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MENGER PROBABILISTIC NORMED SPACE IS A CATEGORY TOPOLOGICAL VECTOR SPACE

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Abstract. In this paper, we formalize the Menger probabilistic normed space as a category in which its objects are the Menger probabilistic normed spaces and its morphisms are fuzzy continuous operators. Then, we show that the category of probabilistic normed spaces is isomorphicly a subcategory of the category of topological vector spaces. So, we can easily apply the results of topological vector spaces in probabilistic normed spaces.

1. Introduction

The notion of a probabilistic metric space has been introduced in 1942 by K. Menger [8]. The first idea of K. Menger was to use distribution functions instead of non-negative real numbers as values of the metric. In [7], it has been shown that a class of probabilistic metric spaces belongs to the class of fuzzy metric spaces which is introduced in [6]. For more information one can see [2, 4, 13, 16, 17]. In [14], the category of Samanta's fuzzy normed linear space is studied and some results are given. In this paper, we will apply the above mentioned results to show that the category of probabilistic normed spaces, is isomorphicly a subcategory of the category of topological vector spaces. So all results such as Hahn-Banach theorem, open mapping theorem and etc. in topological vector spaces can be applied to probabilistic normed spaces in general.

2. Preliminary

Definition 2.1 ([15])**.** A triple $(X, N, *)$ is called a Menger probabilistic normed linear space (briefly, Menger PN space) if X is a real vector

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space, N is a mapping from X into Δ^+ (for $x \in X$, the distribution function $N(x)$ and $N(x,t)$ is the value of $N(x)$ at $t \in \mathbb{R}$) and $*$ is a t-norm satisfying the following conditions:

(PN1) $N(x,t) = \epsilon_0$ for all $t > 0$ iff $x = 0$,
(PN2) $N(cx,t) = N(x, \frac{t}{|c|})$ for all $c \in \mathbb{R}/\{0\}$,
(PN3) $N(x+y, s+t) \geq N(x,s) * N(y,t)$ for all $x, y \in X$, and $s, t > 0$.

We call the mapping $x \to N(x)$ a $*$-norm or a $*$-probabilistic norm on X. If $* = \wedge$, we will denote $(X, N, *)$ by (X, N).

Definition 2.2 ([12]). Suppose τ is a topology on a vector space X such that

(i) every point of X is a closed set, and
(ii) the vector space operations are continuous with respect to τ.

Under these conditions, τ is said to be a vector topology on X, and X is a topological vector space (TVS).

Theorem 2.3 ([15]). *Let $(X, N, *)$ be a Menger PN space. Then, $(X, N, *)$ is a Hausdorff topological vector space, whose local base of origin is $\{B(0, \alpha, t) : \ t > 0 \ , \ \alpha \in (0,1)\}$, where*

$$B(0, \alpha, t) = \{x : \ N(x,t) > 1 - \alpha)\} .$$

Definition 2.4. A triple $(X, \rho, *)$ is called a Menger probabilistic seminorm space (briefly, Menger PSN space) if X is a real vector space, ρ is a mapping from X into Δ^+ (for $x \in X$, the distribution function $\rho(x)$ is denoted by ρ_x, and $\rho_{x(t)}$ is the value of ρ_x at $t \in \mathbb{R}$) and $*$ is a t-norm satisfying the following conditions:

(PSN1) If $x = 0$, then $\rho_x(t) = 1, t > 0$,
(PSN2) $\rho_{cx}(t) = \rho_x(\frac{t}{|c|})$ for all $c \in \mathbb{R}/0$,
(PSN3) $\rho_{x+y}(s+t) \geq \rho_x(s) * \rho_y(t)$ for all $x, y \in X$, and $s, t > 0$.

We call the mapping $x \to \rho_x$ a $*$-seminorm on X.
Moreover, a Menger probabilistic seminorm ρ is called a Menger probabilistic norm if in addition $\rho_x(t) = 1$ for all $t > 0$ implies $x = 0$.

Definition 2.5 ([1]). Let (X, N) and (Y, N) be two fuzzy normed spaces. The operator $T : (X, N) \to (Y, N)$ is said to be fuzzy continuous at point x_0, if for every given $\epsilon > 0$ and $\alpha \in (0,1)$, there exist $\delta = \delta(\alpha, \epsilon)$ and $\beta = \beta(\alpha, \epsilon) \in (0,1)$ such that

$$N(x - x_0, \delta) > 1 - \beta \quad \Rightarrow \quad N\left(T(x) - T(x_0), \epsilon\right) > 1 - \alpha.$$

3. PN is a subcategory of TVS

In this section, we formalize the Menger probabilistic normed spaces PN as a category in which its objects are the Menger probabilistic normed spaces and its morphisms are fuzzy continuous operators. Moreover, we formalize the topological vector spaces TVS as a category whose objects are the topological vector spaces and whose morphisms are topological continuous operators. Then, we define a functor between $\mathcal{C}_{PN}$ and $\mathcal{C}_{TVS}$ while showing that it is morphism-invertible. Therefore, we will have a categorical embedding which is well-behaved. As an important result of this section, we can conclude that all results in topological vector spaces remain true in Menger probabilistic normed linear spaces.

Definition 3.1 ([3])**.** A category $\mathcal{C}$ consists of the following three mathematical entities:

(i) A class of objects $ob(\mathcal{C})$, usually denoted by just $\mathcal{C}$;
(ii) A class $\hom(\mathcal{C})$ of morphisms. Each morphism f has a unique source object A and target object B. We write $f : A \to B$, and we say "f is a morphism from A to B". We write $Hom(A, B)$ to denote the hom-class of all morphisms from A to B;
(iii) A binary operation $\circ$, called composition of morphisms, such that for any three objects A, B and C, we have

$$\hom(A, B) \times \hom(B, C) \to \hom(A, C).$$

The composition of $f : A \to B$ and $g : B \to C$ is written as $g \circ f$ or gf (Some authors write fg), governed by two axioms:

(i) Associativity: If $f : A \to B$, $g : B \to C$ and $h : C \to D$ then $h \circ (g \circ f) = (h \circ g) \circ f$.
(ii) Identity: For every object X, there exists a morphism $id_X : X \to X$ called the identity morphism for X, such that for every morphism $f : A \to B$, we have $id_B \circ f = f = f \circ id_A$.

Example 3.2. It is well known that the composition of two fuzzy continuous operators is a fuzzy continuous operator. Let X, Y, Z, W be PN and $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$ be fuzzy continuous operators, then, $h \circ (g \circ f) = (h \circ g) \circ f$. The identity operator is a fuzzy continuous operator and so it is a morphism in $Hom(\mathcal{C}_{PN})$. Also, $id_Y \circ f = f = f \circ id_X$, where $f : X \to Y$ is a fuzzy continuous operator and $id_X : X \to X$ and $id_Y : Y \to Y$ are identity operators.

By the above discussion, we conclude that fuzzy normed linear spaces (as objects) together with the fuzzy continuous operators (as morphisms) make a category denoted by $\mathcal{C}_{PN}$. By the same way, it is easy to see that the set of all topological vector spaces (as objects) together with topological continuous operators (as morphisms) also is a category, called $\mathcal{C}_{TVS}$.

Definition 3.3 ([3])**.** A functor F from the category $\mathcal{C}$ to the category $\mathcal{D}$;

(i) associates to each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$;
(ii) associates to each morphism $f : X \to Y$ a morphism $F(f) : F(X) \to F(Y)$ such that the following two properties hold:
   (a) $F(id_X) = id_{F(X)}$ for every object X in $\mathcal{C}$;
   (b) $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$.

Definition 3.4 ([3])**.** Let $\mathcal{C}$ be a category and $\{X_i : i \in I\}$ a family of objects of $\mathcal{C}$. A product for family $\{X_i : i \in I\}$ is an object P of $\mathcal{C}$ together with a family of morphisms $\{\pi_i : P \to X_i | \ i \in I\}$ such that for any object Y and family of morphisms $\{f_i : Y \to X_i\}$ there is a unique mediating map $f : Y \to P$ such that $\pi_i \circ f = f_i$. The product is written $X_1 \times X_2 \times \cdots \times X_n$, and the mediator f, is written $f_1 \times f_2 \times \cdots \times f_n$.

Definition 3.5 ([3])**.** Let $X, Y \in ob(\mathcal{C})$. A functor $F : \mathcal{C} \to \mathcal{D}$ preserves the product of X and Y, if whenever $X \leftarrow P \to Y$ obeys the universal property defining a product cone in $\mathcal{C}$, then, so does $F(X) \leftarrow F(P) \to F(Y)$, but in $\mathcal{D}$.

The property of morphism-invertibility is previously known in probabilistic sets for the hypograph functor in lattice-valued topology. In this note, we will define a new functor between $\mathcal{C}_{PN}$ and $\mathcal{C}_{TVS}$, having the same property. The definition of morphism-invertibility and the hypograph functor and related results in below go back to [5] and is used extensively in [9, 10, 11] and studied in greater generality in [6].

Definition 3.6. Let $(L, \preceq)$ be a complete lattice with the universal upper bound 1 and the universal lower bound 0. For a nonempty set X, a subset τ of L^X is called an L-topology on X if τ satisfies the following conditions:

(i) $1_X, 1_0 \in \tau$;
(ii) $\mu_1, \ \mu_2 \in \tau \Rightarrow \mu_1 \wedge \mu_2 \in \tau$;
(iii) $\{\mu_i \mid i \in I\} \subseteq \tau \Rightarrow \bigvee_{i \in I} \mu_i \in \tau$.

A pair (X, τ) is called an L-topological space and members of τ are called open L-sets. A map from X to L is called an L-set. For L-topological spaces (X, τ_1) and (Y, τ_2), a map $f : X \to Y$ is called L-continuous if f satisfies $f^{\leftarrow}(\mu) = \{\mu \circ f \mid \mu \in \tau_2\}$. L-topological spaces and L-continuous maps form a category denoted by $L - \mathbf{Top}$.

Now we briefly deal with the notion of hypograph functor and its properties below.

Definition 3.7. Let L be a complete chain. Let (X, τ) be a L-topological space and let $L_1 = L - \{1\}$. Put $S(X) = X \times L_1$ and for $a \in L^X$, put

$S\left(a\right)=\left\{(x,\alpha)\in X\times L_1 : a\left(x\right)>\alpha\right\}$.
$S\left(a\right)$ is the hypograph of a, though some call it the hypergraph. Further, put $S\left(\tau\right)=\left\{S\left(u\right): u\in\tau\right\}$ and note the linearity of L guarantees that $S\left(\tau\right)$ is a traditional topology on $X\times L_1$. Thus, we have the action $S : |L\text{-}\mathbf{Top}| \rightarrow |\mathbf{Top}|$ by $S\left(X,\tau\right)=\left(S\left(X\right),S\left(\tau\right)\right)$ on objects. For morphisms, let $f : X\rightarrow Y$ in **Set** and put $S\left(f\right): S\left(X\right)\rightarrow S\left(Y\right)$ in **Set** by $\left(S\left(f\right)\right)(x,\alpha)=\left(f\left(x\right),\alpha\right)$.

Note that $S\left(f\right)=f\times id_{L_1}$. It can be shown that we have a functor (hypograph functor) $S : L-\mathbf{Top}\rightarrow\mathbf{Top}$ and that the following theorem holds.

Theorem 3.8. *Let X,Y be sets and τ,σ be L-topologies, respectively,on X,Y, and let $f : X\rightarrow Y$ be a function. The followings are equivalent:*

(i) *$f : (X,\tau)\rightarrow(Y,\sigma)$ is L-continuous (i.e., a morphism in L-**Top**);*
(ii) *$S\left(f\right) : \left(S\left(X\right),S\left(\tau\right)\right)\rightarrow\left(S\left(Y\right),S\left(\sigma\right)\right)$ is continuous (in the traditional sense).*

We further note that S injects both objects and morphisms, therefore, it follows that S is a categorical embedding. However it has the terrible property that it does not preserve products in general.
Now we define the functor between $\mathcal{C}_{PN}$ and $\mathcal{C}_{TVS}$ and verify whether it has morphism-invertibility property, which plays the crucial rule in making a strong relation between Menger PN spaces and TVS.
Let (X,N) be a Menger PN space, put $F(X)=X$ and for $N(.,t) : X\rightarrow[0,1]$, $\alpha\in(0,1)$, $t>0$, put $F(N(.,t),\alpha)=\{x : \ N(x,t)>1-\alpha\}$ where the later set is just a neighborhood in (X,N) which is already denoted by $B(0,\alpha,t)$. Now we define

$$\begin{aligned}F(\tau_N)&=\{F(N(.,t),\alpha)\ ;\ \alpha\in(0,1],\ t>0\}\\&=\left\{B\left(0,\alpha,t\right):\ t>0\ ,\ \alpha\in(0,1)\right\},\end{aligned}$$

where τ_N is the induced topology by N in Menger probabilistic normed space (X,N). By Theorem 2.3, $F(\tau_N)$ is the base of the topology of the topological vector space X, say, τ_{tvs}, so, $F(\tau_N)=\tau_{tvs}$. Hence we have the action $F : \mathcal{C}_{PN}\rightarrow\mathcal{C}_{TVS}$ by $F(X,N)=(F(X),F(\tau_N))=(X,\tau_{tvs})$ on objects.
For morphisms, let $f : X\rightarrow Y$ be a fuzzy continuous operator and define $F(f) : F(X)\rightarrow F(Y)$ as $F(f)=f$. Since, the operator T is a fuzzy continuous operator if and only if T is a topological continuous operator. So it can easily be shown that $F(id_X)=id_{F(X)}$ and $F(g\circ f)=F(g)\circ F(f)$. Therefore, the above action is a well defined identity functor between $\mathcal{C}_{PN}$ and $\mathcal{C}_{TVS}$, and so it can easily be checked that F is injective. As cited before the product in $\mathcal{C}_{PN}$ and $\mathcal{C}_{TVS}$ is usual cartesian product, also since F is an identity functor and the mediator

morphism for cartesian product is unique so $F(X \times Y) = F(X) \times F(Y)$ and F preserves product with the universal property.

Theorem 3.9. *Let (X, N_1) and (Y, N_2) be two objects in $\mathcal{C}_{PN}$ satisfying $(N7)$ and $T : X \to Y$ be an operator. The followings are equivalent:*

(i) *T is a morphism in $\mathcal{C}_{PN}$.*
(ii) *$F(T) : F(X) \to F(Y)$ is a morphism in $\mathcal{C}_{TVS}$.*

Definition 3.10. Let $\mathcal{A}, \mathcal{B}$ be categories over a ground category $\mathcal{X}$. Then, a functor $F : \mathcal{A} \to \mathcal{B}$ is morphism-invertible if for all $X, Y \in ob(\mathcal{A})$ and for all $f \in \hom_{\mathcal{X}}(X, Y)$,

$$f \in \hom_{\mathcal{A}}(X, Y) \quad \Leftrightarrow \quad F(f) \in \hom_{\mathcal{B}}(F(X), F(Y)).$$

Remark 3.11. By Theorem 3.8, hypograph functor is a morphism-invertible and this property is critical in the proofs of continuity and uniform continuity of addition and continuity of multiplication in $\mathbb{R}(L)$ and $\mathbb{E}(L)$ in the Hutton-Gantner-Steinlage-Warren sense when L is a chain.

Also, if we choose the vector spaces as the ground category for probabilistic normed linear spaces and topological vector spaces, by Theorem 3.9, the identity functor from $\mathcal{C}_{PN}$ to $\mathcal{C}_{TVS}$ is morphism-invertible, which is another functor having the same property as hypoghraph functor.

Corollary 3.12. *Let $\mathcal{C}_{PN}$ be the category of Menger probabilistic normed spaces and $\mathcal{C}_{TVS}$ be the category of topological vector spaces. Then, we have a categorical embedding which is well-behaved, therefore, $\mathcal{C}_{PN}$ is isomorphically a subcategory of $\mathcal{C}_{TVS}$ in a well-behaved way (with both preservation of products and morphism-invertibility).*

Remark 3.13. Since we showed that the category of Menger probabilistic normed spaces is isomorphically a subcategory of the category of topological vector spaces, in a well-behaved way (with both preservation of products and morphism-invertibility). So, all results and theorems of topological vector spaces apply to Menger probabilistic normed spaces in general. There are many important well known theorems in topological vector spaces; among them are Hahn-Banach theorem, uniform boundedness theorem, open mapping theorem, closed graph theorem, ..., which remain true for probabilistic normed linear spaces. Therefore, as it is cited above there is no need to check the results which have already been proved in classical analysis, to Menger probabilistic normed spaces.

Conclusion. As it is cited in the introduction of this paper, the category of normed spaces can be embedded in the category of probabilistic normed spaces. Also in [13] the authors showed that $C(\Omega)$ is probabilistic normable but is not normable in classical case. So the spectrum of the category of probabilistic normed spaces is broader than the category of classical normed spaces. Therefore, the study of probabilistic normed spaces is of great importance.

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