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L_k-BIHARMONIC SPACELIKE HYPERSURFACES IN MINKOWSKI 4-SPACE $\mathbb{E}_1^4$

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ABSTRACT. Biharmonic surfaces in Euclidean space $\mathbb{E}^3$ are firstly studied from a differential geometric point of view by Bang-Yen Chen, who showed that the only biharmonic surfaces are minimal ones. A surface $x : M^2 \to \mathbb{E}^3$ is called biharmonic if $\Delta^2 x = 0$, where Δ is the Laplace operator of M^2. We study the L_k-biharmonic spacelike hypersurfaces in the 4-dimentional pseudo-Euclidean space $\mathbb{E}_1^4$ with an additional condition that the principal curvatures of M^3 are distinct. A hypersurface $x : M^3 \to \mathbb{E}^4$ is called L_k-biharmonic if $L_k^2 x = 0$ (for $k = 0, 1, 2$), where L_k is the linearized operator associated to the first variation of $(k+1)$-th mean curvature of M^3. Since $L_0 = \Delta$, the matter of L_k-biharmonicity is a natural generalization of biharmonicity. On any L_k-biharmonic spacelike hypersurfaces in $\mathbb{E}_1^4$ with distinct principal curvatures, by, assuming H_k to be constant, we get that H_{k+1} is constant. Furthermore, we show that L_k-biharmonic spacelike hypersurfaces in $\mathbb{E}_1^4$ with constant H_k are k-maximal.

1. Introduction

In the theory of elastics and also in the fluid mechanics, the concept of biharmonic surfaces play an important role. In [1, 15], one can find the plane elastic problems in terms of the biharmonic equation. The biharmonic maps are appeared in PDE theory as solutions of a fourth order strongly elliptic semilinear PDE and in computational geometry as the biharmonic Bezier surfaces. Clearly, the importance of biharmonic maps

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will be serious where harmonic ones do not exist. Ordinary harmonicity implies biharmoniciy, but not vise versa. Biharmonic non-harmonic maps are called proper-biharmonic. For instance, in the homotopy class of Brower degree ± 1, one may not find a harmonic map as $\mathbb{T}^2 \to \mathbb{S}^2$, so, the role of a proper-biharmonic map from $\mathbb{T}^2$ into $\mathbb{S}^2$ will be appeared ([10]). From a geometric point of view, the variational problem associated to the bienergy functional on the set of Riemannian metrics on a domain has given rise to the biharmonic stress-energy tensor. This is useful to obtain a new example of proper-biharmonic maps for the study of submanifolds with certain geometric properties, like pseudo-umbilical and parallel submanifolds.

A differential geometric motivation of the subject of biharmonic maps is a well-known conjecture of Bang-Yen Chen (in 1987) which says that the biharmonic surfaces in Euclidean 3-spaces are minimal ones. Later on, Dimitrić proved that any biharmonic hypersurface in $\mathbb{E}^m$ with at most two distinct principal curvatures is minimal ([9]). In 1995, Hasanis and Vlachos proved an extension of Chen's result to the hypersurfaces in Euclidean 4-spaces ([11]). Under the assumption of completeness, Akutagawa and Maeta ([2]) gave a generalization of the result to the global version of Chen's conjecture on biharmonic submanifolds in Euclidean spaces. On the other hand, Chen has found a good relation between the finite type hypersurfaces and biharmonic ones. The theory of finite type hypersurfaces is a well-known subject interested by Chen and also L.J. Alias, S.M.B. Kashani and others (see for instance, [7, 12, 16, 17]). One can see main results in Chapter 11 of Chen's book ([6]). In [12], Kashani has introduced the notion of L_k-finite type hypersurfaces as an extension of finite type ones in the Euclidean space, which is followed in the doctoral thesis second author.

The map L_k, as an extension of the Laplacian operator $L_0 = \Delta$, stands for the linearized operator of the first variation of the $(k+1)$th mean curvature of the hypersurface (see, for instance, [20]). This operator is given by $L_k(f) = \operatorname{tr}(P_k \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where P_k denotes the k-th Newton transformation associated to the second fundamental from of the hypersurface and $\nabla^2 f$ is the hessian of f.

It is interesting to generalize the definition of biharmonic hypersurface replacing Δ by L_k. Recently, in [18], we have studied the L_k-biharmonic hypersurfaces in 4-dimentional Euclidean space $\mathbb{E}^4$. In this paper, we pay attention to isometric immersed spacelike hypersurfaces $x : M^3 \to \mathbb{E}^4_1$ with three mutually distinct principal curvatures which are L_k-biharmonic, i.e. $L_k^2 x = 0$. Since k-maximal immersions are L_k-biharmonic, a naturally question is "what about the vise versa?"

Our following theorems give partial positive answer to the question.

Theorem 1.1. *Let a spacelike hypersurface $x : M^3 \to \mathbb{E}_1^4$ with three mutually distinct principal curvatures be L_k-biharmonic for a nonnegative integer $k \leq 2$. If H_k be constant on M^3, then H_{k+1} is also constant.*

Theorem 1.2. *Let M^3 be a L_k-biharmonic spacelike hypersurface of the Minkowski space $\mathbb{E}_1^4$, which has three mutually distinct principal curvatures and constant k-th mean curvature (where k is a nonnegative integer less than 3). Then, M^3 is k-maximal*

2. Preliminaries

In this section, we recall some prerequisites from [5, 14, 19]. By $\mathbb{E}^m$, we mean the m-dimensional Euclidean space and by $\mathbb{E}_1^m$ we denote the m-dimensional Lorentz-Minkowski spacetime. For an isometrically immersed spacelike hypersurface $x : M^n \to \mathbb{E}_1^{n+1}$, the symbols ∇ and $\bar{\nabla}$ denote the Levi-Civita connections on M^n and $\mathbb{E}_1^{n+1}$, respectively. The Weingarten formula for a spacelike hypersurface $x : M^n \to \mathbb{E}_1^{n+1}$ is $\bar{\nabla}_V W = \nabla_V W - \langle SV, W\rangle \mathbf{N}$, for $V, W \in \chi(M)$, where, S is the shape operator of M associated to a unit normal vector field $\mathbf{N}$ on M with $\langle \mathbf{N}, \mathbf{N}\rangle = -1$. Since M is spacelike, S can be diagonalized. Denote its eigenvalues (the principal curvatures of M) by the functions $\lambda_1, \ldots, \lambda_n$ on M, define the elementary symmetric function as $s_k := \sum_{1\leq i_1<\cdots<i_j\leq n} \lambda_{i_1} \ldots \lambda_{i_j}$, and the j-th mean curvature of M by $\binom{n}{k} H_k = (-1)^k s_k$, as in [5]. The hypersurface M^n in $\mathbb{E}_p^{n+1}$ is called *k-maximal*, if its $(k+1)$-th mean curvature H_{k+1} is identically zero. A 0-maximal hypersurface is a maximal hypersurface in $\mathbb{E}_1^4$. As is well-known, for every point $p \in M^3$, S defines a linear self-adjoint endomorphism on the tangent space T_pM^3, and its eigenvalues $\lambda_1(p)$, $\lambda_2(p)$ and $\lambda_3(p)$ are the principal curvatures of the hypersurface. The characteristic polynomial $Q_S(t)$ of S is defined by

$$\begin{aligned} Q_S(t) &= \det(tI - S) \\ &= (t-\lambda_1)(t-\lambda_2)(t-\lambda_3) \\ &= t^3 + a_1t^2 + a_2t + a_3, \end{aligned}$$

where the coefficients of $Q_S(t)$ are given by

$$a_1 = -(\lambda_1 + \lambda_2 + \lambda_3), \qquad a_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \qquad a_3 = -\lambda_1\lambda_2\lambda_3.$$

We apply the Newton transformations on M^3 by expression

$$P_0 = I, \qquad P_1 = -s_1 I + S, \qquad P_2 = s_2 I - s_1 S + S^2, \tag{2.1}$$

where I is the identity on $\chi(M)$. Let us recall that, for each point $p \in M^3$, each $P_k(p)$ is a self-adjoint linear operator on the tangent hyperplane T_pM which commutes with $S(p)$. Indeed, $S(p)$ and $P_k(p)$ can be simultaneously diagonalized. If e_1, e_2 and e_3 are the eigenvectors of $S(p)$ corresponding to the eigenvalues $\lambda_1(p), \lambda_2(p)$ and $\lambda_3(p)$, respectively, then they are also the eigenvectors of $P_k(p)$ with corresponding eigenvalues given by $\mu_{1,1} = -\lambda_2 - \lambda_3$, $\mu_{2,1} = -\lambda_1 - \lambda_3$, $\mu_{3,1} = -\lambda_1 - \lambda_2$, $\mu_{1,2} = \lambda_2\lambda_3$, $\mu_{2,2} = \lambda_1\lambda_3$ and $\mu_{3,2} = \lambda_1\lambda_2$.

We have the following formulae for the Newton transformations:

$$\begin{aligned} &\mathrm{tr}(P_k) = c_k H_k, \\ &\mathrm{tr}(S \circ P_k) = -c_k H_{k+1}, \\ &\mathrm{tr}(S^2 \circ P_1) = 9HH_2 - 3H_3, \\ &\mathrm{tr}(S^2 \circ P_2) = 3HH_3, \end{aligned} \tag{2.2}$$

where $k = 1, 2$, $c_1 = 6$ and $c_0 = 3$.

Associated to each Newton transformation P_k, we consider the second-order linear differential operator $L_k : C^\infty(M^3) \to C^\infty(M^3)$ given by $L_k(f) = tr(P_k \circ \nabla^2 f)$, where, $\nabla^2 f : \chi(M) \to \chi(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f which is given for every vector fields $X, Y \in \chi(M^3)$, by

$$\left\langle \nabla^2 f(X), Y \right\rangle = \left\langle \nabla_X(\nabla f), Y \right\rangle .$$

Therefore by considering the local orthonormal frame $\{e_1, e_2, e_3\}$, $L_k(f)$ is given by

$$L_k(f) = \sum_{i=1}^{3} \mu_{i,k}(e_i e_i f - \nabla_{e_i} e_i f). \tag{2.3}$$

3. L_k-BIHARMONIC HYPERSURFECES IN $\mathbb{E}_1^4$

Consider $x : M^3 \to \mathbb{E}_1^4$ be a connected spacelike hypersurface isometrically immersed into the Minkowski space $\mathbb{E}_1^4$, with a timelike Gauss map $\mathbf{N}$. By definition, M^3 is called an L_k-*biharmonic* hypersurface if its position vector field satisfies the condition $L_k^2 x = 0$. By the equality $L_k x = c_k H_{k+1}\mathbf{N}$ from [3, 13], the condition $L_k^2 x = 0$ has another equivalent expression $L_k(H_{k+1}\mathbf{N}) = 0$. It is clear that, k-maximal spacelike hypersurfaces are k-biharmonic. Also from [4, 13], by a similar calculation we obtain the value of $L_k^2 x$, which obviously results that, M^3 is k-biharmonic if and only if it satisfies conditions

$$L_k H_{k+1} = \mathrm{tr}\left(S^2 \circ P_k\right) H_{k+1} \tag{3.1}$$

and

$$\left(S \circ P_k\right)\left(\nabla H_{k+1}\right) = \frac{1}{2}\left(-1\right)^{k+1}\binom{3}{k+1} H_{k+1}\nabla H_{k+1}. \tag{3.2}$$

Proof. (Proof of Theorem 1.1). In the case $k = 0$, it is enough to see the works of Defever et al., such as in [8]. We prove the claim in two remained cases $k = 1, 2$. In order to show that H_{k+1} is constant, we put $\mathcal{U} := \{p \in M^3 : \nabla H_2^2(p) \neq 0\}$ and prove that $\mathcal{U}$ is empty. The method of reasoning is on Reductio ad absurdum with assumption $\mathcal{U} \neq \emptyset$.

Assume that $\{e_1, e_2, e_3\}$ be a local orthonormal frame of principal directions of the shape operator S on $\mathcal{U}$ such that $Se_i = \lambda_i e_i$ $(i = 1, 2, 3)$. Then we have $P_{k+1}e_i = \mu_{i,k+1}e_i$, for every i.

In case $k = 1$, by direct computing we get

$$P_2(\nabla H_2) = \frac{9}{2} H_2 \nabla H_2 \quad \text{on} \quad \mathcal{U}. \tag{3.3}$$

On the other hand, we have

$$\nabla H_2 = \sum_{i=1}^{3} e_i\,(H_2)\,e_i. \tag{3.4}$$

From (3.4) and (3.3) we get

$$e_i(H_2)\left(\mu_{i,2} - \frac{9}{2}H_2\right) = 0 \quad \text{on} \quad \mathcal{U}, \tag{3.5}$$

By definition of $\mathcal{U}$, there exists at least one i $(1 \leq i \leq 3)$ that $e_i(H_2) \neq 0$. So, we can assume that $e_1(H_2) \neq 0$, then we have $\mu_{1,2} = \frac{9}{2}H_2$ (locally) on $\mathcal{U}$, which gives $\lambda_2\lambda_3 = \frac{9}{2}H_2 \neq 0$ on U. Then, by distinctness of λ_i's, we get $\mu_{2,2} - \mu_{1,2} = \lambda_3(\lambda_1 - \lambda_2) \neq 0$, then $\mu_{2,2} - \frac{9}{2}H_2 \neq 0$, which by (3.5) gives $e_2(H_2) = 0$. Similarly, we can get that $e_3(H_2) = 0$.

Now, in order to prove the main claim, $\mathcal{U} = \emptyset$, firstly, we show that $e_2(\lambda_3) = e_3(\lambda_2) = 0$. Since H is constant, using the results $e_2(H_2) = e_3(H_2) = 0$ one can easily get that $e_2(\lambda_1) = e_3(\lambda_1) = 0$.

Using the well-known notation

$$\nabla_{e_i} e_j = \sum_{k=1}^{3} \omega_{ij}^k e_k$$

for $i, j = 1, 2, 3$, the compatibility conditions $e_k \langle e_i, e_i \rangle = 0$ and $e_k \langle e_i, e_j \rangle = 0$ imply that $\omega_{ki}^i = 0$ and $\omega_{ki}^j + \omega_{kj}^i = 0$, for $i, j, k = 1, 2, 3$ where $i \neq j$. Furthermore, it follows from the Codazzi equation that

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \tag{3.6}$$

$$(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j \tag{3.7}$$

for distinct $i, j, k = 1, 2, 3$.

We take into account the action of S on the basis $\{e_1, e_2, e_3\}$, and use the Codazzi equations. The relations

$$\begin{aligned}
&\langle(\nabla_{e_1}S)\,e_2,e_1\rangle=\langle(\nabla_{e_2}S)\,e_1,e_1\rangle\,,\ \langle(\nabla_{e_2}S)\,e_3,e_3\rangle=\langle(\nabla_{e_3}S)\,e_2,e_3\rangle\,,\\
&\langle(\nabla_{e_1}S)\,e_3,e_3\rangle=\langle(\nabla_{e_3}S)\,e_1,e_3\rangle\,,\ \langle(\nabla_{e_2}S)\,e_3,e_2\rangle=\langle(\nabla_{e_3}S)\,e_2,e_2\rangle\,,\\
&\langle(\nabla_{e_1}S)\,e_2,e_3\rangle=\langle(\nabla_{e_2}S)\,e_1,e_3\rangle\,,\ \langle(\nabla_{e_1}S)\,e_3,e_2\rangle=\langle(\nabla_{e_3}S)\,e_1,e_2\rangle\,,\\
&\langle(\nabla_{e_2}S)\,e_3,e_1\rangle=\langle(\nabla_{e_3}S)\,e_2,e_1\rangle\,,\ [e_2,\ e_3]\,(H_2)=0,
\end{aligned}\tag{3.8}$$

imply that

$$\begin{aligned}
&\omega_{12}^1=\omega_{13}^1=\omega_{13}^2=\omega_{21}^3=\omega_{32}^1=0,\\
&\omega_{21}^2=\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2},\qquad \omega_{31}^3=\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3},\\
&\omega_{23}^2=\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2},\qquad \omega_{32}^3=\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}.
\end{aligned}\tag{3.9}$$

Therefore, the covariant derivatives $\nabla_{e_i}e_j$ simplify to

$$\begin{aligned}
&\nabla_{e_1}e_1=0, && \nabla_{e_2}e_2=\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1}e_1,\\
&\nabla_{e_1}e_2=\nabla_{e_1}e_3=0, && \nabla_{e_2}e_3=\tfrac{e_3(\lambda_2)}{\lambda_3-\lambda_2}e_2,\\
&\nabla_{e_2}e_1=\tfrac{e_1(\lambda_2)}{\lambda_1-\lambda_2}e_2, && \nabla_{e_3}e_2=\tfrac{e_2(\lambda_3)}{\lambda_2-\lambda_3}e_3,\\
&\nabla_{e_3}e_1=\tfrac{e_1(\lambda_3)}{\lambda_1-\lambda_3}e_3, && \nabla_{e_3}e_3=\tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1}e_1+\tfrac{e_2(\lambda_3)}{\lambda_3-\lambda_2}e_2.
\end{aligned}\tag{3.10}$$

Now, the Gauss equation for $\langle R\,(e_2,e_3)\,e_1,\ e_2\rangle$ and $\langle R\,(e_2,e_3)\,e_1,\ e_3\rangle$ show that

$$e_3\left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)=\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}-\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right),\tag{3.11}$$

$$e_2\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right)=\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}-\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right).\tag{3.12}$$

Also, we can use the Gauss equation for

$$\langle R\,(e_1,e_2)\,e_1,\ e_2\rangle$$

and

$$\langle R\,(e_3,e_1)\,e_1,\ e_3\rangle\,,$$

which give the following relations

$$e_1\left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)+\left(\frac{e_1\,(\lambda_2)}{\lambda_1-\lambda_2}\right)^2=\lambda_1\lambda_2,\tag{3.13}$$

$$e_1\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right)+\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\right)^2=\lambda_1\lambda_3. \tag{3.14}$$

Finally, we obtain from the Gauss equation for $\langle R(e_3,e_1)e_2,e_3\rangle$, that

$$e_1\left(\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\right)=\frac{e_1(\lambda_3)\,e_2(\lambda_3)}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_3)}. \tag{3.15}$$

On the other hand, from conditions (3.1) and (2.3), by using the result $e_2(H_2)=e_3(H_2)=0$, we obtain

$$\begin{aligned}-\mu_{1,1}e_1e_1(H_2)+\left(\frac{\mu_{2,1}e_1(\lambda_2)}{\lambda_2-\lambda_1}+\frac{\mu_{3,1}e_1(\lambda_3)}{\lambda_3-\lambda_1}\right)e_1(H_2)\\ -9H_2^2\left(H-\frac{3}{2}\lambda_1\right)=0.\end{aligned} \tag{3.16}$$

By differentiating (3.16) along on e_2 respectively e_3, and using (3.11) and (3.12) we obtain

$$e_2\left(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}\right)=\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}-\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right), \tag{3.17}$$

$$e_3\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\right)=\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}-\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right). \tag{3.18}$$

Using (3.10), we find that

$$[e_1,\ e_2]=\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}e_2. \tag{3.19}$$

Applying both sides of the equality (3.19) on $\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}$, using (3.17), (3.13), (3.14), and (3.15), we deduce that

$$\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)=0. \tag{3.20}$$

The equation (3.20) shows that $e_2(\lambda_3)=0$ or

$$\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}=\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}. \tag{3.21}$$

In the case

$$\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}=\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}, \tag{3.22}$$

we will derive a contradiction. By differentiating on both sides of (3.22) along on e_1, in view of (3.13) and (3.14), gives $\lambda_2=\lambda_3$, this is a contradiction.

Hence, we conclude that $e_2(\lambda_3)=0$.

Analogously, using (3.10), we find that $[e_1,\ e_3]=\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}e_3$. Applying both sides of this equality on $\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}$, using (3.18), (3.13), (3.14), and (3.15), we deduce that

$$\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right)=0. \tag{3.23}$$

In a similar way as above, one can show that $e_3(\lambda_2)$ necessarily has to vanish.

Hence, get

$$e_2(\lambda_3)=0 \text{ and } e_3(\lambda_2)=0. \tag{3.24}$$

In view of (3.24), the Gauss equation for $\langle R(e_2,e_3)e_1,e_3\rangle$, gives the following relation

$$\frac{e_1(\lambda_3)\,e_1(\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)}=\lambda_2\lambda_3. \tag{3.25}$$

By differentiating (3.25) along on e_1, using (3.13) and (3.14) gives

$$\lambda_2\lambda_3\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)=0. \tag{3.26}$$

Hence, we conclude that $\lambda_2\lambda_3=0$, therefore $H_2=0$ on $\mathcal{U}$, which is a contradiction. Hence H_2 is constant on M^3.

For the case $k=2$, by using formulaes (3.1), (3.2) and (2.2)(d) on $\mathcal{U}$ we get

$$(SoP_2)\nabla H_3=-\frac{1}{2}H_3\nabla H_3, \tag{3.27}$$

$$L_2H_3=3HH_3^2. \tag{3.28}$$

But by the Cayley-Hamilton theorem we have $P_3=0$, so

$$S\circ P_2=H_3I, \qquad (S\circ P_2)\nabla H_3=H_3\nabla H_3,$$

which jointly with (3.27) yields $\nabla H_3^2=0$ on $\mathcal{U}$, which is a contradiction. □

Now, We turn to the question whether there are non k-maximal L_k-biharmonic spacelike hypersurfaces of $\mathbb{E}_1^4$. In [18], we proved that every L_k-biharmonic hypersurface of $\mathbb{E}_1^4$ with at most two principal curvatures is in fact k-maximal. In this paper, all principal curvatures have to be mutually different. Otherwise, M^3 would be a L_k-biharmonic hypersurface with at most two different principal curvatures.
In the proof of Theorem 1.2, we follow Defever's techniques (see [8]).

Proof. (Proof of Theorem 1.2). In case $k=1$, if $H_2\neq 0$, by using (3.1) we obtain that H_3 is constant. Therefore all the mean curvatures H_i are constant functions, this is equivalent to M^3 is isoparametric. An isoparametric hypersurface of Minkowski space can have at most two distinct principal curvatures ([21]), which is a contradiction. So, $H_2 = 0$.

Similarly, in case $k = 2$, If $H_3 \neq 0$, we obtain that all the mean curvatures H_i are constant functions, this is equivalent to M^3 is isoparametric, which implies the same contradiction as the case $k = 1$. So, $H_3 = 0$. This finishes the proof. □

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