1}{2}$. Also let $q(z_1) = 2p(z_1) - 1$. Then $\mathrm{Re}q(z_1) > 0$ for $|z_1| < 1$, and thus (see e.g. [3], Theorem 2.1.3)

$$\mathrm{Re}\left\{z_1 q'(z_1)\right\} \geq \frac{-2|z_1|}{1 - |z_1|^2}\mathrm{Re}q(z_1), \qquad |z_1| < 1.$$

Therefore, we deduce that

$$\operatorname{Re}\left\{z_1 p'(z_1)\right\} \geq \frac{-2|z_1|}{1-|z_1|^2}\operatorname{Re}p(z_1) + \frac{|z_1|}{1-|z_1|^2}. \tag{2.3}$$

On the other hand, since

$$\frac{f''(z_1)f(z_1)}{(f'(z_1))^2} = 1 - z_1 p'(z_1) - p(z_1),$$

we deduce from (2.3) that

$$\begin{aligned}
&|z_1|^2\operatorname{Re}\left\{\frac{f(z_1)}{z_1 f'(z_1)}\right\} + \left(1-|z_1|^2\right) - \gamma\left(1-|z_1|^2\right)\operatorname{Re}\left\{\frac{f(z_1)f''(z_1)}{(f'(z_1))^2}\right\}\\
&\quad = |z_1|^2\operatorname{Re}p(z_1) + \left(1-|z_1|^2\right) - \gamma\left(1-|z_1|^2\right)\operatorname{Re}\left\{1 - z_1 p'(z_1) - p(z_1)\right\}\\
&\quad = \left(|z_1|^2 + \gamma\left(1-|z_1|^2\right)\right)\operatorname{Re}p(z_1) + \gamma\left(1-|z_1|^2\right)\operatorname{Re}\left\{z_1 p'(z_1)\right\}\\
&\qquad + \left(1-|z_1|^2\right)(1-\gamma)\\
&\quad \geq \left(|z_1|^2 + \gamma\left(1-|z_1|^2\right)\right)\operatorname{Re}p(z_1) + \left(1-|z_1|^2\right)(1-\gamma)\\
&\qquad + \gamma\left(1-|z_1|^2\right)\left(\frac{-2|z_1|}{1-|z_1|^2}\operatorname{Re}p(z_1) + \frac{|z_1|}{1-|z_1|^2}\right)\\
&\quad = \left(|z_1|^2 + \gamma\left(1-|z_1|^2\right)\right)\operatorname{Re}p(z_1) - 2\gamma|z_1|\operatorname{Re}p(z_1) + \gamma|z_1|\\
&\qquad + \left(1-|z_1|^2\right)(1-\gamma)\\
&\quad \geq \frac{1}{2}\left(\gamma + (1-\gamma)|z_1|^2\right) + \left(1-|z_1|^2\right)(1-\gamma)\\
&\quad = \frac{1}{2}\gamma + \frac{1-\gamma}{2}|z_1|^2 + \left(1-|z_1|^2\right)(1-\gamma)\\
&\quad = \frac{1}{2}\gamma + (1-\gamma)\left(1 - \frac{1}{2}|z_1|^2\right)\\
&\quad \geq \frac{1}{2}\gamma + \frac{1-\gamma}{2} = \frac{1}{2}.
\end{aligned}$$

Hence the relation (2.2) holds, as desired. This completes the proof. □

In view of Theorem 2.1, we obtain the following particular cases. This result, was obtained in [5], in the case $\gamma = \frac{1}{2}$.

Corollary 2.2. *Let $f : U \to C$ be a normalized locally univalent function, which satisfies the condition*

$$\left|\frac{z_1 f'(z_1)}{f(z_1)} - 1\right| < 1, \qquad z_1 \in U.$$

Also, let $F = \Phi_n(f)$ where

$$\Phi_n(f)(z) = (f(z_1), \sqrt{f'(z_1)}\hat{z}), \qquad z = (z_1, \hat{z}) \in B^n.$$

then

$$\left|\frac{\|z\|^2}{\langle DF^{-1}(z)F(z), z\rangle} - 1\right| < 1, \qquad z \in B^n\backslash\{0\},$$

and hence F is a parabolic starlike mapping on B^n.

In the next Theorem, through a different method, we show that $\Psi_{n,\beta}(f)$ is also a parabolic starlike mapping on B^n.

Theorem 2.3. *Assume that f be a parabolic starlike function on U. For any $\beta \in [0,1]$, let $F = \Psi_{n,\beta}(f)$. Then $\Psi_{n,\beta}(f)$ is a parabolic starlike mapping on B^n.*

Proof. Let $f \in S$ be a parabolic starlike mapping on U, since parabolic starlike mappings are starlike mappings with respect to 1 (Lemma 1.3) then we must show that

$$F(z) = \Phi_{n,\beta}(f)(z) = \begin{bmatrix} f(z_1) \\ \left(\frac{f(z_1)}{z_1}\right)^{\beta} \hat{z} \end{bmatrix},$$

is a parabolic starlike map with respect to 1 on B^n. For this purpose, through simple calculations we have

$$DF(z) = \begin{bmatrix} f'(z_1) & 0 \\ \beta \left(\frac{f(z_1)}{z_1}\right)^{\beta-1} \left(\frac{f'(z_1)}{z_1} - \frac{f(z_1)}{z_1^2}\right) \hat{z} & \left(\frac{f(z_1)}{z_1}\right)^{\beta} \end{bmatrix},$$

and

$$DF^{-1}(z)F(z) = \begin{bmatrix} \frac{f(z_1)}{f'(z_1)} \\ \beta \left(\frac{f(z_1)}{z_1 f'(z_1)} - 1\right) \hat{z} + \hat{z} \end{bmatrix},$$

therefore

$$\langle DF(z)^{-1}F(z), z\rangle = \frac{f(z_1)}{f'(z_1)}\bar{z}_1 + \beta\|\hat{z}\|^2 \left(\frac{f(z_1)}{z_1 f'(z_1)} - 1\right) + \|\hat{z}\|^2.$$

Now we will show that

$$\left|\frac{1}{\|z\|^2}\langle DF(z)^{-1}F(z), z\rangle - 1\right| < 1.$$

For this purpose we have

$$\left|\frac{1}{\|z\|^2}\left\{\frac{|z_1|^2 f(z_1)}{z_1 f'(z_1)} + \beta\|\hat{z}\|^2 \left(\frac{f(z_1)}{z_1 f'(z_1)} - 1\right) + \|\hat{z}\|^2\right\} - 1\right|$$

$$= \left| \frac{1}{\|z\|^2} \left\{ (|z_1|^2 + \beta\|\hat{z}\|^2) \frac{f(z_1)}{z_1 f'(z_1)} - \beta\|\hat{z}\|^2 + \|\hat{z}\|^2 \right\} - 1 \right|$$
$$= \left| \frac{|z_1|^2 + \beta\|\hat{z}\|^2}{\|z\|^2} \left(\frac{f(z_1)}{z_1 f'(z_1)} - 1 \right) + \frac{|z_1|^2 + \beta\|\hat{z}\|^2}{\|z\|^2} + \frac{(1-\beta)\|\hat{z}\|^2}{\|z\|^2} - 1 \right|$$
$$= \left| \frac{|z_1|^2 + \beta\|\hat{z}\|^2}{\|z\|^2} \left(\frac{f(z_1)}{z_1 f'(z_1)} - 1 \right) + \frac{(1-\beta)\|\hat{z}\|^2}{\|z\|^2} - \frac{(1-\beta)\|\hat{z}\|^2}{\|z\|^2} \right|$$
$$= \frac{|z_1|^2 + \beta\|\hat{z}\|^2}{\|z\|^2} \left| \frac{f(z_1)}{z_1 f'(z_1)} - 1 \right|$$
$$\leq \frac{|z_1|^2 + \beta\|\hat{z}\|^2}{\|z\|^2} \leq 1,$$

where $0 \leq \beta \leq 1$. This completes the proof. □

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