

SOME PROPERTIES OF FUZZY REAL NUMBERS

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ABSTRACT. In the mathematical analysis, there are some theorems and definitions that established for both real and fuzzy numbers. In this study, we try to prove Bernoulli's inequality in fuzzy real numbers with some of its applications. Also, we prove two other theorems in fuzzy real numbers which are proved before, for real numbers.

1. INTRODUCTION

According to [5] if η is a non-negative fuzzy real number, then for any real number $p \neq 0$, η^p is a fuzzy real number, with some of their properties were proved before. (This paper is trying to study some properties which are proved in [5] for two real fuzzy numbers η^p and δ^p). Also in [4], Bernoulli's inequality states that, if α is a real number whenever $1 + \alpha \geq 0$, then we have $(1 + \alpha)^n \geq 1 + n\alpha$. Furthermore $(1 + \alpha)^n = 1 + n\alpha$ if and only if $\alpha = 0$ or $n = 1$, and this theorem indicates for every positive x and positive integer $n > 0$, there is one and only one positive real y such that $y^n = x$ (in this study we try to prove the above theorems in fuzzy real numbers). Section 2 contains basic definitions and theorems. In Section 3, we prove Bernoulli's inequality and two other theorems in fuzzy real numbers.

2. PRELIMINARIES AND BASIC FACTS

In this section, we introduce some preliminaries in the theory of fuzzy real numbers. Let η be a fuzzy subset on \mathbb{R} , i.e., a mapping $\eta : \mathbb{R} \rightarrow [0, 1]$ which associates with each real number t , its grade of membership $\eta(t)$. [3]

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Definition 2.1 ([2]). A fuzzy subset η on \mathbb{R} is called a fuzzy real number, whose its α -level set is denoted by $[\eta]_\alpha$, i.e., $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies two axioms:

(N₁) There exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.

(N₂) For each $\alpha \in (0, 1]$, $[\eta]_\alpha = [\eta_\alpha^1, \eta_\alpha^2]$ where $-\infty < \eta_\alpha^1 \leq \eta_\alpha^2 < +\infty$.

The set of all fuzzy real numbers denoted by $F(\mathbb{R})$. If $\eta \in F(\mathbb{R})$ and $\eta(t) = 0$ whenever $t < 0$, then η is called a non-negative fuzzy real number and $F^*(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers. The number $\bar{0}$ stands for the fuzzy real number as:

$$\bar{0}(t) = \begin{cases} 1, & t = 0, \\ 0, & t \neq 0. \end{cases}$$

Clearly, $\bar{0} \in F^*(\mathbb{R})$. Also the set of all real numbers can be embedded in $F(\mathbb{R})$, because, if $r \in (-\infty, \infty)$, then $\bar{r} \in F(\mathbb{R})$ satisfies $\bar{r}(t) = \bar{0}(t - r)$.

Theorem 2.2 ([1]). Let $\{[a_\alpha, b_\alpha] : \alpha \in (0, 1]\}$ be a family of nested bounded closed intervals and $\eta : \mathbb{R} \rightarrow [0, 1]$ be a function defined by

$$\eta(t) = \sup\{\alpha \in (0, 1] : t \in [a_\alpha, b_\alpha]\}.$$

Then η is a fuzzy real number.

Theorem 2.3 ([1]). Let $[a_\alpha, b_\alpha]$, $0 < \alpha \leq 1$, be a family of non-empty intervals. If

- (i) $[a_{\alpha_1}, b_{\alpha_1}] \supset [a_{\alpha_2}, b_{\alpha_2}]$ for all $0 < \alpha_1 \leq \alpha_2 \leq 1$,
- (ii) $[\lim_{k \rightarrow \infty} a_{\alpha_k}, \lim_{k \rightarrow \infty} b_{\alpha_k}] = [a_\alpha, b_\alpha]$ whenever $\{\alpha_k\}$ is an increasing sequence in $(0, 1]$ converging to α ,

then the family $[a_\alpha, b_\alpha]$ represents the α -level sets of a fuzzy real number η such that $\eta(t) = \sup\{\alpha \in (0, 1] : t \in [a_\alpha, b_\alpha]\}$ and $[\eta]_\alpha = [\eta_{\alpha_2}, \eta_{\alpha_2}] = [a_\alpha, b_\alpha]$.

Definition 2.4 ([2]). Fuzzy arithmetic operations \oplus , \ominus , \otimes and \oslash on $F(\mathbb{R}) \times F(\mathbb{R})$ can be defined as:

- (i) $(\eta \oplus \delta)(t) = \sup_{s \in \mathbb{R}} \{\eta(s) \wedge \delta(t - s)\}$, $t \in \mathbb{R}$,
- (ii) $(\eta \ominus \delta)(t) = \sup_{s \in \mathbb{R}} \{\eta(s) \wedge \delta(s - t)\}$, $t \in \mathbb{R}$,
- (iii) $(\eta \otimes \delta)(t) = \sup_{s \in \mathbb{R}, s \neq 0} \{\eta(s) \wedge \delta(t/s)\}$, $t \in \mathbb{R}$,
- (iv) $(\eta \oslash \delta)(t) = \sup_{s \in \mathbb{R}} \{\eta(st) \wedge \delta(s)\}$, $t \in \mathbb{R}$.

Definition 2.5 ([2]). For $k \in \mathbb{R} \setminus 0$, fuzzy scalar multiplication $k \odot \eta$ is defined as $(k \odot \eta) = \eta(t/k)$ and $0 \odot \eta$ is defined to be $\bar{0}$.

Lemma 2.6 ([5]). *Let η and δ be fuzzy real numbers. Then*

$$\forall t \in \mathbb{R}, \quad \eta(t) = \delta(t) \quad \Leftrightarrow \quad \forall \alpha \in (0, 1], \quad [\eta]_\alpha = [\delta]_\alpha.$$

Lemma 2.7 ([2]). *Let $\eta, \delta \in F(\mathbb{R})$ and $[\eta]_\alpha = [\eta_\alpha^1, \eta_\alpha^2]$ and $[\delta]_\alpha = [\delta_\alpha^1, \delta_\alpha^2]$. Then*

- (i) $[\eta \oplus \delta]_\alpha = [\eta_\alpha^1 + \delta_\alpha^1, \eta_\alpha^2 + \delta_\alpha^2]$,
- (ii) $[\eta \ominus \delta]_\alpha = [\eta_\alpha^1 - \delta_\alpha^2, \eta_\alpha^2 - \delta_\alpha^1]$,
- (iii) $[\eta \otimes \delta]_\alpha = [\eta_\alpha^1 \delta_\alpha^1, \eta_\alpha^2 \delta_\alpha^2]$, $\eta, \delta \in F^*(\mathbb{R})$,
- (iv) $[1 \otimes \delta]_\alpha = [1/\delta_\alpha^2, 1/\delta_\alpha^1]$, $\delta_\alpha^1 > 0$.

Definition 2.8 ([5]). *Let η be a non-negative fuzzy real number and $p \neq 0$ be a real number. We define η^p as:*

$$\eta^p(t) = \begin{cases} \eta(t^{1/p}), & t \geq 0, \\ \eta^p(t) = 0, & t < 0. \end{cases}$$

Also we set $\eta^p = \bar{1}$, in the case $p = 0$.

In [5], Sadeqi and al. showed that η^p is a non-negative fuzzy real number, i.e., $\eta^p \in F^*(R)$, $\forall p \in R$. We need to investigate conditions (N_1) and (N_2) in the definition of fuzzy real numbers. For condition (N_1) , since η is a fuzzy real number, there exists $t_0 \in [0, \infty)$, such that $\eta(t_0) = 1$. Set $t' = t_0^p$. Then

$$\eta^p(t') = \eta((t')^{1/p}) = \eta((t_0^p)^{1/p}) = \eta(t_0) = 1.$$

For condition (N_2) , since $[\eta]_\alpha = [\eta_\alpha^1, \eta_\alpha^2]$, for all $\alpha \in (0, 1]$, we have

$$\begin{aligned} [\eta^p]_\alpha &= \{t; \eta^p(t) \geq \alpha\} \\ &= \{t; \eta(t^{1/p}) \geq \alpha\} \\ &= \{s^p; \eta(s) \geq \alpha\} \\ &= (\{s; \eta(s) \geq \alpha\})^p \\ &= ([\eta_\alpha^1, \eta_\alpha^2])^p. \end{aligned}$$

Therefore, η^p is a fuzzy real number. Also it is clear that if $p > 0$, then $[\eta^p]_\alpha = [(\eta_\alpha^1)^p, (\eta_\alpha^2)^p]$ and if $p < 0$, then $[\eta^p]_\alpha = [(\eta_\alpha^2)^p, (\eta_\alpha^1)^p]$.

Theorem 2.9 ([5]). *Let η be a non-negative fuzzy real number and p, q be non-zero integers. Then*

- (i) $p > 0 \Rightarrow \eta^p = \otimes_{i=1}^p \eta$,
- (ii) $p < 0 \Rightarrow \eta^p = \bar{1} \otimes (\otimes_{i=1}^{-p} \eta)$,
- (iii) $\eta^p \otimes \eta^q = \eta^{p+q}$; $pq > 0$,
- (iv) $(\eta^p)^q = \eta^{pq}$.

Definition 2.10 ([2]). *Define a partial ordering \preceq in $F(R)$ by $\eta \preceq \delta$ if and only if $\eta_\alpha^1 \leq \delta_\alpha^1$ and $\eta_\alpha^2 \leq \delta_\alpha^2$ for all $\alpha \in (0, 1]$. The strict inequality*

in $F(R)$ is defined by $\eta \prec \delta$ if and only if $\eta_\alpha^1 < \delta_\alpha^1$ and $\eta_\alpha^2 < \delta_\alpha^2$ for all $\alpha \in (0, 1]$.

3. MAIN RESULTS

Using results of the last section and an idea of W. Rudin [4], we prove the following theorems.

Proposition 3.1. *Let η, δ be fuzzy real numbers and p be a non-zero integer. Then*

- (i) $\eta^p \otimes \delta^p = (\eta \otimes \delta)^p = \otimes_{i=1}^p (\eta \otimes \delta)$ if $p > 0$,
- (ii) $\eta^p \otimes \delta^p = 1 \odot (\otimes_{i=1}^{-p} (\eta \otimes \delta))$ if $p < 0$,
- (iii) $\eta^p \odot \delta^p = (\eta \odot \delta)^p = \otimes_{i=1}^p (\eta \odot \delta)$.

Proof. We prove this proposition by using Lemmas 2.6, 2.7 and Theorem 2.9

- (i) Let $p > 0$ and $\alpha \in (0, 1]$. Then

$$\begin{aligned}
 (3.1) \quad [\eta^p \otimes \delta^p]_\alpha &= [(\eta_\alpha^1)^p (\delta_\alpha^1)^p, (\eta_\alpha^2)^p (\delta_\alpha^2)^p] \\
 &= \left[\prod_{i=1}^p (\eta_\alpha^1 \delta_\alpha^1), \prod_{i=1}^p (\eta_\alpha^2 \delta_\alpha^2) \right] \\
 &= [\otimes_{i=1}^p \eta \otimes \delta]_\alpha.
 \end{aligned}$$

- (ii) For all $\alpha \in (0, 1]$ and $p < 0$, we have

$$\begin{aligned}
 (3.2) \quad [\eta^p \otimes \delta^p]_\alpha &= ([\eta_\alpha^1 \delta_\alpha^1, \eta_\alpha^2 \delta_\alpha^2])^p \\
 &= [(\eta_\alpha^2 \delta_\alpha^2)^p, (\eta_\alpha^1 \delta_\alpha^1)^p] \\
 &= [1/(\eta_\alpha^2 \delta_\alpha^2)^{-p}, 1/(\eta_\alpha^1 \delta_\alpha^1)^{-p}] \\
 &= \left[1/\prod_{i=1}^{-p} (\eta_\alpha^2 \delta_\alpha^2), 1/\prod_{i=1}^{-p} (\eta_\alpha^1 \delta_\alpha^1) \right] \\
 &= [\bar{1} \odot (\otimes_{i=1}^{-p} \eta \otimes \delta)]_\alpha.
 \end{aligned}$$

(iii) Let $p > 0$ and $\alpha \in (0, 1]$. Then

$$\begin{aligned}
 (3.3) \quad [\eta^p \circledast \delta^p]_\alpha &= [(\eta_\alpha^1/\delta_\alpha^2)^p, (\eta_\alpha^2/\delta_\alpha^1)^p] \\
 &= \left[\prod_{i=1}^p (\eta_\alpha^1/\delta_\alpha^2), \prod_{i=1}^p (\eta_\alpha^2/\delta_\alpha^1) \right] \\
 &= [\otimes_{i=1}^p (\eta \circledast \delta)]_\alpha.
 \end{aligned}$$

In case $p < 0$, the proof is similar. It is easy to check the other cases. □

Theorem 3.2. *For any non-negative fuzzy real number $\eta > \bar{0}$ and any integer $n > 0$, there is one and only one fuzzy real number $\delta > 0$ such that $\delta^n = \eta$.*

Proof. Let δ be a non-negative fuzzy real number. Then, according to Definition 2.8, δ^n is a fuzzy real number and also, according to the Theorem 2.9, $[\delta^n]_\alpha = [(\delta_\alpha^1)^n, (\delta_\alpha^2)^n]$, since both δ_α^1 and δ_α^2 are non-negative real numbers. Using [4], we have $(\delta_\alpha^1)^n = a_\alpha$ and $(\delta_\alpha^2)^n = b_\alpha$. Therefore, we have $[\delta^n]_\alpha = [a_\alpha, b_\alpha] = [\eta]_\alpha$. Now, let $\delta_1, \delta_2 \in F^*(\mathbb{R})$, $(\delta_1)^n = \eta$ and $(\delta_2)^n = \eta$, then $\delta_1 = \delta_2$. Hence δ is a unique fuzzy real number. □

Corollary 3.3. *If a and b are two positive fuzzy real numbers and n is a positive integer, then $(a \otimes b)^{1/n} = (a^{1/n}) \otimes (b^{1/n})$.*

Proof. Put $\eta = a^{1/n}$ and $\delta = b^{1/n}$. According to Proposition 3.1 we have $a \otimes b = \eta^n \otimes \delta^n = (\eta \otimes \delta)^n$, so, according to Theorem 3.2, we have:

$$\begin{aligned}
 (a \otimes b)^{1/n} &= \eta \otimes \delta \\
 &= (a^{1/n}) \otimes (b^{1/n}).
 \end{aligned}$$
□

Theorem 3.4. *(Bernoulli's inequality) Let α be a fuzzy real number such that $1 \oplus \alpha \succeq 0$. Then, for all $n \in \mathbb{N}$, we have $(1 \oplus \alpha)^n \succeq 1 \oplus n\alpha$. Furthermore $(1 \oplus \alpha)^n = 1 \oplus n\alpha \iff \alpha = 0$ or $n = 1$.*

Proof. We prove this theorem by using Definition 2.10 and Theorem 2.9. Using mathematical induction, for $n = 1$, we have $(1 \oplus \alpha) \succeq 1 \oplus \alpha$. If

the theorem is true for $n = k$, then for $n = k + 1$ we have

$$\begin{aligned}
 (3.4) \quad (1 \oplus \alpha)^{k+1} &= (1 \oplus \alpha)^k \otimes (1 \oplus \alpha) \\
 &\succeq (1 \oplus k\alpha) \otimes (1 \oplus \alpha) \\
 &= 1 \oplus \alpha \oplus k\alpha \oplus k\alpha^2 \\
 &\succeq 1 \oplus \alpha \oplus k\alpha \\
 &= 1 \oplus (1 + k)\alpha,
 \end{aligned}$$

hence, the theorem is true for $n = k + 1$ too.

If $\alpha = 0$ or $n = 1$, the proof is clear. Conversely, let $(1 \oplus n\alpha) = (1 \oplus \alpha)^n$ and $n \neq 0$, then we have $1 \oplus n\alpha = (1 \oplus \alpha)^n \succeq 1 \oplus n\alpha \oplus (n - 1)\alpha^2$ and hence

$$1 \succeq 1 \oplus (n - 1)\alpha^2 \quad \Rightarrow \quad (n - 1)\alpha^2 \preceq 0,$$

we know that $n - 1 > 0$, so we conclude that $\alpha^2 \preceq 0$. It follows that $\alpha^2 = 0$, hence $\alpha = 0$. Then equality is established. \square

Corollary 3.5. *Let β be a fuzzy real number such that $1 \oplus \beta \succeq 0$. Then for all $n \in \mathbb{N}$, we have $\sqrt[n]{1 \oplus \beta} \preceq 1 \oplus \beta/n$. Therefore $\sqrt[n]{1 \oplus \beta} = 1 \oplus \beta/n$ if and only if $\beta = 0$ or $n = 1$.*

Proof. Let $\beta \otimes n = \alpha$. Using of Bernoulli's inequality, we have $(1 \oplus \alpha)^n \succeq 1 \oplus n\alpha$. So $\sqrt[n]{1 \oplus n\alpha} \preceq 1 \oplus \alpha$ or $\sqrt[n]{1 \oplus \beta} \preceq 1 \oplus \beta/n$. \square

Theorem 3.6. *Let x, y be fuzzy real numbers and $x \oplus y \succeq 0$. Then for all $n \in \mathbb{N}$, we have*

$$(x \oplus y/2)^n \preceq x^n \oplus y^n/2.$$

Proof. We prove this theorem by using Definition 2.10 and Theorem 2.9. Using mathematical induction, for $n = 1$, we have $(x \oplus y/2) \leq x \oplus y/2$. If theorem is true for $n = k$, then for $n = k + 1$ we have

$$\begin{aligned}
 (3.5) \quad (x_\alpha^1 + y_\alpha^1/2)^{k+1} &\leq (x_\alpha^1)^k + (y_\alpha^1)^k/2 \times (x_\alpha^1 + y_\alpha^1/2) \\
 &= (x_\alpha^1)^{k+1} + (x_\alpha^1)^k y_\alpha^1 + x_\alpha^1 (y_\alpha^1)^k + (y_\alpha^1)^{k+1}/4 \\
 &\leq (x_\alpha^1)^{k+1} + (y_\alpha^1)^{k+1}/2.
 \end{aligned}$$

Similarly we have $(x_\alpha^2 + y_\alpha^2/2)^{k+1} \leq (x_\alpha^2)^{k+1} + (y_\alpha^2)^{k+1}/2$. Hence from the above inequality and Definition 2.10, we have $(x \oplus y/2)^n \preceq x^n \oplus y^n/2$. \square

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