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r-FUZZY REGULAR SEMI OPEN SETS IN SMOOTH TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce and study the concept of r-fuzzy regular semi open (closed) sets in smooth topological spaces. By using r-fuzzy regular semi open (closed) sets, we define a new fuzzy closure operator namely r-fuzzy regular semi interior (closure) operator. Also, we introduce fuzzy regular semi continuous and fuzzy regular semi irresolute mappings. Moreover, we investigate the relationship among fuzzy regular semi continuous and fuzzy regular semi irresolute mappings. Finally, we have given some counter examples to show that these types of mappings are not equivalent.

1. Introduction and Preliminaries

Šostak [11] introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and Chang's fuzzy topology [2]. Subsequently, Badard [1] introduced the concept of smooth topological space. Chattopadhyay et al. [4] have redefined the same concept under the name gradation of openness. Ramadan [9] introduced a similar definition, namely, smooth topological space for lattice $L = [0, 1]$. Thus, the term 'fuzzy topology' in Šostak sense, 'gradation of openness' and 'smooth topology' are essentially referring to the same concept. In our paper, we adopt the term smooth topology. It has been developed in many direction [3–5, 10]. Mashhour et. al., [8], Zahran [13] and Kerre et. al., [6] introduced the notion of fuzzy regular semi open and fuzzy regular semi closed sets and investigate the relationship among fuzzy regular semi continuity and fuzzy regular semi irresolute mappings.

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In this paper, we introduce and study the concept of r-fuzzy regular semi open (closed) sets in smooth topological spaces. By using r-fuzzy regular semi open (closed) sets, we define a new fuzzy closure operator namely r-fuzzy regular semi interior (closure) operator. Also, we generalize the notions of fuzzy regular semi continuous (open, closed) and fuzzy regular semi irresolute (irresolute open, irresolute closed) mappings in smooth topological spaces. We show that the fuzzy regular continuous mapping is fuzzy regular semi continuous, however, the converse is not true. Also, we show that the fuzzy regular semi irresolute mapping is fuzzy regular semi continuous, but the converse is not true. Therefore, we have given characterizations of all mentioned types of mappings. Finally, we establish some counter examples to show that these types are not equivalent.

Throughout this paper, let X be a non-empty set, $I = [0,1]$, $I_0 = (0,1]$. A fuzzy set λ of X is a mapping $\lambda : X \to I$, and I^X be the family of all fuzzy sets on X. The complement of a fuzzy set λ is denoted by $\overline{1} - \lambda$. For $\lambda \in I^X$, $\overline{\lambda}(x) = \lambda$ for all $x \in X$. For each $x \in X$ and $t \in I_0$, a fuzzy point x_t is defined by

$$x_t(y) = \begin{cases} t & \text{if } y = x; \\ 0 & \text{if } y \neq x. \end{cases}$$

Let $Pt(X)$ be the family of all fuzzy points in X. All other notations and definitions are standard in the fuzzy set theory.

Definition 1.1 ([11])**.** A function $\tau : I^X \to I$ is called a smooth topology on X if it satisfies the following conditions:

(1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$,
(2) $\tau(\bigvee_{i\in J} \mu_i) \geq \bigwedge_{i\in J} \tau(\mu_i)$, for any $\{\mu_i : i \in J\} \subseteq I^X$,
(3) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, for all $\mu_1,\ \mu_2 \in I^X$.

The pair $(X,\ \tau)$ is called a smooth topological space. A fuzzy set λ is called an r-fuzzy open (r-fo, for short) if $\tau(\lambda) \geq r$. A fuzzy set λ is called an r-fuzzy closed (r-fc, for short) set if $\tau(\overline{1} - \lambda) \geq r$.

Theorem 1.2 ([3])**.** *Let (X, τ) be a smooth topological space. Then for each $\lambda \in I^X$ and $r \in I_0$, we define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows:*

$$C_\tau(\lambda,\ r) = \bigwedge \left\{\mu \in I^X : \lambda \leq \mu,\ \tau(\overline{1} - \mu) \geq r\right\}.$$

For $\lambda,\ \mu \in I^X$ and $r, s \in I_0$, the operator C_τ satisfies the following statements:

(C1) $C_\tau(\overline{0}, r) = \overline{0}$,
(C2) $\lambda \leq C_\tau(\lambda,\ r)$,

(C3) $C_\tau(\lambda,\ r) \vee C_\tau(\mu,\ r) = C_\tau(\lambda \vee \mu,\ r)$,
(C4) $C_\tau(\lambda,\ r) \leq C_\tau(\lambda,\ s)$ *if* $r \leq s$,
(C5) $C_\tau(C_\tau(\lambda,\ r),\ r) = C_\tau(\lambda,\ r)$.

Theorem 1.3 ([3]). *Let (X, τ) be a smooth topological space. Then for each $\lambda \in I^X$ and $r \in I_0$, we define an operator $I_\tau : I^X \times I_0 \to I^X$ as follows:*

$$I_\tau(\lambda,\ r) = \bigvee \left\{ \mu \in I^X : \mu \leq \lambda,\ \tau(\mu) \geq r \right\}.$$

For $\lambda,\ \mu \in I^X$ and $r, s \in I_0$, the operator I_τ satisfies the following statements:

(I1) $I_\tau(\overline{1}, r) = \overline{1}$,
(I2) $I_\tau(\lambda,\ r) \leq \lambda$,
(I3) $I_\tau(\lambda,\ r) \wedge I_\tau(\mu,\ r) = I_\tau(\lambda \wedge \mu,\ r)$,
(I4) $I_\tau(\lambda,\ r) \leq I_\tau(\lambda,\ s)$ *if* $s \leq r$,
(I5) $I_\tau(I_\tau(\lambda,\ r),\ r) = I_\tau(\lambda,\ r)$,
(I6) $I_\tau(\overline{1} - \lambda, r) = \overline{1} - C_\tau(\lambda, r)$ *and* $C_\tau(\overline{1} - \lambda, r) = \overline{1} - I_\tau(\lambda,\ r)$.

Definition 1.4 ([7]). Let $(X,\ \tau)$ be a smooth topological space, $\lambda \in I^X$ and $r \in I_0$. Then

(1) A fuzzy set λ is called r-fuzzy regular open (for short, r-fro) if $\lambda = I_\tau(C_\tau(\lambda, r), r)$.
(2) A fuzzy set λ is called r-fuzzy regular closed (for short, r-frc) if $\lambda = C_\tau(I_\tau(\lambda, r), r)$.

Definition 1.5 ([12]). Let $(X,\ \tau)$ be a smooth topological space. For $\lambda, \mu \in I^X$ and $r \in I_0$.

(1) The r-fuzzy regular closure of λ, denoted by $RC_\tau(\lambda, r)$, and is defined by
$$RC_\tau(\lambda, r) = \bigwedge \left\{ \mu \in I^X | \mu \geq \lambda, \mu \text{ is } r\text{-frc} \right\}.$$
(2) The r-fuzzy regular interior of λ, denoted by $RI_\tau(\lambda, r)$, and is defined by
$$RI_\tau(\lambda, r) = \bigvee \left\{ \mu \in I^X | \mu \leq \lambda, \mu \text{ is } r\text{-fro} \right\}.$$

Definition 1.6 ([10]). Let $(X,\ \tau)$ be a smooth topological space and $\lambda \in I^X,\ r \in I_0$. Then

(1) λ is called r-fuzzy semi open (for short, r-fso) if there exists $\mu \in I^X$ with $\tau(\mu) \geq r$ and $\mu \leq \lambda \leq C_\tau(\mu, r)$.
(2) λ is called r-fuzzy semi closed (for short, r-fsc) if there exists $\mu \in I^X$ with $\tau(\overline{1} - \mu) \geq r$ and $I_\tau(\mu, r) \leq \lambda \leq \mu$.

2. r-fuzzy regular semi open and r-fuzzy regular semi closed sets

Definition 2.1. Let $(X,\ \tau)$ be a smooth topological space and $\lambda \in I^X,\ r \in I_0$. Then

(1) λ is called r-fuzzy regular semi open (for short, r-frso) if there exists r-fro set $\mu \in I^X$ such that $\mu \le \lambda \le C_\tau(\mu, r)$.

(2) λ is called r-fuzzy regular semi closed (for short, r-frsc) if there exists r-frc set $\mu \in I^X$ such that $I_\tau(\mu, r) \le \lambda \le \mu$.

(3) The r-fuzzy regular semi interior of λ, denoted by $RSI_\tau(\lambda, r)$, is defined by

$$RSI_\tau(\lambda, r) = \bigvee \left\{\mu \in I^X \mid \mu \le \lambda,\ \mu \text{ is } r\text{-frso}\right\}.$$

(4) The r-fuzzy regular semi closure of λ, denoted by $RSC_\tau(\lambda, r)$ is defined by

$$RSC_\tau(\lambda, r) = \bigwedge \left\{\mu \in I^X \mid \mu \ge \lambda,\ \mu \text{ is } r\text{-frsc}\right\}.$$

Theorem 2.2. *Let $(X,\ \tau)$ be a smooth topological space. For $\lambda \in I^X,\ r \in I_0$, the following statements are equivalent:*

(1) *λ is r-frso.*

(2) $I_\tau(\lambda, r) = I_\tau\left(C_\tau(\lambda, r), r\right)$.

(3) *$\overline{1} - \lambda$ is r-frso.*

(4) $C_\tau(\overline{1} - \lambda, r) = C_\tau\left(I_\tau(\overline{1} - \lambda, r), r\right)$.

Proof. (1)⇔(2) Let λ be an r-frso set. Then there exists an r-fro set μ such that $\mu \le \lambda \le C_\tau(\mu, r)$. Hence $C_\tau(\lambda, r) = C_\tau(\mu, r)$. Since $I_\tau(C_\tau(\lambda, r), r) = \mu$,

$$\begin{aligned} I_\tau(C_\tau(\lambda, r), r) &= \mu \le I_\tau(\lambda, r) \\ &\le I_\tau\left(C_\tau(\mu, r), r\right) \\ &= I_\tau\left(C_\tau(\lambda, r), r\right). \end{aligned}$$

Thus, $I_\tau(\lambda, r) = I_\tau(C_\tau(\lambda, r), r)$.

(1)⇔(3) Let λ be an r-frso set. Then there exists an r-fro set μ such that $\mu \le \lambda \le C_\tau(\mu, r)$. Since $\overline{1} - \mu$ is an r-frc set, $I_\tau(\overline{1} - \mu, r)$ is an r-fro set such that

$$\begin{aligned} I_\tau(\overline{1} - \mu, r) &\le \overline{1} - \lambda \\ &\le \overline{1} - \mu \\ &= C_\tau(I_\tau\left(\overline{1} - \mu, r), r\right). \end{aligned}$$

Thus $\overline{1} - \lambda$ is r-frso.

The implication (1)⇔(4) follow immediately by taking the complement of the two sides. □

Theorem 2.3. *Let (X, τ) be a smooth topological space. Then*

1. *any union of r-frso sets is an r-frso set.*
2. *any intersection of r-frsc sets is an r-frsc set.*

Proof. (1) Let $\{\lambda_\alpha : \alpha \in \Gamma\}$ be a family of r-frso. Then for each $\alpha \in \Gamma$ there exists r-fro set $\nu_\alpha \in I^X$ such that

$$\nu_\alpha \leq \lambda_\alpha \leq C_\tau(\nu_\alpha, r). \tag{2.1}$$

Also, $\nu_\alpha \leq \vee\nu_\alpha$ implies that $C_\tau(\nu_\alpha, r) \leq C_\tau(\vee\nu_\alpha, r)$. Then, from (2.1) we have:

$$\vee\nu_\alpha \leq \vee\lambda_\alpha \leq C_\tau(\nu_\alpha, r) \leq C_\tau(\vee\nu_\alpha, r).$$

Since, $\tau(\vee\nu_\alpha) \geq \wedge(\nu_\alpha)$ is r-fro set, then $\vee\lambda_\alpha$ is r-frso.

(2) Similar to (1).

□

Theorem 2.4. *Let (X, τ) be an smooth topological space, $\lambda, \mu \in I^X$ and $r \in I_0$.*

1. *If λ is an r-fro set, then λ is an r-frso.*
2. *If λ is an r-frso and $I_\tau(\lambda, r) \leq \mu \leq C_\tau(\lambda, r)$, then μ is an r-frso.*
3. *If λ is an r-frsc and $I_\tau(\lambda, r) \leq \mu \leq C_\tau(\lambda, r)$, then μ is an r-frsc.*

Proof. (1) Let λ is an r-fro, since $\lambda \leq \lambda \leq C_\tau(\lambda, r)$. Then λ is r-frso.

(2) Let λ be r-frso such that $I_\tau(\lambda, r) \leq \mu \leq C_\tau(\lambda, r)$. Then there exists r-fro set $\nu \in I^X$ such that $\nu \leq \lambda \leq C_\tau(\nu, r)$. It implies $C_\tau(\lambda, r) \leq C_\tau(\nu, r)$ and therefore $\mu \leq C_\tau(\nu, r)$. Also, $\nu \leq I_\tau(\lambda, r) \leq \mu$. It follows that, $\nu \leq \mu \leq C_\tau(\nu, r)$. So, μ is r-frso.

(3) It is similar to part (2).

□

Remark 2.5. (1) Every r-fro set is an r-frso set.

(2) Every r-frso set is an r-fso set.

The converse of the above Remark 2.5 is not true in general, as shown from the following examples.

Example 2.6. Let $X = \{a, b, c\}$ and $\lambda, \mu, \delta \in I^X$ are defined as

$$\begin{array}{lll} \lambda(a) = 0.5, & \lambda(b) = 0.5, & \lambda(c) = 0.6, \\ \mu(a) = 0.4, & \mu(b) = 0.5, & \mu(c) = 0.6, \\ \delta(a) = 0.4, & \delta(b) = 0.5, & \delta(c) = 0.4. \end{array}$$

We define smooth topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \mu, \\ \frac{1}{3} & \text{if } \lambda = \delta, \\ 0 & \text{otherwise.} \end{cases}$$

For $r = \frac{1}{3}$, then the fuzzy set λ is an r-frso set, because there exists an r-fro set $\mu \in I^X$ such that $\mu \leq \lambda \leq C_\tau(\mu, r)$. But λ is not an r-fro set.

Example 2.7. Let $X = \{a, b, c\}$, $\lambda, \mu, \delta \in I^X$ are defined as

$$\begin{array}{lll} \lambda(a) = 0.4, & \lambda(b) = 0.5, & \lambda(c) = 0.6, \\ \mu(a) = 0.3, & \mu(b) = 0.5, & \mu(c) = 0.6, \\ \delta(a) = 0.6, & \delta(b) = 0.5, & \delta(c) = 0.5. \end{array}$$

We define smooth topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \mu, \\ \frac{1}{2} & \text{if } \lambda = \delta, \\ \frac{1}{2} & \text{if } \lambda = \mu \vee \delta, \\ \frac{1}{2} & \text{if } \lambda = \mu \wedge \delta, \\ 0 & \text{otherwise.} \end{cases}$$

For $r = \frac{1}{2}$, then the fuzzy set λ is an r-fso set, because there is an r-fuzzy open set $\mu \in I^X$ such that $\mu \leq \lambda \leq C_\tau(\mu, r)$. But λ is not an r-frso set.

Theorem 2.8. *Let (X, τ) be a smooth topological space. For $\lambda, \mu \in I^X$ and $r \in I_0$, it satisfies the following statements:*

1. *λ is r-frso $\Leftrightarrow \lambda = RSI_\tau(\lambda, r)$.*
2. *λ is r-frsc $\Leftrightarrow \lambda = RSC_\tau(\lambda, r)$.*
3. *$RSC_\tau(\overline{0}, r) = \overline{0}$.*
4. *$RI_\tau(\lambda, r) \leq RSI_\tau(\lambda, r) \leq \lambda \leq RSC_\tau(\lambda, r) \leq RC_\tau(\lambda, r)$.*
5. *$RSC_\tau(\lambda, r) \vee RSC_\tau(\mu, r) \leq RSC_\tau(\lambda \vee \mu, r)$.*
6. *$RC_\tau(RSC_\tau(\lambda, r), r) = RSC_\tau(RC_\tau(\lambda, r), r) = RC_\tau(\lambda, r)$.*

Proof. (1) Let λ be r-frso. Then

$$RSI_\tau(\lambda, r) = \vee \left\{ \rho \in I^X : \rho \leq \lambda,\ \rho \text{ is } r\text{ -frso} \right\} = \lambda.$$

Conversely, let $\lambda = RSI_\tau(\lambda, r)$. Since $RSI_\tau(\lambda, r)$ is the arbitrary union of r-frso's, then λ is r-frso.

(2) It is similar to part (1).

(3) It is easily obtained from Definition 2.1.

(4) Since

$$\begin{aligned} RI_\tau(\lambda, r) &= \bigvee \left\{\rho \in I^X : \rho \leq \lambda,\ \rho \text{ is } r\text{ -fro}\right\} \\ &\leq \bigvee \left\{\rho \in I^X : \rho \leq \lambda, \rho \text{ is } r\text{ -frso}\right\} \\ &= RSI_\tau(\lambda, r), \end{aligned}$$

it follows that, $RI_\tau(\lambda, r) \leq RSI_\tau(\lambda, r)$. Also,

$$\begin{aligned} RSC_\tau(\lambda, r) &= \bigwedge \left\{\rho \in I^X : \rho \geq \lambda,\ \rho \text{ is } r\text{ -frsc}\right\} \\ &\leq RC_\tau(\lambda, r). \end{aligned}$$

Finally, we have

$RI_\tau(\lambda, r) \leq RSI_\tau(\lambda, r) \leq \lambda \leq RSC_\tau(\lambda, r) \leq RC_\tau(\lambda, r).$

(5) Since, $\mu \leq \mu \vee \rho,\ \rho \leq \mu \vee \rho$, then

$$RSC_\tau(\mu, r) \leq RSC_\tau(\mu \vee \rho, r)$$

and

$$RSC_\tau(\rho, r) \leq RSC_\tau(\mu \vee \rho, r).$$

Hence,

$$RSC_\tau(\mu, r) \vee RSC_\tau(\rho, r) \leq RSC_\tau(\mu \vee \rho, r).$$

(6) Since $RC_\tau(\lambda, r)$ is r-frsc set, then

$$RSC_\tau\left(RC_\tau(\lambda, r), r\right) = RC_\tau(\lambda, r). \tag{2.2}$$

Now it remains to prove only the relation:

$$RC_\tau\left(RSC_\tau(\lambda, r), r\right) = RC_\tau(\lambda, r).$$

Since, $\lambda \leq RSC_\tau(\lambda, r)$, then $RC_\tau(\lambda, r) \leq RC_\tau(RSC_\tau(\lambda, r)$. It remains to prove:

$$RC_\tau(RSC_\tau(\lambda, r), r) \leq RC_\tau(\lambda, r).$$

Let the contrary, that is,

$$RC_\tau(RSC_\tau(\lambda, r), r) \nleq RC_\tau(\lambda, r).$$

Then

$$RC_\tau(RSC_\tau(\lambda, r), r) > RC_\tau(\lambda, r).$$

So, there exists an r-frc set $\rho \in I^X$, $\rho \geq \lambda$ such that

$$RC_\tau(\lambda, r)(x) < \rho(x) < RC_\tau(RSC_\tau(\lambda, r), r)(x). \tag{2.3}$$

Since,

$$\begin{aligned} \lambda \leq \rho \quad \Rightarrow \quad RSC_\tau(\lambda, r) &\leq RSC_\tau(\rho, r) \\ &= RSC_\tau(RC_\tau(\lambda, r), r) \\ &= RC_\tau(\rho, r). \end{aligned}$$

Then, $RSC_\tau(\lambda, r) \leq RC_\tau(\rho, r)$ and this implies

$$RC_\tau(RSC_\tau(\lambda, r), r) \leq RC_\tau(\lambda, r),$$

which contradicts to the relation (2.3). Hence the result. $\square$

3. Fuzzy regular semi continuous mappings

Definition 3.1. Let $(X,\ \tau)$ and $(Y,\ \eta)$ be smooth topological spaces. Let $f : X \to Y$ be a mapping. Then f is said to be:

(1) fuzzy regular semi irresolute (resp. fuzzy regular semi continuous) iff $f^{-1}(\mu)$ is r-frso for each r-frso set $\mu \in I^Y$ (resp. $\mu \in I^Y,\ \eta(\mu) \geq r$).
(2) fuzzy regular semi irresolute open (resp. fuzzy regular semi open) iff $f(\lambda)$ is r-frso in Y for each r-frso set $\lambda \in I^X$ (resp. $\lambda \in I^X,\ \tau(\lambda) \geq r$).
(3) fuzzy regular semi irresolute closed (resp. fuzzy regular semi closed) iff $f(\lambda)$ is r-frsc in Y for each r-frsc set $\lambda \in I^X$ (resp. $\lambda \in I^X,\ \tau(\overline{1} - \lambda) \geq r$).
(4) fuzzy regular semi irresolute homeomorphism iff f is bijective, f and f^{-1} are fuzzy regular semi irresolute.

Remark 3.2. Every fuzzy regular continuous (resp. fuzzy regular semi irresolute) mapping is fuzzy regular semi continuous. However, the converse is not true as shown in the following example.

Example 3.3. Let $X = \{a, b, c\} = Y$, $\lambda_1, \lambda_2 \in I^X,\ \lambda_3$ and $\lambda_4 \in I^Y$ are defined as

$$\begin{array}{lll} \lambda_1(a) = 0.4, & \lambda_1(b) = 0.5, & \lambda_1(c) = 0.6, \\ \lambda_2(a) = 0.4, & \lambda_2(b) = 0.5, & \lambda_2(c) = 0.4, \\ \lambda_3(a) = 0.5, & \lambda_3(b) = 0.5, & \lambda_3(c) = 0.6, \\ \lambda_4(a) = 0.4, & \lambda_4(b) = 0.5, & \lambda_4(c) = 0.6. \end{array}$$

We define smooth topologies $\tau, \eta : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ \frac{1}{2} & \text{if } \lambda = \lambda_2, \\ 0 & \text{otherwise,} \end{cases} \qquad \eta(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_3, \\ \frac{1}{2} & \text{if } \lambda = \lambda_4, \\ 0 & \text{otherwise.} \end{cases}$$

For $r = \frac{1}{2}$ and $\eta(\lambda_3) \geq r$, there exist an r-fro set $\lambda_1 \in I^X$ such that $\lambda_1 \leq \lambda_3 \leq C_\tau(\lambda_1, r)$ and $f^{-1}(\lambda_3)$ is an r-frso set in X, but $f^{-1}(\lambda_3)$ is not r-fro set in X. Then the identity mapping f is fuzzy regular semi continuous but not fuzzy regular continuous.

Example 3.4. Let $X = \{a, b, c\} = Y$, $\lambda_1, \lambda_2 \in I^X$, λ_3 and $\lambda_4 \in I^Y$ are defined as

$$\begin{array}{lll} \lambda_1(a) = 0.4, & \lambda_1(b) = 0.5, & \lambda_1(c) = 0.6, \\ \lambda_2(a) = 0.4, & \lambda_2(b) = 0.5, & \lambda_2(c) = 0.4, \\ \lambda_3(a) = 0.6, & \lambda_3(b) = 0.5, & \lambda_3(c) = 0.4, \\ \lambda_4(a) = 0.4, & \lambda_4(b) = 0.5, & \lambda_4(c) = 0.6. \end{array}$$

We define smooth topologies $\tau, \eta : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ \frac{1}{2} & \text{if } \lambda = \lambda_2, \\ 0 & \text{otherwise,} \end{cases} \qquad \eta(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_3, \\ \frac{1}{2} & \text{if } \lambda = \lambda_4, \\ \frac{1}{2} & \text{if } \lambda = \lambda_3 \vee \lambda_4, \\ \frac{1}{2} & \text{if } \lambda = \lambda_3 \wedge \lambda_4, \\ 0 & \text{otherwise,} \end{cases}$$

For $r = \frac{1}{2}$ and $\eta(\lambda_4) \geq r$, there exist an r-fro set λ_1 such that $\lambda_1 \leq \lambda_4 \leq C_\tau(\lambda_1, r)$ and $f^{-1}(\lambda_4)$ is an r-frso set in X. But λ_3 is an r-frso set in Y, $f^{-1}(\lambda_3)$ is not an r-frso set in X. Then the identity mapping f is fuzzy regular semi continuous but not fuzzy regular semi irresolute.

Theorem 3.5. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be smooth topological spaces. Let $f : X \to Y$ be a mapping. The following statements are equivalent:*

(1) *f is fuzzy regular semi irresolute.*
(2) *For each r-frsc set $\mu \in I^Y$, $f^{-1}(\mu)$ is an r-frsc set in X.*
(3) *$f(RSC_\tau(\lambda, r)) \leq RSC_\eta(f(\lambda), r)$, for each $\lambda \in I^X$ and $r \in I_0$.*
(4) *$RSC_\tau(f^{-1}(\mu), r) \leq f^{-1}(RSC_\eta(\mu, r))$, for each $\mu \in I^Y$ and $r \in I_0$.*
(5) *$f^{-1}(RSI_\eta(\mu, r)) \leq RSI_\tau(f^{-1}(\mu), r)$, for each $\mu \in I^Y$ and $r \in I_0$.*

Proof. (1)$\Leftrightarrow$(2) Obvious.

(2)$\Rightarrow$(3) Let (2) holds and let

$$f(RSC_\tau(\lambda, r)) \nleq RSC_\eta(f(\lambda), r),$$

for some $\lambda \in I^X$ and $r \in I_0$. So there exists $y \in Y$ and $t \in (0, 1]$ such that:

$$f(RSC_\tau(\lambda, r))(y) > t > RSC_\eta(f(\lambda), r)(y).$$

If $f^{-1}(y) = \emptyset$, it is a contradiction, because

$$f(RSC_\tau(\lambda, r))(y) = \overline{0}.$$

So, if $f^{-1}(y) \neq \emptyset$ and there exists $x \in f^{-1}(y)$ such that

$$f(RSC_\tau(\lambda, r))(y) \geq RSC_\tau(\lambda, r)(x) > t > RSC_\eta(f(\lambda), r)(f(x)). \tag{3.1}$$

Also, $RSC_\eta(f(\lambda), r)(f(x)) < t$, implies that there exists an r-frsc set μ with $f(\lambda) = \mu$ such that

$$RSC_\eta(f(\lambda), r)(f(x)) \leq \mu(f(x)) < t.$$

Moreover, $f(\lambda) \leq \mu$ implies $\lambda \leq f^{-1}(\mu)$. From (2), $f^{-1}(\mu)$ is r-frsc. Thus $RSC_\tau(\lambda, r) \leq f^{-1}(\mu)$ and this implies:

$$RSC_\tau(\lambda, r)(x) \leq f^{-1}(\mu)(x) = \mu(f(x)) < t. \tag{3.2}$$

From relations (3.1) and (3.2) we see that: $RSC_\tau(\lambda, r)(x) > t$ and $RSC_\tau(\lambda, r)(x) \leq t$ which is a contradiction. Hence the result.

(3)⇒(4) Put $\lambda = f^{-1}(\mu)(\mu \in I^Y)$ and apply (3) we have

$$f(RSC_\tau(f^{-1}(\mu, r)) \leq RSC_\eta(f(f^{-1}(\mu), r)) \leq RSC_\eta(\mu, r).$$

Hence,

$$RSC_\tau(f^{-1}(\mu), r) \leq f^{-1}(RSC_\eta(\mu, r)).$$

(4)⇒(5) It follows immediately by taking the complement of (4).

(5)⇒(1) Let (5) holds and let μ be r-frso in Y. By (5)

$$f^{-1}(RSI_\eta(\mu, r)) \leq RSI_\tau(f^{-1}(\mu), r).$$

Then,

$$f^{-1}(\mu) \leq RSI_\tau(f^{-1}(\mu), r).$$

But

$$RSI_\tau(f^{-1}(\mu), r) \leq f^{-1}(\mu).$$

Hence

$$RSI_\tau(f^{-1}(\mu), r) = f^{-1}(\mu).$$

So, $f^{-1}(\mu)$ is r-frso and there by f is fuzzy regular semi irresolute. □

Theorem 3.6. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be smooth topological spaces. Let $f : X \to Y$ be a mapping. The following statements are equivalent:*

1. *f is fuzzy regular semi continuous.*
2. *$f(RSC_\tau(\lambda, r)) \leq RC_\eta(f(\lambda), r)$, for each $\lambda \in I^X$.*
3. *$RSC_\tau(f^{-1}(\mu), r) \leq f^{-1}(RC_\eta(\mu, r))$, for each $\mu \in I^Y$.*
4. *$f^{-1}(RI_\eta(\mu, r)) \leq RSI_\tau(f^{-1}(\mu), r)$, for each $\mu \in I^Y$.*

Proof. (1)⇒(2) Let (1) holds and let $\lambda \in I^X$. Suppose that there exists $\lambda \in I^X$ and $r \in I_0$ such that

$$f(RSC_\tau(\lambda, r)) \not\leq RC_\eta(f(\lambda), r).$$

So there exists $y \in Y$ and $t \in (0, 1]$ such that

$$f(RSC_\tau(\lambda, r))(y) > t > RC_\eta(f(\lambda), r)(y).$$

If $f^{-1}(y) = \emptyset$, then, $f(RSC_\tau(\lambda, r))(y) = \overline{0}$. So, there exists $x \in f^{-1}(y)$ such that:

$$f(RSC_\tau(\lambda, r))(y) > RSC_\tau(\lambda, r))(x) > t > RC_\eta(f(\lambda), r)(y). \tag{3.3}$$

Since $RC_\eta(f(\lambda), r)(y) < t$, there exists an r-frc set μ such that $f(\lambda) = \mu$. Then, $RC_\eta(f(\lambda), r)(y) \leq \mu$ and this implies

$$RC_\eta(f(\lambda), r)(f(x)) \leq \mu(f(x)) < t.$$

Moreover, $f(\lambda) \leq \mu$ implies $\lambda \leq f^{-1}(\mu)$. So, $\overline{1} - \lambda \geq \overline{1} - f^{-1}(\mu)$. Since $f^{-1}(\overline{1} - \mu)$ is r-frso, then from (1),

$$f^{-1}(\overline{1} - \mu) \leq RSI_\tau(\overline{1} - \lambda, r),$$

or

$$\overline{1} - RSI_\tau(\overline{1} - \lambda, r) \leq \overline{1} - f^{-1}(\overline{1} - \mu).$$

Then, $RSC_\tau(\lambda, r) \leq f^{-1}(\mu)$. This gives

$$RSC_\tau(\lambda, r)(x) \leq f^{-1}(\mu)(x) = \mu(f(x)) < t.$$

But the last relation contradicts the relation (3.3) above. Hence, the result in (2) is true.

(2)⇒(3) Take $\lambda = f^{-1}(\mu)(\mu \in I^Y)$ and apply (2) we have the required result.

(3)⇒(4) Taking the complement of (3), we have the result.

(4)⇒(1) Let (4) holds and let $\mu \in I^Y$ be an r-fro set, since every r-fro set is r-fuzzy open. So $\eta(\mu) \geq r$. By (4),

$$f^{-1}(RI_\eta(\mu, r)) \leq RSI_\tau(f^{-1}(\mu), r),$$

or

$$f^{-1}(\mu) \leq RSI_\tau(f^{-1}(\mu), r),$$

(Since $\eta(\mu) \geq r$). But $RSI_\tau(f^{-1}(\mu), r) \leq f^{-1}(\mu)$. Then, $f^{-1}(\mu) = RSI_\tau(f^{-1}(\mu), r)$ and this means that $f^{-1}(\mu)$ is r-frso. Hence, f is fuzzy regular semi continuous.

□

Theorem 3.7. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be smooth topological spaces. Let $f : X \to Y$ be a mapping. The following statements are equivalent:*

(1) *A map f is fuzzy regular semi irresolute open.*
(2) *$f(RSI_\tau(\lambda, r)) \leq RSI_\eta(f(\lambda), r)$, for each $\lambda \in I^X$ and $r \in I_0$.*
(3) *$RSI_\tau(f^{-1}(\mu), r) \leq f^{-1}(RSI_\eta(\mu, r))$, for each $\mu \in I^Y$ and $r \in I_0$.*
(4) *For any $\lambda \in I^Y$ and any r-frso set $\mu \in I^X$ such that $\mu \geq f^{-1}(\lambda)$, there exists an r-frsc set $\rho \in I^Y$ with $\lambda \leq \rho$ such that $f^{-1}(\rho) \leq \lambda$.*

Proof. (1)⇒(2) Since $RSI_\tau(\lambda, r) \leq \lambda$, then $RSI_\tau(\lambda, r) \leq f(\lambda)$. By (1),

$$f(RSI_\tau(\lambda, r)) \leq RSI_\eta(f(\lambda), r).$$

(2)⇒(3) Let (2) holds. Take $\lambda = f^{-1}(\mu),\ \lambda \in I^Y$ and apply part (2).

(3)⇒(4) Let $\mu \in I^X$ be an r-frsc set. Since $f^{-1}(\lambda) \leq \mu$, then

$$\overline{1} - \mu \leq \overline{1} - f^{-1}(\lambda) = f^{-1}(\overline{1} - \lambda).$$

It follows,

$$RSI_\tau(\overline{1} - \mu, r) \leq \overline{1} - \mu \leq RSI_\tau(f^{-1}(\overline{1} - \lambda), r).$$

By (3),

$$\overline{1} - \mu \leq RSI_\tau(f^{-1}(\overline{1} - \lambda), r) \leq f^{-1}(RSI_\eta(\overline{1} - \lambda, r)).$$

It implies

$$\begin{aligned}\mu \geq \overline{1} - f^{-1}(RSI_\eta(\overline{1} - \lambda, r)) &= f^{-1}(\overline{1} - RSI_\eta(\overline{1} - \lambda, r)) \\ &= f^{-1}(RSC_\eta(\lambda, r)).\end{aligned}$$

So, $\mu \geq f^{-1}(RSC_\tau(\lambda, r))$. Take $\rho = RSC_\tau(\lambda, r))$. Then ρ is r-frsc such that $\mu \geq f^{-1}(\rho)$ and $\rho \geq \lambda$. Hence the result.

(4)⇒(1) Let ω be r-frso in X. Put $\lambda = \overline{1} - f(\omega)$ and $\mu = \overline{1} - \omega$. It is easy to see that $\mu \geq f^{-1}(\lambda)$. By part (4), there exists an r-frsc set $\rho \in I^Y$ such that $\rho \geq \lambda$ and $\mu \geq f^{-1}(\rho)$ or $\overline{1} - \omega \geq f^{-1}(\rho)$. It implies

$$\omega \leq \overline{1} - f^{-1}(\rho) = f^{-1}(\overline{1} - \rho).$$

Thus $f(w) \leq ff^{-1}(\overline{1} - \rho) \leq \overline{1} - \rho$. On the other hand, $\lambda \leq \rho$, $f(\omega) = \overline{1} - \lambda$ implies $f(\omega) = \overline{1} - \lambda \geq \overline{1} - \rho$. Finally, we have $f(\omega) = \overline{1} - \rho$ and therefore $f(\omega)$ is r-frso. Hence f is fuzzy regular semi irresolute open.

□

Theorem 3.8. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be smooth topological spaces. Let $f : X \to Y$ be a mapping. The following statements are equivalent:*

(1) *f is fuzzy regular semi open.*
(2) *$f(I_\tau(\lambda, r)) \leq RSI_\eta(f(\lambda), r)$, for each $\lambda \in I^X$.*
(3) *$I_\tau(f^{-1}(\mu), r) \leq f^{-1}(RSI_\eta(\mu, r))$, for each $\mu \in I^Y$.*
(4) *For any $\lambda \in I^Y$ and any $\mu \in I^X$ such that $\tau(\overline{1} - \mu) \geq r,\ \mu \geq f^{-1}(\lambda)$, there exists an r-frsc set $\rho \in I^Y$ with $\lambda \leq \rho$ and $\mu \geq f^{-1}(\rho)$.*

Proof. (1)⇒(2) Since $I_\tau(\lambda, r) \leq \lambda,\ (\lambda \in I^X)$, then $f(I_\tau(\lambda, r)) \leq f(\lambda)$. But $f(I_\tau(\lambda, r))$ is r-frso (by (1)), then $f(I_\tau(\lambda, r)) \leq RSI_\eta(f(\lambda), r)$.

(2)⇒(3) Let (2) holds. Put $\lambda = f^{-1}(\mu),\ \mu \in I^Y$ and apply (2), we have

$$\begin{aligned} f(I_\tau(f^{-1}(\mu), r)) &\leq RSI_\eta(ff^{-1}(\mu), r), \\ &\leq RSI_\eta(\mu, r). \end{aligned}$$

Hence, $I_\tau(f^{-1}(\mu), r) \leq f^{-1}(RSI_\eta(\mu, r))$.

(3)⇒(4) Let (3) holds and let $\lambda \in I^Y$ and $\mu \in I^X$ such that $\tau(\overline{1} - \mu) \geq r$ and $\mu \geq f^{-1}(\lambda)$. Since $\overline{1} - \mu \leq \overline{1} - f^{-1}(\lambda) = f^{-1}(\overline{1} - \lambda)$, it implies

$$I_\tau(\overline{1} - \mu) = \overline{1} - \mu \leq I_\tau(f^{-1}(\overline{1} - \lambda), r)$$

or

$$\overline{1} - \mu \leq f^{-1}(RSI_\eta(\overline{1} - \lambda, r)).$$

Then,

$$\begin{aligned} \mu &\geq \overline{1} - f^{-1}(RSI_\eta(\overline{1} - \lambda, r)) \\ &= f^{-1}(RSC_\eta(\lambda, r)). \end{aligned}$$

Take $\rho = RSC_\eta(\lambda, r)$. Hence ρ is r-frsc and $\rho \geq \lambda$ such that $\mu \geq f^{-1}(\rho)$.

(4)⇒(1) Let ω be a fuzzy set such that $\omega \in I^X$ and $\tau(\omega) \geq r$. Put $\mu = \overline{1} - \omega$ and $\lambda = \overline{1} - f(\omega)$. Then $\mu \geq f^{-1}(\lambda)$. So there exists an r-frsc set ρ such that $\rho \geq \lambda$ and $\mu \geq f^{-1}(\rho)$. Then $\overline{1} - \omega \geq f^{-1}(\rho)$ implies $\omega \leq f^{-1}(\overline{1} - \rho)$. Thus $f(w) \leq \overline{1} - \rho$. Also, $\lambda \leq \rho$, $f(\omega) = \overline{1} - \lambda \geq \overline{1} - \rho$. Hence $f(\omega) = \overline{1} - \rho$ and therefore $f(\omega)$ is r-frso. So f is fuzzy regular semi open.

□

Theorem 3.9. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be smooth topological spaces. Let $f : X \to Y$ be a mapping. The following statements are equivalent:*

(1) *f is fuzzy regular semi irresolute closed.*
(2) *$f(RSC_\tau(\lambda, r)) \geq RSC_\eta(f(\lambda), r)$, for each $\lambda \in I^X$.*

Proof. (1)⇒(2) Let $\lambda \in I^X$. Since $\lambda \leq RSC_\tau(\lambda, r)$, then

$$f(\lambda) \leq f(RSC_\tau(\lambda, r)).$$

It implies $RSC_\eta(f(\lambda), r) \leq f(RSC_\tau(\lambda, r))$.

(2)⇒(1) Let (2) holds and $\lambda \in I^X$ such that λ is r-frsc. Then

$$RSC_\eta(f(\lambda), r) \leq f(\lambda).$$

But $f(\lambda) \leq RSC_\eta(f(\lambda), r)$. Hence $f(\lambda)$ is r-frsc and therefore f is fuzzy regular semi irresolute closed.

□

Theorem 3.10. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be smooth topological spaces. Let $f : X \to Y$ be a mapping. The following statements are equivalent:*

(1) *f is fuzzy regular semi closed.*
(2) $RSC_\eta(f(\lambda), r)) \leq f(RC_\tau(\lambda, r))$, *for each* $\lambda \in I^X$.

Proof. (1)⇒(2) Let (1) holds and let $\lambda \in I^X$. Since $\lambda \leq RC_\tau(\lambda, r)$, then $f(\lambda) \leq f(RC_\tau(\lambda, r))$ and therefore, $RSC_\eta(f(\lambda), r) \leq f(RC_\tau(\lambda, r))$.
(2)⇒(1) Let (2) holds and let $\lambda \in I^X$ be an r-frc set.

Then $RSC_\eta(f(\lambda), r) \leq f(RC_\tau(\lambda, r))$. It implies $RSC_\eta(f(\lambda), r) \leq f(\lambda)$. But $RSC_\eta(f(\lambda), r) \geq f(\lambda)$. Then, $f(\lambda) = RSC_\eta(f(\lambda), r)$ and it follows that $f(\lambda)$ is r-frsc. Hence, f is fuzzy regular semi closed. □

Theorem 3.11. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be smooth topological spaces. Let $f : X \to Y$ is bijective. Then :*

(1) *f is fuzzy regular semi irresolute closed iff*
$$f^{-1}(RSC_\eta(\mu, r)) \leq RSC_\tau(f^{-1}(\mu), r),$$
for each $\mu \in I^Y$ and $r \in I_0$.
(2) *f is fuzzy regular semi irresolute closed iff f is fuzzy regular semi irresolute open.*

Proof. (1)(⇒): Let f be fuzzy regular semi irresolute closed. From Theorem 3.9, we have:
$$f(RSC_\tau(\lambda, r)) \geq RSC_\eta(f(\lambda), r)), \quad \lambda \in I^X.$$
Let $\mu \in I^Y$ and put $\lambda = f^{-1}(\mu)$, we have
$$\begin{aligned} f(RSC_\tau(f^{-1}(\mu), r)) &\geq RSC_\eta(ff^{-1}(\mu), r) \\ &= RSC_\eta(\mu, r). \end{aligned}$$
It implies $RSC_\tau(f^{-1}(\mu), r) \geq f^{-1}(RSC_\eta(\mu, r))$.

(⇐): On the other hand let the condition is satisfied and let $\mu \in I^X$ such that μ is r-frsc. Then $f(\mu) \in I^Y$. Apply the condition we have:
$$RSC_\tau(f^{-1}f(\mu), r) \geq f^{-1}(RSC_\eta(f(\mu), r)).$$
It implies that
$$RSC_\tau(\mu, r) \geq f^{-1}(RSC_\eta(f(\mu), r)).$$
Then, $f(RSC_\tau(\mu, r)) \geq RSC_\eta(f(\mu), r)$. So by Theorem 3.9, f is fuzzy regular semi irresolute closed.

(2) Apply Theorem 3.9 and taking the complement we have the required result.

□

From Theorems 3.5, 3.7, 3.9, 3.11, we obtain the following Theorem.

Theorem 3.12. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be smooth topological spaces. Let $f : X \to Y$ be a mapping. The following statements are equivalent:*

(1) *f is fuzzy regular semi irresolute homeomorphism.*
(2) *f is fuzzy regular semi irresolute and fuzzy regular semi irresolute open.*
(3) *f is fuzzy regular semi irresolute and fuzzy regular semi irresolute closed.*
(4) $f(RSI_\tau(\lambda, r)) = RSI_\eta(f(\lambda), r)$, *for each* $\lambda \in I^X,\ r \in I_0$.
(5) $f(RSC_\tau(\lambda, r)) = RSC_\eta(f(\lambda), r)$, *for each* $\lambda \in I^X,\ r \in I_0$.
(6) $RSI_\tau(f^{-1}(\mu), r) = f^{-1}(RSI_\eta(\mu, r))$.
(6) $RSC_\tau(f^{-1}(\mu), r) = f^{-1}(RSC_\eta(\mu, r))$, $\mu \in I^Y,\ r \in I_0$.

Note that the composition of two fuzzy regular semi irresolute mappings is fuzzy regular semi irresolute. In general, the composition of two fuzzy regular semi continuous mappings is not fuzzy regular semi continuous.

Example 3.13. Let $X = \{a, b, c\} = Y = Z$, $\lambda_1, \lambda_2 \in I^X$, $\lambda_3, \lambda_4 \in I^Y$ and $\lambda_5 \in I^Z$ are defined as

$$\begin{array}{lll} \lambda_1(a) = 0.4, & \lambda_1(b) = 0.5, & \lambda_1(c) = 0.6, \\ \lambda_2(a) = 0.4, & \lambda_2(b) = 0.5, & \lambda_2(c) = 0.4, \\ \lambda_3(a) = 0.6, & \lambda_3(b) = 0.5, & \lambda_3(c) = 0.4, \\ \lambda_4(a) = 0.4, & \lambda_4(b) = 0.5, & \lambda_4(c) = 0.6, \\ \lambda_5(a) = 0.6, & \lambda_5(b) = 0.5, & \lambda_5(c) = 0.4. \end{array}$$

We define smooth topologies $\tau, \sigma, \eta : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ \frac{1}{2} & \text{if } \lambda = \lambda_2, \\ 0 & \text{otherwise}, \end{cases} \qquad \sigma(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_3, \\ \frac{1}{2} & \text{if } \lambda = \lambda_4, \\ \frac{1}{2} & \text{if } \lambda = \lambda_3 \vee \lambda_4, \\ \frac{1}{2} & \text{if } \lambda = \lambda_3 \wedge \lambda_4, \\ 0 & \text{otherwise}, \end{cases}$$

$$\eta(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_5, \\ 0 & \text{otherwise}. \end{cases}$$

For $r = \frac{1}{2}$ and $\eta(\lambda_5) \geq r$, there exist an r-fro set $\lambda_3 \in I^Y$ such that $\lambda_3 \leq \lambda_5 \leq C_\sigma(\lambda_3, r)$ and $f^{-1}(\lambda_5)$ is an r-frso set in Y. $\sigma(\lambda_4) \geq r$, there exist an r-fro set $\lambda_1 \in I^X$ such that $\lambda_1 \leq \lambda_4 \leq C_\tau(\lambda_1, r)$ and $f^{-1}(\lambda_4)$ is an r-frso set in X. But $\eta(\lambda_5) \geq r$, $f^{-1}(\lambda_5)$ is not r-frso set in X. The identity mapping $id : (X,\ \tau) \to (Y,\ \sigma)$ and $id : (Y,\ \sigma) \to (Z,\ \eta)$ are fuzzy regular semi continuous, but $id : (X,\ \tau) \to (Z,\ \eta)$ is not fuzzy regular semi continuous.

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References

1. R. Badard, *Smooth axiomatics*, First IFSA Congress Palma de Mallorca, 1986.
2. C.L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., 24 (1968) 182-190.
3. K.C. Chattopadhyay and S.K. Samanta, *Fuzzy topology*, Fuzzy Sets and Systems, 54 (1993) 207-212.
4. K.C. Chattopadhyay, R.N. Hazra, and S.K. Samanta, *Gradation of openness*, Fuzzy Sets and Systems, 49 (2) (1992) 237-242.
5. R.N. Hazra, S.K. Samanta, and K.C. Chattopadhyay, *Fuzzy topology redefined*, Fuzzy Sets and Systems, 45 (1992) 79-82.
6. E.E. Kerre and M.A. Fath Alla, *On fuzzy regular semi open sets and αS^*-closed fuzzy topological spaces*, The Journal of Fuzzy Mathematics, 11 (1) (2003) 225-235.
7. S.J. Lee and E.P. Lee, *Fuzzy r-regular open sets and fuzzy almost r-continuous maps*, Bull. Korean Math. Soc., 39(3) (2002) 441-453.
8. A.S. Mashhour, M.H. Ghanim, and M.A. Fath Alla, *α-separation axioms and α-compactness in fuzzy topological spaces*, Rocky Moutain Journal of Mathematics, 16 (3) (1986) 591-600.
9. A.A. Ramadan, *Smooth topological spaces*, Fuzzy Sets and Systems, 48 (1992) 371-375.
10. A.A. Ramadan, S.E. Abbas, and Y.C. Kim, *Fuzzy irresolute mappings in smooth fuzzy topological spaces*, The Journal of Fuzzy Mathematics, 9 (4) (2001) 865-877.
11. A.P. Šostak, *On a fuzzy topological structure*, Rend. Circ. Matem. Palermo Ser II, 11 (1986), 89-103.
12. A. Vadivel and E. Elavarasan, *Applications of r-generalized regular fuzzy closed sets*, Annals of Fuzzy Mathematics and Informatics, (Accepted).
13. A.M. Zahran, *Fuzzy regular semi open sets and αS-closed spaces*, The Journal of Fuzzy Mathematics, 2 (4) (1997), 579-586.

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