Sahand Communications in Mathematical Analysis (SCMA) Vol. 5 No. 1 (2017), 9-20
http://scma.maragheh.ac.ir

EXTENSION OF KRULL'S INTERSECTION THEOREM FOR FUZZY MODULE

ALI REZA SEDIGHI[1] * AND MOHAMMAD HOSSEIN HOSSEINI[2]

Abstract. In this article we introduce μ-filtered fuzzy module with a family of fuzzy submodules. It shows the relation between μ-filtered fuzzy modules and crisp filtered modules by level sets. We investigate fuzzy topology on the μ-filtered fuzzy module and apply that to introduce fuzzy completion. Finally we extend Krull's intersection theorem of fuzzy ideals by using concept μ-adic completion.

---

1. Introduction

In [10] Murali and Makamba study the concepts of primary decompositions of fuzzy ideals and the radicals of such ideals over a commutative ring. Using such decompositions and a form of Nakayamas lemma, they prove Krulls intersection theorem on fuzzy ideals. In fact they show that if μ be a finitely generated fuzzy ideal of R such that μ is contained in the fuzzy Jacobson radical of R, Then $\bigwedge_{n=0}^{\infty} \mu^n = 0$.
In this article we effort to introduce filtered fuzzy ring and μ-filtered fuzzy module in order to extend the subjects of commutative algebra. In the other part of the paper, the relationship between μ-filtered fuzzy module and its crisp form are investigated. We prove a fuzzy module is μ-filtered if and only if all its level sets be filtered module. We process topology on the μ-filtered fuzzy module and apply that to introduce fuzzy completion. In Lemma 3.12 we show that topology induced by fuzzy filtration is a fuzzy topological group and with the help of that we express a condition for making Hausdorff topology. Then theorem

---

2010 *Mathematics Subject Classification.* 22A10, 16W70, 16W80.
Key words and phrases. μ-Fuzzy filtered module, Fuzzy inverse system, Fuzzy topological group, Krull's intersection theorem.
Received: 21 February 2016, Accepted: 02 June 2016.
* Corresponding author.

3.18 is proved for exact sequence of limit inverse system of fuzzy modules associated to this topology. Finally we extend Krull's intersection theorem of fuzzy ideals by using concept μ-adic completion. Our main goals are theorem 3.26.

2. Priliminaries

In the all parts of this article, R is a commutative ring and unital. Let μ be a fuzzy subset in R that $\mu(0) = 1$. If for every $x, y \in R$, $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$, then μ is a fuzzy group of R and we call fuzzy group μ a fuzzy ring (fuzzy ideal) of R if $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ $(\mu(xy) \geq \max\{\mu(x), \mu(y)\})$.
Let R be an ordinary ring and M be a R-module. we adopt the concept of fuzzy modules, which was introduce by Negoita and Ralescu [11], as follows.
(M, ν) is called a fuzzy R-module if the is a map

$$\nu : M \to [0, 1]$$

satisfying the following conditions:

(i) $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}, (\forall a, b \in M)$;
(ii) $\nu(0) = 1$;
(iii) $\nu(ra) \geq \nu(a), (\forall a \in M, r \in R)$.

.

Definition 2.1 ([7])**.** Let μ be a fuzzy subset of R. We define,

$$\mu^* = \{x \in R \mid \mu(x) > 0\},$$
$$\mu^{x>\alpha} = \{x \in R \mid \mu(x) > \alpha\}.$$

Proposition 2.2 ([7])**.** *Let μ be a fuzzy ring (fuzzy ideal), then μ^* are subrings (ideals) of R.*

Definition 2.3 ([8])**.** Let ζ and ν be fuzzy module on M such that $\zeta \subseteq \nu$. Then obviously ζ^* and ν^* are submodule of M and $\zeta^* \subseteq \nu^*$. Thus ζ^* is a submodule of ν^*. Now define ν/ζ on quotient module ν^*/ζ^* as follow:

$$\nu/\zeta(x + \zeta^*) = \sup\{\nu(y) \mid y \in x + \zeta^*\} \quad x \in \nu^*.$$

Then ν/ζ is fuzzy module on ν^*/ζ^* and called the quotient of ν with respect to ζ.

Definition 2.4 ([8])**.** Let f be a mapping from X into Y, and μ, ν fuzzy subsets on X, Y, respectively. The fuzzy subset $f(\mu)$ on y, defined by $\forall y \in Y$,

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) \mid x \in X, f(x) = y\} & f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.5 ([7]). Let μ and ν be two fuzzy subsets on R. Fuzzy subset $\mu\nu$ on R is defined as following,

$$\mu\nu(x) = \sup\{\inf\{\mu(y), \nu(z)\} \,|yz = x\}.$$

Proposition 2.6 ([7]). *Let μ and ν be fuzzy rings (fuzzy ideal) on R. Then $\mu\nu$ is fuzzy ring (fuzzy ideal) on R.*

Definition 2.7. Let μ be a fuzzy ring on R and ν is fuzzy module on M then $\mu\nu$ is defined as

$$\mu\nu(x) = \sup\{\inf\{\mu(r), \nu(y)\} \,|ry = x\}, \quad r \in R, x, y \in M.$$

Proposition 2.8. *If μ is a fuzzy ring on R and ν is fuzzy module on M, then $\mu\nu$ is a fuzzy module.*

Proof. It is clear that $\mu\nu$ is fuzzy group by proof of proposition above. Now, for every $x \in M$ and $r \in R$ we have,

$$\begin{aligned}\mu\nu(rx) &= \sup\{\inf\{\mu(s), \nu(y)\} \,|sy = rx\} \\ &= \sup\left\{\inf\left\{\mu(s^{'}), \nu(ry^{'})\right\} |s^{'}y^{'} = x\right\} \\ &\geq \sup\left\{\inf\left\{\mu(s^{'}), \nu(y^{'})\right\} |s^{'}y^{'} = x\right\} \\ &= \mu\nu(x).\end{aligned}$$

This show that $\mu\nu$ is a fuzzy module. □

Definition 2.9 ([4]). Let μ be a fuzzy ideal of R and let $x \in R$. Then the fuzzy subset $x + \mu$ of R defined by

$$(x + \mu)(r) = \mu(r - x) \quad \text{for all} \quad r \in R$$

is termed as the fuzzy coset determined by x and μ. The set of all fuzzy cosets of μ in R is a ring under the binary operations

$$(x + \mu) + (y + \mu) = (x + y) + \mu$$

and

$$(x + \mu)(y + \mu) = (xy + \mu) \quad \forall x, y \in R$$

and it is denoted by R/μ. We call it the fuzzy quotient ring of R induced by the fuzzy ideal μ.

Definition 2.10 ([2]). A filtered ring R is a ring R together with a family $\{R_n\}_{n\geq 0}$ of subgroups of R satisfying the conditions

(i) $R_0 = R$;
(ii) $R_{n+1} \subseteq R_n$ for all $n \geq 0$;
(iii) $R_nR_m \subseteq R_{n+m}$ for all $m, n \geq 0$.

Example 2.11 ([2]). Let I be an ideal in R and let $R_n = I^n$, $n \geq 0$. Then $\{R_n\}$ is a filtration on R called I-adic filtration.

Definition 2.12 ([2])**.** Let R be a filtered ring. A filtered R-module M is an R-module M together with a family $\{M_n\}_{n\geq 0}$ of R-submodules of M satisfying

(i) $M_0 = M$;
(ii) $M_{n+1} \subseteq M_n$ for all $n \geq 0$;
(iii) $R_m M_n \subseteq M_{n+m}$ for all $m, n \geq 0$.

Example 2.13 ([2])**.** Let M be a filtered R-module and N an R-submodule of M. The filtration $\{M_n\}$ on M induces a filtration $\{N_n\}$ on N where $N_n = N \cap M_n, n \geq 0$.

3. Main Result

Definition 3.1. Let R be an unital ring and μ a fuzzy ring of R. μ is called filtered fuzzy ring, if μ with $\{\mu_n\}_{n\geq 0}$ of fuzzy groups on R include the following conditions,

(i) $\mu_0 = \mu$;
(ii) $\mu_{n+1} \subseteq \mu_n(\mu_n \subseteq \mu_{n+1})$, for all $n \geq 0$;
(iii) $\mu_n\mu_m \subseteq \mu_{n+m}$, for all $n, m \geq 0$.

Example 3.2. Suppose μ be a fuzzy ring on R. Then μ is filtered by $\{\mu_n\}_{n\geq 0}$ filtration such that $\mu_n = \mu^n$. This filtration is called μ-adic filtration.

Example 3.3. Let μ be a filtered fuzzy ring on R. Then $\nu \subseteq \mu$ fuzzy ring of μ is filtered by $\nu_n = \mu_n \cap \nu$ filtration.

Example 3.4. Let ν be a discrete valuation on quotient field of R. Then $\mu_n(x) = 1 - n^{-\nu(x)}$ is an increasing filtration on fuzzy ring 1_R.

Definition 3.5. Let μ be a fuzzy filtered ring by $\{\mu_n\}$ on ring R and ν a fuzzy module on R-module M. ν is called μ-filtered fuzzy module, if ν with $\{\nu_n\}_{n\geq 0}$ of fuzzy module on M include the following conditions,

(i) $\nu_0 = \nu$;
(ii) $\nu_{n+1} \subseteq \nu_n$, for all $n \geq 0$;
(iii) $\mu_n\nu_m \subseteq \nu_{n+m}$, for all $n, m \geq 0$.

Example 3.6. Let μ be a filtered fuzzy ideal by μ-adic filtration. Fuzzy module ν of M is μ-filtered with $\{\nu_n = \mu^n\nu\}$ that is called μ-adic filtration.

Example 3.7. Let ν be a μ-filtered fuzzy module by $\{\nu_n\}_{n\geq 0}$ in M and μ is fuzzy ring filtered by $\{\mu_n\}_{n\geq 0}$ and ζ is a fuzzy submodule of ν. Then fuzzy module $\dfrac{\nu}{\zeta}$ of $\dfrac{\nu^*}{\zeta^*}$ is μ-filtered with $\left\{\dfrac{\nu}{\zeta}\right\}_n (x + \zeta^*) = \sup\{\nu_n(y) \mid y \in x + \zeta^*\}$, for all $x \in \nu^*$. Because, for every $x \in \nu^*$,

(i)

$$\left\{\frac{\nu}{\zeta}\right\}_0 (x+\zeta^*) = \sup\{\nu_0(y) \mid y \in x+\zeta^*\}$$
$$= \sup\{\nu(y) \mid y \in x+\zeta^*\}$$
$$= \left\{\frac{\nu}{\zeta}\right\} (x+\zeta^*);$$

(ii)

$$\left\{\frac{\nu}{\zeta}\right\}_{n+1} (x+\zeta^*) = \sup\{\nu_{n+1}(y) \mid y \in x+\zeta^*\}$$
$$\leq \sup\{\nu_n(y) \mid y \in x+\zeta^*\}$$
$$= \left\{\frac{\nu}{\zeta}\right\}_n (x+\zeta^*), \quad \text{for every } n \geq 0;$$

(iii) for all $n, m \geq 0$,

$$\mu_n \left(\frac{\nu}{\zeta}\right)_m (x+\zeta^*) = \sup\left\{\inf\left\{\mu_n(r), \left(\frac{\nu}{\zeta}\right)_m (y+\zeta^*)\right\} |ry+\zeta^* = x+\zeta^*\right\}$$
$$\leq \sup\{\sup\{\inf\{\mu_n(r), \nu_m(z)\} |z \in y+\zeta^*\} |ry = x\}$$
$$\leq \sup\{\nu_{n+m}(y)|y \in x+\zeta^*\} = \left(\frac{\nu}{\zeta}\right)_{n+m} (x+\zeta^*).$$

Hence $\mu_n \left(\frac{\nu}{\zeta}\right)_m \subseteq \left(\frac{\nu}{\zeta}\right)_{n+m}$.

Example 3.8. Let ν be a μ-filtered fuzzy module on M. Then fuzzy submodule $\zeta \subseteq \nu$ of ν is μ-filtered by filtration $\zeta_n = \nu_n \cap \zeta$.

Proposition 3.9. *Let μ be a fuzzy filtered ring with filtration $\{\mu_n\}_{n\geq 0}$. Then ν is a μ-filtered fuzzy module with filtration $\{\nu_n\}_{n\geq 0}$ if and only if $\nu^{x>\alpha}$ be a filtered module with $\{\nu_n^{x>\alpha}\}$ for every $\alpha \in [0,1]$.*

Proof. Let ν be a μ-filtered fuzzy module by filtration $\{\nu_n\}_{n\geq 0}$ and $\alpha \in [0,1]$. Then it is clear that $\nu_0^{x>\alpha} = \nu^{x>\alpha}$. Since $\nu_{n+1} \subseteq \nu_n$ for every $n \geq 0$,

$$x \in \nu_{n+1}^{x>\alpha} \implies \alpha < \nu_{n+1}(x) \leq \nu_n(x)$$
$$\implies x \in \nu_n^{x>\alpha}.$$

Therefore $\nu_{n+1}^{x>\alpha} \subseteq \nu_n^{x>\alpha}$.
Finally for every $n, m \geq 0$, $r \in \mu_n^{x>\alpha}$ and $x \in \nu_m^{x>\alpha}$, we have

$$\nu_{n+m}(rx) \geq \mu_n\nu_m(rx) \geq \inf\{\mu_n(r), \nu_m(x)\} > \alpha.$$

This show that $rx \in \nu_{n+m}^{x>\alpha}$ and so $\mu_n^{x>\alpha}\nu_m^{x>\alpha} \subseteq \nu_{n+m}^{x>\alpha}$.
Conversly we suppose $\nu^{x>\alpha}$ is filtered module for all $\alpha \in [0,1]$. Then for every $x \in M$, if $\nu_0(x) = \beta > 0$, then $x \in \nu_0^{x>0} = \nu^{x>0}$ and so $\nu = \nu_0$. In otherwise it is clear that $\nu = \nu_0$.
Now for every $n \geq 0$, let $\nu_{n+1}(x) = \alpha$, then $x \in \nu_{n+1}^{x>\beta}$ for every $\beta \in [0,\alpha)$. Therefore $x \in \nu_n^{x>\beta}$. This result $\nu_n(x) \geq \alpha = \nu_{n+1}(x)$ and hence $\nu_{n+1} \subseteq \nu_n$.
We choose $n, m \geq 0$, arbitrary. For every $x \in R$, let $\mu_n\nu_m(x) = \alpha$. Then there exist $r \in R$ and $y \in M$ such that $ry = x$ and $\alpha = \inf\{\mu_n(r), \nu_m(y)\}$. Therefore $r \in \mu_n^{x>\alpha}, y \in \nu_m^{x>\alpha}$.
By above discussion we result that $ry \in \nu_{n+m}^{x>\alpha}$ and we have

$$\nu_{n+m}(x) = \mu_{n+m}(ry) \geq \alpha = \mu_n\nu_m(x).$$

This complete the proof. □

Corollary 3.10. *Let ν be a μ-filtered fuzzy module on M. Then fuzzy submodule ζ is μ-filtered by induced filtration if and only if $\zeta^{x>\alpha}$ is a filtered module by induced filtration of $\nu^{x>\alpha}$ for every $\alpha \in [0,1]$.*

Proof. It is enough we show $\zeta_n = \zeta \cap \nu_n$ if and only if $\zeta_n^{x>\alpha} = \zeta^{x>\alpha} \cap \nu_n^{x>\alpha}$ for all $\alpha \in [0,1]$ and $n \geq 0$.
In fact, by this work we prove level sets of elements of induced filtration $\{\zeta_n\}_{n\geq 0}$ are induced filtration of level sets ν. The proof of be filtered, gained by last proposition. If $\zeta_n = \zeta \cap \nu_n$ for $n \geq 0$, then for $\alpha \in [0,1]$,

$$\begin{aligned} x \in \zeta_n^{x>\alpha} \quad &\Longleftrightarrow \quad \zeta_n(x) = \min\{\zeta(x), \nu_n(x)\} > \alpha \\ &\Longleftrightarrow \quad x \in \zeta^{x>\alpha} \cap \nu_n^{x>\alpha}. \end{aligned}$$

Conversly, if for every $\alpha \in [0,1]$, $\zeta_n^{x>\alpha} = \zeta^{x>\alpha} \cap \nu_n^{x>\alpha}$. Then for every $x \in M$, let $\zeta_n(x) = \eta$. Since $\eta = \sup\{\beta \in [0,1] \mid \zeta_n(x) > \beta\}$, for every $\beta \in [0,\alpha)$, $x \in \zeta^{x>\beta} \cap \nu_n^{x>\beta}$. This prove that $(\zeta \cap \nu_n)(x) > \beta$ and so $(\zeta \cap \nu_n)(x) \geq \alpha = \zeta_n(x)$. Similarly $\zeta_n(x) \geq (\zeta \cap \nu_n)(x)$. Hence the proof is completed. □

Given two fuzzy R-modules (M,μ) and (N,η) and an R-homomorphism $f : M \to N$ we say that f is a fuzzy R-homomorphism from (M,μ) into (N,η) if f is a fuzzy map relative to the grade function μ and η, i.e. if for all $x \in M$, $\eta(f(x)) \geq \mu(x)$.
Let

$$(\underline{M},\mu) = \Big(\{(M_i,\mu_i)\}_{i\geq 0}, \{\overline{P_i} : (M_i,\mu_i) \to (M_{i-1},\mu_{i-1})\}_{i\geq 1}\Big) \tag{3.1}$$

be inverse system of fuzzy modules, $A = \Big\{P_n : \prod_{i\geq 0} M_i \to M_n\Big\}_{n\geq 0}$ be a family of projections, and $(\prod_{i\geq 0} M_i, \mu_A)$ be the direct product of the fuzzy modules, then we get the fuzzy module $(\overset{\lim}{\leftarrow}_n M_n, \mu_A |_{\overset{\lim}{\leftarrow}_n M_n})$.[5] The following theorem talks that mentioned fuzzy module is limit of inverse system, exactly.

Theorem 3.11. [3]*Every inverse system in representation 3.1 has limit in the category of R-fuzzy modules, and this limit is equal to the fuzzy module* $(\overset{\lim}{\leftarrow}_n M_n, \mu_A |_{\overset{\lim}{\leftarrow}_n M_n})$.

Suppose that ν be a fuzzy R-module. The filtration $(M, \nu_n)_{n\geq 0}$ on ν be generated topology by the neighbourhood resulted from fuzzy groups structure which $\{\nu_n\}$ is a fundemental system of neighbourhood (0). This is called fuzzy topology induced by filtration $\{\nu_n\}$ and is presented by τ_ν.

Lemma 3.12. *The topology induced by filtration* $\{\nu_n\}$ *with* ν *is a fuzzy topological group.*

Proof. We have to show that maps $f : (\nu, \tau_\nu) \to (\nu, \tau_\nu)$, $g : (\nu, \tau_\nu) \times (\nu, \tau_\nu) \to (\nu, \tau_\nu)$ by $f(x) = x^{-1}$, $g(x, y) = x + y$,respectively, are fuzzy continuous. For this work let $x + \nu_n \in \tau_\nu$, then for each $t \in M$ we have,

$$\begin{aligned} f^{-1}(x+\nu_n) \cap \nu(t) &= \min\left\{f^{-1}(x+\nu_n)(t), \nu(t)\right\} \\ &= \min\left\{x+\nu_n(f(t)), \nu(t)\right\} \\ &= \min\left\{x+\nu_n(-t), \nu(t)\right\} \\ &= \min\left\{x+\nu_n(t), \nu(t)\right\} \\ &= x+\nu_n \cap \nu(t) \end{aligned}$$

and so $f^{-1}(x+\nu_n) \cap \nu = x + \nu_n \cap \nu \in \tau_\nu$. In the other hand, for each $(t, s) \in M \times M$ and without loss of generality suppose that $\nu(t) \leq \nu(s)$. Then

$$\begin{aligned} f^{-1}(x+\nu_n) \cap \nu \times \nu(t,s) &= \min\left\{x+\nu_n(f(t,s)), \nu(t), \nu(s)\right\} \\ &= \min\left\{x+\nu_n(t+s), \nu(t), \nu(s)\right\} \\ &= \min\left\{(x-t)+\nu_n(s), \nu(t)\right\} \\ &= \nu \times [(x-t)+\nu_n](t,s) \end{aligned}$$

and hence $f^{-1}(x+\nu_n) \cap \nu \times \nu = \nu \times [(x-t)+\nu_n] \in \tau_\nu \times \tau_\nu$. The proof is completed. □

Lemma 3.13 ([1])**.** *Let* μ *be a fuzzy topological group in a group* G *and let* $\{U\}$ *be the system of all fuzzy open neighborhoods of* e *in a fuzzy topological group* μ *such that* $U(e) = 1$*, where* e *is the identity of* G*. Then for any fuzzy subset* A *of* μ*,* $x \in \overline{A}$ *if and only if* $x \in \bigcap AU$*, where* $x \in \{w : \mu(w) = \mu(e)\}$*.*

Theorem 3.14. *Let η be a fuzzy submodule of μ-filtered fuzzy module ν. In the topology induced by filtration, we have*

$$x \in \overline{\eta} \iff x \in \bigwedge_{n=0}^{\infty}(\eta + \nu_n) \qquad \forall x \in \eta_*.$$

Proof. The proof is clear by lemma 3.12 and lemma 3.13. □

Corollary 3.15. *The fuzzy topology defined by the fuzzy filtration is fuzzy Hausdorff iff $\bigwedge_{n=0}^{\infty} \nu_n = 0$.*

Proof. The topology is fuzzy Hausdorff iff $0 = \overline{0} = \bigwedge_{n=0}^{\infty}(\nu_n)$. □

Let ν be a μ-filtered fuzzy module on M with filtration $\{\nu_n\}_{n\geq 0}$. By the defined topology of filtration, the following equivalent relation on cauchy sequences of elements of ν is defined as

$$(x_n) \sim (y_n) \iff \forall m \geq 0 \exists n_0 \geq 0; \quad \forall n \geq n_0, \nu_m(x_n - y_n) > 0.$$

The collection of above equivalent classes is called completion of fuzzy module (M, ν) and is presented by $\widehat{M_\nu}$.

Remark 3.16. Let ν be a μ-filtered fuzzy module on module M with filtration $\{\nu_n\}$. Then by Theorem 1 page 138 of [2]

$$\widehat{M_\nu} = \overset{lim}{\leftarrow}_n \frac{\nu*}{\nu_n*}.$$

Now if we apply definition of limit of direct system for fuzzy modules, then we define

$$\widehat{\nu} : \widehat{M_\nu} \to [0,1]$$
$$\widehat{\nu} = \frac{\nu}{\nu_n}{}_A\Big|_{\overset{lim}{\leftarrow}_n \frac{\nu*}{\nu_n*}}$$

such that for every $\{x + \nu_n^*\} \in \widehat{M_\nu}$

$$\widehat{\nu}(\{x + \nu_n^*\}) = \inf_n\{\sup\{\nu(y_n) | \nu_n(x - y_n) > 0\}\}, \quad x \in \nu^*$$

and

$$A = \left\{ P_i : \prod_{i=0}^{\infty} \frac{\nu*}{\nu_i*} \to \frac{\nu*}{\nu_i*} \right\}.$$

$\widehat{\nu}$ is called fuzzy completion of fuzzy module ν.

Lemma 3.17 ([3])**.** *Limit of inverse system on exact sequence of fuzzy modules is exact.*

Theorem 3.18. *Let* $0 \to (M', \nu') \xrightarrow{f} (M, \nu) \xrightarrow{g} (M'', \nu'') \to 0$ *be an exact sequence of R-fuzzy modules and* $\{\nu_n\}_{n\geq 0}$ *be a filtration on* ν *with induced filtration* $\nu^{'} \cap \underline{f}^{-1}(\nu_n)$ *on* $\nu^{'}$ *and* $\{\underline{g}(\nu_n)\}$ *on* $\nu^{''}$*. If we apply fuzzy completion on sequence,*

$$0 \to (\widehat{M'}_{\nu'}, \widehat{\nu'}) \xrightarrow{\widehat{f}} (\widehat{M}_{\nu}, \widehat{\nu}) \xrightarrow{\widehat{g}} (\widehat{M''}_{\nu''}, \widehat{\nu''}) \to 0$$

is exact.

Proof. By definition of exact sequence of fuzzy modules mentioned in [14], if $0 \to (M', \nu') \to (M, \nu) \to (M'', \nu'') \to 0$ be an exact sequence, then $0 \to \nu'^* \to \nu^* \to \nu''^* \to 0$ is exact and by proposition 3.9 since $\{\nu_n\}_{n\geq 0}$ is a filtration on ν, then $\{\nu_n^*\}$ is a filtration on module ν^*. Now by corollary 3.10 $\nu'^* \cap \underline{f}^{-1}(\nu_n^*)$ is induced filtration on ν'^* and $\{\underline{g}(\nu_n^*)\}$ induced filtration on ν''^*. Therefore for each n, sequence $0 \to \dfrac{\nu'^*}{\nu'^* \cap \underline{f}^{-1}(\nu_n^*)} \to \dfrac{\nu^*}{\nu_n^*} \to \dfrac{\nu''^*}{\underline{g}(\nu_n^*)} \to 0$ is exact. As regard for every $\eta \subseteq \nu$ we have $\left(\dfrac{\nu}{\eta}\right)^* = \dfrac{\nu^*}{\eta^*}$, sequence

$$0 \to \left(\frac{\nu'}{\nu' \cap \underline{f}^{-1}(\nu_n)}, \frac{\nu'^*}{\nu'^* \cap \underline{f}^{-1}(\nu_n^*)}\right) \to \left(\frac{\nu}{\nu_n}, \frac{\nu^*}{\nu_n^*}\right) \to \left(\frac{\nu''}{\underline{g}(\nu_n)}, \frac{\nu''^*}{\underline{g}(\nu_n^*)}\right) \to 0$$

is exact and hence by lemma 3.17, $0 \to \widehat{\nu'} \to \widehat{\nu} \to \widehat{\nu''} \to 0$ is exact too. □

Lemma 3.19. *Let* ν *be a* μ*-filtered fuzzy module by filtration* $\{\nu_n\}$ *and fuzzy completion* $\widehat{\nu}$*. Then* $\dfrac{\widehat{\nu}}{\widehat{\nu_n}} = \dfrac{\nu}{\nu_n}$ *such that* $\widehat{\nu_n}$ *is fuzzy completion respect to filtration induced on* ν_n*.*

Proof. It is clear that $\widehat{\nu}^* = \widehat{\nu^*}$ and by corollary 2 page 141 of [2],

$$\frac{\widehat{\nu}^*}{\widehat{\nu_n}^*} = \frac{\widehat{\nu^*}}{\widehat{\nu_n^*}} \simeq \frac{\nu^*}{\nu_n^*}.$$

This show that definition domains of desired fuzzy quotient modules is equal. Now we set $\widehat{x} = \{x + \nu_n^*\}$ then we have

$$\begin{aligned}\frac{\widehat{\nu}}{\widehat{\nu_n}}(\widehat{x} + \widehat{\nu_n}^*) &= \sup\{\widehat{\nu}(\widehat{y}) | \widehat{\nu_n}(\widehat{x} - \widehat{y}) > 0\} \\ &= \widehat{\nu}(\{x + \nu_m^*\}_{m \leq n}) \\ &= \inf_{m \leq n}\{\sup\{\nu(y_m) | \nu_m(x - y_m) > 0\}\} \\ &= \sup\{\nu(y_n) | \nu_n(x - y_n) > 0\} \\ &= \frac{\nu^*}{\nu_n^*}(x + \nu_n^*).\end{aligned}$$

This complete the proof. □

Corollary 3.20. *Let ν be a μ-filtered fuzzy module with filtration $\{\nu_n\}$. This induced a filtration $\{\widehat{\nu_n}\}$ on $\widehat{\nu}$ and $\widehat{\widehat{\nu}} = \widehat{\nu}$.*

Proof. Clearly $\{\widehat{\nu_n}\}$ is a filtration on $\widehat{\nu}$ as $\{\nu_n\}$ is a filtration on ν. By corollary 3 page 141 of [2] and lemma 3.19 $\widehat{M_\nu} = \widehat{\widehat{M_{\widehat{\nu}}}}$ and $\frac{\nu}{\nu_n} = \frac{\widehat{\nu}}{\widehat{\nu_n}}$. Therefore we have $\frac{\nu}{\nu_n}|_{\widehat{M_\nu}} = \frac{\widehat{\nu}}{\widehat{\nu_n}}|_{\widehat{\widehat{M_{\widehat{\nu}}}}}$ and we result $\widehat{\widehat{\nu}} = \widehat{\nu}$. □

The topology defined on ν by the μ-adic filtration for fuzzy ideal μ is called the μ-adic fuzzy topology and the fuzzy completion is called the μ-adic completion ν.

Remark 3.21. Suppose that ν be a μ-filtered fuzzy module with μ-adic filtration for fuzzy ideal μ. Then the completion $\widehat{M_\nu}$ is equal with μ^*-completion.

This remark leads us to the following conclusions.

Corollary 3.22. *$\widehat{M_\nu}$ is a fuzzy $\widehat{R}$-module such that $\widehat{R}$ is μ^*-completion.*

Proof. It is clear by page 143 of [2]. □

Proposition 3.23. *Let μ be a fuzzy ideal on R and ν be a μ-filtered fuzzy module by μ-adic filtration, then $\widehat{\nu}$ is an fuzzy $\widehat{R}$-module on $\widehat{M_\nu}$ such that R is filtered by μ^*-adic filtration.*

Proof. Since $\dfrac{\nu}{\mu^{*n}\nu}$ is fuzzy group, $\widehat{\nu}$ is fuzzy group. In other hand we have,

$$\begin{aligned}\widehat{\nu}\left(\{r+\mu^{*n}\}\{x+\mu^{*n}\nu\}\right) &= \widehat{\nu}\left(\{rx+\mu^{*n}\nu\}\right)\\ &= \inf_n \sup\{\mu(y_n)|\mu^n(rx-y_n)>0\}\\ &\geq \inf_n \sup\{\mu(ry_n)|\mu^n(rx-ry_n)>0\}\\ &\geq \inf_n \sup\{\mu(y_n)|\mu^n(x-y_n)>0\}\\ &= \widehat{\nu}(x+\mu^{*n}\nu).\end{aligned}$$

The proof is complete. □

Corollary 3.24. *Let*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

be an exact sequence of finitely generated R-modules on Noetherian ring R and μ be a fuzzy ideal. Then μ-adic completion

$$0 \to \widehat{\mu'} \xrightarrow{\widehat{f}} \widehat{\mu} \xrightarrow{\widehat{g}} \widehat{\mu''} \to 0$$

is exact sequence of fuzzy $\widehat{R}$-modules.

Proof. By proposition1 page 144 of [2] and theorem 3.18 , the proof is clear. □

Corollary 3.25. *Let R be a Noetherian domain and μ be a fuzzy ideal of R that $\mu(x)=0$ for some $x \in R$, then for every $x \in R$, $\bigwedge_{n=0}^{\infty} \mu^n(x) = 0$.*

Proof. If we apply ideal μ^* on corollary1 page 147 of [2] then we have $\bigcap_{n=0}^{\infty} \mu^{*n} = 0$ and it is equal with $\bigwedge_{n=0}^{\infty} \mu^n(x) = 0$, for every $x \in R$. □

Theorem 3.26 (Extention of Krull's intersection theorem)**.** *Let ν be a finitely generated fuzzy module on Noetherian ring R and μ be a fuzzy ideal with $\mu^* \subseteq J(R)$. Then $\bigwedge_{n=0}^{\infty} \mu^n\nu = 0$ i.e. μ-adic topology on ν is fuzzy Hausdorff.*

Proof. If we apply μ^* on crisp Krull's intersection theorem, we have $\bigcap_{n=0}^{\infty} \mu^{*n}\nu^* = 0$ and the proof is complete because $\mu^{*n}\nu^* = (\mu^n\nu)^*$. □

By corollary 3.8 of [6] it is remind that if $I \neq R$ be an ideal of R. Then I is a maximal ideal of R if and only if χ_I, is a fuzzy maximal left ideal of R. This show that if μ be an ideal such that μ is contained in the fuzzy Jacobson radical of R, then μ^* is contained in the Jacobson radical of R. Hence theorem 3.26 extend the conditions of Krull's intersection theorem of fuzzy ideals.

Acknowledgment. The authors would like to thank the referee for the valuable suggestions and comments.

References

1. I. Chon, *Properties of fuzzy topological groups and semigroups*, Kangweon-Kyungki Math. Jour, 8, (2000), 103-110.
2. N.S. Gopalakrishnan, *Commutative algebra*, Oxonian Press Pvt, University of Poona,1984.
3. C. Gunduz(Aras) and S. Bayramov, *Inverse and direct system in category of fuzzy modules*, Fuzzy Sets, Rough Sets and Multivalued Operations and Applications, 3, (2011), 11-25.
4. K.H. Lee, *On fuzzy quotient rings and chain conditions*, J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math, 7(1), (2000) 33-40.
5. S.R. Lopez-Permouth and D.S. Malik, *On Categories of Fuzzy Modules*, Information Sciences, 52, (1990), 211-220.
6. D.S. Malik, *Fuzzy maximal, radical, and primary ideals of a ring*, Information science, 53, (1991), 237-250.
7. D.S. Malik and John.N. Mordeson, *Fuzzy direct sums of fuzzy rings*, Fuzzy Sets and Systems, 45, (1992), 83-91.
8. J.N. Mordeson and D.S. Malik, *Fuzzy Commutative Algebra*, World Scientic, 1998.
9. G.C. Muganda, *Free fuzzy modules and their bases*, Inform. Sci, 72, (1993), 65-82.
10. V. Murali and B. Makamba, *On Krull's intersection theorem of fuzzy ideals*, International journal of mathematics and mathematical sciences, 4, (2003), 251-262.
11. C.V. Negoita and D.A. Ralescu, *Applications of Fuzzy Sets and System Analysis*, Birkhauser, Basel, 1975.
12. A. Rosenfeld, *Fuzzy groups*, J1. Math. Anal. Appl., 35, (1971), 512-517.
13. L.A. Zadeh, *Fuzzy sets*, Inform. Control, 81, (1965), 338-353.
14. M.M. Zahedi and A. Ameri, *Fuzzy exact sequences in category of fuzzy modules*, Jl. Fuzzy Math., 2(2), (1994), 409-424.

---

[1] Department of Mathematics, Faculty of mathematics and statistics, University of Birjand, Birjand, Iran.

E-mail address: `sedighi.phdbirjand.ac.ir`

[2] Department of Mathematics, Faculty mathematics and statistics, University of Birjand, Birjand, Iran.

E-mail address: `mhhosseinibirjand.ac.ir`