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# Generalized Ritt type and generalized Ritt weak type connected growth properties of entire functions represented by vector valued Dirichlet series

Sanjib Kumar Datta[1*], Tanmay Biswas[2], and Jinarul Haque Shaikh[3]

Abstract. In this paper, we introduce the idea of generalized Ritt type and generalised Ritt weak type of entire functions represented by a vector valued Dirichlet series. Hence, we study some growth properties of two entire functions represented by a vector valued Dirichlet series on the basis of generalized Ritt type and generalised Ritt weak type.

## 1. Introduction

Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ ($\sigma$ and $t$ are real variables) defined by everywhere absolutely convergent vector valued Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}, \tag{1.1}$$

where $a_n$'s are belong to a Banach space $(E, \|.\|)$ and $\lambda_n$'s are non-negative real numbers such that $0 < \lambda_n < \lambda_{n+1} \, (n \geq 1), \lambda_n \to \infty$ as $n \to \infty$ and satisfy the conditions

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D < \infty,$$

and

$$\limsup_{n\to\infty} \frac{\log \|a_n\|}{\lambda_n} = -\infty.$$

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* Corresponding author.

If $\sigma_a$ and $\sigma_c$ are denoted respectively the abscissa of convergence and absolute convergence of (1.1), then in this case clearly $\sigma_a = \sigma_c = \infty$.

The function $M(\sigma, f)$ known as maximum modulus function corresponding to the entire function $f(s)$ defined by (1.1), is written as follows

$$M(\sigma, f) = \underset{-\infty<t<\infty}{l.u.b.} |f(\sigma + it)|.$$

Then the Ritt order [1] of $f(s)$, denoted by $\rho_f$ is given by

$$\rho_f = \inf\{\mu > 0 : \log M(\sigma, f) < \exp(\sigma\mu) \text{ for all } \sigma > R(\mu)\}$$
$$= \limsup_{\sigma\to\infty} \frac{\log^{[2]} M(\sigma, f)}{\sigma},$$

where

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \quad \text{for} \quad k = 1, 2, 3, \ldots;$$
$$\log^{[0]} x = x.$$

Similarly, one can define the Ritt lower order of $f(s)$, denoted by $\lambda_f$ in the following manner:

$$\lambda_f = \liminf_{\sigma\to\infty} \frac{\log^{[2]} M(\sigma, f)}{\sigma}.$$

To compare the relative growth of two entire functions having same non zero finite Ritt orders, one may introduce the definition of Ritt-type in the following manner:

**Definition 1.1.** The Ritt type denoted by $\Delta_f$ of an entire function $f$ is defined as follows:

$$\Delta_f = \limsup_{r\to\infty} \frac{\log M(\sigma, f)}{\exp[\rho_f \cdot \sigma]}, \quad 0 < \rho_f < \infty.$$

During the past decades, several authors (see for example [1–3, 5, 7]) made close investications on the properties of entire Dirichlet series related to Ritt order. Further, B. L. Srivastava [6] defined different growth parameters such as order and lower order of entire functions represented by vector valued Dirichlet series. He also obtained the results for coefficient characterization of order. Now in the line of Sato [4], it therefore seems reasonable to define suitably the generalized Ritt order denoted as $\rho_f^{[l]}$ of an entire function represented by a vector valued Dirichlet series which is as follows:

$$\rho_f^{[l]} = \limsup_{\sigma\to\infty} \frac{\log^{[l]} M(\sigma, f)}{\sigma},$$

where $l$ is any positive integer.

Likewise, one can define the generalized Ritt lower order of $f(s)$, denoted by $\lambda_f^{[l]}$ for any integer $l \geq 1$ in the following manner:

$$\lambda_f^{[l]} = \liminf_{\sigma \to \infty} \frac{\log^{[l]} M(\sigma, f)}{\sigma}.$$

Further, to compare the relative growth of two entire functions represented by vector valued Dirichlet series having same non zero finite generalized Ritt orders, one may introduce the definitions of generalized Ritt-type and generalized Ritt-lower type in the following manner:

**Definition 1.2.** The generalized Ritt-type and generalized Ritt-lower type denoted respectively by $\Delta_f^{[l]}$ and $\overline{\Delta}_f^{[l]}$ of an entire function $f$ represented by vector valued Dirichlet series are respectively defined as follows:

$$\Delta_f^{[l]} = \limsup_{r \to \infty} \frac{\log^{[l-1]} M(\sigma, f)}{\exp\left[\rho_f^{[l]} \cdot \sigma\right]},$$

and

$$\overline{\Delta}_f^{[l]} = \liminf_{r \to \infty} \frac{\log^{[l-1]} M(\sigma, f)}{\exp\left[\rho_f^{[l]} \cdot \sigma\right]}, \quad 0 < \rho_f^{[l]} < \infty,$$

where $l$ is any positive integer.

Analogusly to determine the relative growth of two entire functions represented by vector valued Dirichlet series having same non zero finite generalized Ritt-lower order one may introduce the definition of generalized Ritt-weak type in the following way:

**Definition 1.3.** The generalized Ritt-weak type denoted by $\tau_f^{[l]}$ of an entire function $f$ represented by vector valued Dirichlet series is defined as follows:

$$\tau_f^{[l]} = \liminf_{r \to \infty} \frac{\log^{[l-1]} M(\sigma, f)}{\exp\left[\lambda_f^{[l]} \cdot \sigma\right]}, \quad 0 < \lambda_f^{[l]} < \infty$$

where $l$ is any positive integer.

Also one may define the growth indicator $\overline{\tau}_f^{[l]}$ of an entire function $f$ represented by vector valued Dirichlet series in the following manner:

$$\overline{\tau}_f^{[l]} = \limsup_{r \to \infty} \frac{\log^{[l-1]} M(\sigma, f)}{\exp\left[\lambda_f^{[l]} \cdot \sigma\right]}, \quad 0 < \lambda_f^{[l]} < \infty,$$

where $l$ is any positive integer.

In the paper, we establish some new results depending on the comparative growth properties of two entire functions represented by vector valued Dirichlet series using generalized Ritt-order, generalized Ritt-type and generalized Ritt-weak type as compared to the corresponding left and right factors.

## 2. Theorems

In this section, we present the main results of the paper.

**Theorem 2.1.** *If $f$, $g$ be any two entire vector valued Dirichlet series such that $0<\overline{\Delta}_f^{[m]}\leq\Delta_f^{[m]}<\infty$, $0<\overline{\Delta}_g^{[n]}\leq\Delta_g^{[n]}<\infty$ and $\rho_f^{[m]}=\rho_g^{[n]}$ where $m$ and $n$ are any two pisitive integers, then*

$$\begin{aligned}\frac{\overline{\Delta}_f^{[m]}}{\Delta_g^{[n]}} &\leq \liminf_{\sigma\to\infty}\frac{\log^{[m-1]}M(\sigma,f)}{\log^{[n-1]}M(\sigma,g)}\\ &\leq \frac{\overline{\Delta}_f^{[m]}}{\overline{\Delta}_g^{[n]}}\\ &\leq \limsup_{\sigma\to\infty}\frac{\log^{[m-1]}M(\sigma,f)}{\log^{[n-1]}M(\sigma,g)}\\ &\leq \frac{\Delta_f^{[m]}}{\overline{\Delta}_g^{[n]}}.\end{aligned}$$

*Proof.* From the definition of $\Delta_g^{[n]}$ and $\overline{\Delta}_f^{[m]}$, we have for any arbitrary positive $\varepsilon$ and for all sufficiently large values of $\sigma$ that

$$\log^{[m-1]}M(\sigma,f)\geq\left(\overline{\Delta}_f^{[m]}-\varepsilon\right)\exp\left[\rho_f^{[m]}\cdot\sigma\right], \tag{2.1}$$

and

$$\log^{[n-1]}M(\sigma,g)\leq\left(\Delta_g^{[n]}+\varepsilon\right)\exp\left[\rho_g^{[n]}\cdot\sigma\right]. \tag{2.2}$$

Now from (2.1), (2.2) and the condition $\rho_f^{[m]}=\rho_g^{[n]}$, it follows for all sufficiently large values of $\sigma$ that

$$\frac{\log^{[m-1]}M(\sigma,f)}{\log^{[n-1]}M(\sigma,g)}\geq\frac{\overline{\Delta}_f^{[m]}-\varepsilon}{\Delta_g^{[n]}+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from above that

$$\liminf_{\sigma\to\infty}\frac{\log^{[m-1]}M(\sigma,f)}{\log^{[n-1]}M(\sigma,g)}\geq\frac{\overline{\Delta}_f^{[m]}}{\Delta_g^{[n]}}. \tag{2.3}$$

Again for a sequence of values of $\sigma$ tending to infinity,

$$\log^{[m-1]} M(\sigma, f) \leq \left(\overline{\Delta}_f^{[m]} + \varepsilon\right) \exp\left[\rho_f^{[m]} \cdot \sigma\right], \tag{2.4}$$

and for all sufficiently large values of $\sigma$,

$$\log^{[n-1]} M(\sigma, g) \geq \left(\overline{\Delta}_g^{[n]} - \varepsilon\right) \exp\left[\rho_g^{[n]} \cdot \sigma\right]. \tag{2.5}$$

Combining (2.4) and (2.5) and the condition $\rho_f^{[m]} = \rho_g^{[n]}$, we get for a sequence of values of $\sigma$ tending to infinity that

$$\frac{\log^{[m-1]} M(\sigma, f)}{\log^{[n-1]} M(\sigma, g)} \leq \frac{\overline{\Delta}_f^{[m]} + \varepsilon}{\overline{\Delta}_g^{[n]} - \varepsilon}.$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\liminf_{\sigma\to\infty} \frac{\log^{[m-1]} M(\sigma, f)}{\log^{[n-1]} M(\sigma, g)} \leq \frac{\overline{\Delta}_f^{[m]}}{\overline{\Delta}_g^{[n]}}. \tag{2.6}$$

Also for a sequence of values of $r$ tending to infinity, it follows that

$$\log^{[n-1]} M(\sigma, g) \leq \left(\overline{\Delta}_g^{[n]} + \varepsilon\right) \exp\left[\rho_g^{[n]} \cdot \sigma\right]. \tag{2.7}$$

Now from (2.1), (2.7) and the condition $\rho_f^{[m]} = \rho_g^{[n]}$, we obtain for a sequence of values of $\sigma$ tending to infinity that

$$\frac{\log^{[m-1]} M(\sigma, f)}{\log^{[n-1]} M(\sigma, g)} \geq \frac{\overline{\Delta}_f^{[m]} - \varepsilon}{\overline{\Delta}_g^{[n]} + \varepsilon}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we get from above that

$$\limsup_{\sigma\to\infty} \frac{\log^{[m-1]} M(\sigma, f)}{\log^{[n-1]} M(\sigma, g)} \geq \frac{\overline{\Delta}_f^{[m]}}{\overline{\Delta}_g^{[n]}}. \tag{2.8}$$

Also for all sufficiently large values of $\sigma$,

$$\log^{[m-1]} M(\sigma, f) \leq \left(\Delta_f^{[m]} + \varepsilon\right) \exp\left[\rho_f^{[m]} \cdot \sigma\right]. \tag{2.9}$$

In view of the condition $\rho_f^{[m]} = \rho_g^{[n]}$, it follows from (2.5) and (2.9) for all sufficiently large values of $\sigma$ that

$$\frac{\log^{[m-1]} M(\sigma, f)}{\log^{[n-1]} M(\sigma, g)} \leq \frac{\Delta_f^{[m]} + \varepsilon}{\overline{\Delta}_g^{[n]} - \varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{\sigma\to\infty}\frac{\log^{[m-1]}M(\sigma,f)}{\log^{[n-1]}M(\sigma,g)}\leq\frac{\Delta_f^{[m]}}{\Delta_g^{[n]}}. \tag{2.10}$$

Thus the theorem follows from (2.3), (2.6), (2.8) and (2.10). □

**Theorem 2.2.** *If $f$, $g$ be any two entire vector valued Dirichlet series such that $0<\Delta_f^{[m]}<\infty$, $0<\Delta_g^{[n]}<\infty$ and $\rho_f^{[m]}=\rho_g^{[n]}$ where $m$ and $n$ are any two pisitive integers, then*

$$\liminf_{\sigma\to\infty}\frac{\log^{[m-1]}M(\sigma,f)}{\log^{[n-1]}M(\sigma,g)}\leq\frac{\Delta_f^{[m]}}{\Delta_g^{[n]}}\leq\limsup_{r\to\infty}\frac{\log^{[m-1]}M(\sigma,f)}{\log^{[n-1]}M(\sigma,g)}.$$

*Proof.* From the definition of $\Delta_g^{[n]}$, we get for a sequence of values of $\sigma$ tending to infinity that

$$\log^{[n-1]}M(\sigma,g)\geq\left(\Delta_g^{[n]}-\varepsilon\right)\exp\left[\rho_g^{[n]}\cdot\sigma\right]. \tag{2.11}$$

Now from (2.9), (2.11) and the condition $\rho_f^{[m]}=\rho_f^{[n]}$, it follows for a sequence of values of $\sigma$ tending to infinity that

$$\frac{\log^{[m-1]}M(\sigma,f)}{\log^{[n-1]}M(\sigma,g)}\leq\frac{\Delta_f^{[m]}+\varepsilon}{\Delta_g^{[n]}-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{\sigma\to\infty}\frac{\log^{[m-1]}M(\sigma,f)}{\log^{[n-1]}M(\sigma,g)}\leq\frac{\Delta_f^{[m]}}{\Delta_g^{[n]}}. \tag{2.12}$$

Again for a sequence of values of $\sigma$ tending to infinity, we have

$$\log^{[m-1]}M(\sigma,f)\geq\left(\Delta_f^{[m]}-\varepsilon\right)\exp\left[\rho_f^{[m]}\cdot\sigma\right]. \tag{2.13}$$

So combining (2.2) and (2.13) and in view of the condition $\rho_f^{[m]}=\rho_g^{[n]}$, we get for a sequence of values of $\sigma$ tending to infinity that

$$\frac{\log^{[m-1]}M(\sigma,f)}{\log^{[n-1]}M(\sigma,g)}\geq\frac{\Delta_f^{[m]}-\varepsilon}{\Delta_g^{[n]}+\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{\sigma\to\infty}\frac{\log^{[m-1]}M(\sigma,f)}{\log^{[n-1]}M(\sigma,g)}\geq\frac{\Delta_f^{[m]}}{\Delta_g^{[n]}}. \tag{2.14}$$

Thus the theorem follows from (2.12) and (2.14). □

The following theorem is a natural consequence of Theorem 2.1 and Theorem 2.2.

**Theorem 2.3.** *If $f$, $g$ be any two entire vector valued Dirichlet series such that $0 < \overline{\Delta}_f^{[m]} \leq \Delta_f^{[m]} < \infty$, $0 < \overline{\Delta}_g^{[n]} \leq \Delta_g^{[n]} < \infty$ and $\rho_f^{[m]} = \rho_g^{[n]}$ where $m$ and $n$ are any two pisitive integers, then*

$$\liminf_{\sigma\to\infty}\frac{\log^{[m-1]} M(\sigma,f)}{\log^{[n-1]} M(\sigma,g)} \leq \min\left\{\frac{\overline{\Delta}_f^{[m]}}{\overline{\Delta}_g^{[n]}}, \frac{\Delta_f^{[m]}}{\Delta_g^{[n]}}\right\}$$
$$\leq \max\left\{\frac{\overline{\Delta}_f^{[m]}}{\overline{\Delta}_g^{[n]}}, \frac{\Delta_f^{[m]}}{\Delta_g^{[n]}}\right\}$$
$$\leq \limsup_{\sigma\to\infty}\frac{\log^{[m-1]} M(\sigma,f)}{\log^{[n-1]} M(\sigma,g)}.$$

Now in the line of Theorem 2.1, Theorem 2.2 and Theorem 2.3, one can easily prove the following three theorems using the notion of generalized Ritt weak type and therefore their proofs are omitted.

**Theorem 2.4.** *If $f$, $g$ be any two entire vector valued Dirichlet series such that $0 < \tau_f^{[m]} \leq \overline{\tau}_f^{[m]} < \infty$, $0 < \tau_g^{[n]} \leq \overline{\tau}_g^{[n]} < \infty$ and $\lambda_f^{[m]} = \lambda_g^{[n]}$ where $m$ and $n$ are any two pisitive integers, then*

$$\frac{\tau_f^{[m]}}{\overline{\tau}_g^{[n]}} \leq \liminf_{\sigma\to\infty}\frac{\log^{[m-1]} M(\sigma,f)}{\log^{[n-1]} M(\sigma,g)}$$
$$\leq \frac{\tau_f^{[m]}}{\tau_g^{[n]}}$$
$$\leq \limsup_{\sigma\to\infty}\frac{\log^{[m-1]} M(\sigma,f)}{\log^{[n-1]} M(\sigma,g)}$$
$$\leq \frac{\overline{\tau}_f^{[m]}}{\tau_g^{[n]}}.$$

**Theorem 2.5.** *If $f$, $g$ be any two entire vector valued Dirichlet series with $0 < \overline{\tau}_f^{[m]} < \infty$, $0 < \overline{\tau}_g^{[n]} < \infty$ and $\lambda_f^{[m]} = \lambda_g^{[n]}$ where $m$ and $n$ are any two pisitive integers, then*

$$\liminf_{\sigma\to\infty}\frac{\log^{[m-1]} M(\sigma,f)}{\log^{[n-1]} M(\sigma,g)} \leq \frac{\overline{\tau}_f^{[m]}}{\overline{\tau}_g^{[n]}}$$
$$\leq \limsup_{\sigma\to\infty}\frac{\log^{[m-1]} M(\sigma,f)}{\log^{[n-1]} M(\sigma,g)}.$$

**Theorem 2.6.** *If $f$, $g$ be any two entire vector valued Dirichlet series such that $0 < \tau_f^{[m]} \leq \overline{\tau}_f^{[m]} < \infty$, $0 < \tau_g^{[n]} \leq \overline{\tau}_g^{[n]} < \infty$ and $\lambda_f^{[m]} = \lambda_g^{[n]}$ where $m$ and $n$ are any two pisitive integers, then*

$$\liminf_{\sigma\to\infty} \frac{\log^{[m-1]} M(\sigma,f)}{\log^{[n-1]} M(\sigma,g)} \leq \min\left\{ \frac{\tau_f^{[m]}}{\tau_g^{[n]}}, \frac{\overline{\tau}_f^{[m]}}{\overline{\tau}_g^{[n]}} \right\}$$

$$\leq \max\left\{ \frac{\tau_f^{[m]}}{\tau_g^{[n]}}, \frac{\overline{\tau}_f^{[m]}}{\overline{\tau}_g^{[n]}} \right\}$$

$$\leq \limsup_{\sigma\to\infty} \frac{\log^{[m-1]} M(\sigma,f)}{\log^{[n-1]} M(\sigma,g)}.$$

We may now state the following theorems without their proofs based on generalized Ritt type and generalized Ritt weak type.

**Theorem 2.7.** *If $f$, $g$ be any two entire vector valued Dirichlet series such that $0 < \overline{\Delta}_f^{[m]} \leq \Delta_f^{[m]} < \infty$, $0 < \tau_g^{[n]} \leq \overline{\tau}_g^{[n]} < \infty$ and $\rho_f^{[m]} = \lambda_g^{[n]}$ where $m$ and $n$ are any two pisitive integers, then*

$$\frac{\overline{\Delta}_f^{[m]}}{\overline{\tau}_g^{[n]}} \leq \liminf_{\sigma\to\infty} \frac{\log^{[m-1]} M(\sigma,f)}{\log^{[n-1]} M(\sigma,g)}$$

$$\leq \frac{\overline{\Delta}_f^{[m]}}{\tau_g^{[n]}}$$

$$\leq \limsup_{\sigma\to\infty} \frac{\log^{[m-1]} M(\sigma,f)}{\log^{[n-1]} M(\sigma,g)}$$

$$\leq \frac{\Delta_f^{[m]}}{\tau_g^{[n]}}.$$

**Theorem 2.8.** *If $f$, $g$ be any two entire vector valued Dirichlet series with $0 < \Delta_f^{[m]} < \infty$, $0 < \overline{\tau}_g^{[n]} < \infty$ and $\rho_f^{[m]} = \lambda_g^{[n]}$ where $m$ and $n$ are any two pisitive integers, then*

$$\liminf_{\sigma\to\infty} \frac{\log^{[m-1]} M(\sigma,f)}{\log^{[n-1]} M(\sigma,g)} \leq \frac{\Delta_f^{[m]}}{\overline{\tau}_g^{[n]}}$$

$$\leq \limsup_{\sigma\to\infty} \frac{\log^{[m-1]} M(\sigma,f)}{\log^{[n-1]} M(\sigma,g)}.$$

**Theorem 2.9.** *If $f$, $g$ be any two entire vector valued Dirichlet series such that $0 < \overline{\Delta}_f^{[m]} \leq \Delta_f^{[m]} < \infty$, $0 < \tau_g^{[n]} \leq \overline{\tau}_g^{[n]} < \infty$ and $\rho_f^{[m]} = \lambda_g^{[n]}$*

*where $m$ and $n$ are any two pisitive integers, then*

$$\begin{aligned}\liminf_{\sigma\to\infty}\frac{\log^{[m-1]}M(\sigma,f)}{\log^{[n-1]}M(\sigma,g)} &\leq \min\left\{\frac{\overline{\Delta}_f^{[m]}}{\tau_g^{[n]}},\frac{\Delta_f^{[m]}}{\overline{\tau}_g^{[n]}}\right\}\\ &\leq \max\left\{\frac{\overline{\Delta}_f^{[m]}}{\tau_g^{[n]}},\frac{\Delta_f^{[m]}}{\overline{\tau}_g^{[n]}}\right\}\\ &\leq \limsup_{\sigma\to\infty}\frac{\log^{[m-1]}M(\sigma,f)}{\log^{[n-1]}M(\sigma,g)}.\end{aligned}$$

**Theorem 2.10.** *If $f$, $g$ be any two entire vector valued Dirichlet series with $0<\tau_f^{[m]}\leq\overline{\tau}_f^{[m]}<\infty$, $0<\overline{\Delta}_g^{[n]}\leq\Delta_g^{[n]}<\infty$ and $\lambda_f^{[m]}=\rho_g^{[n]}$ where $m$ and $n$ are any two pisitive integers, then*

$$\begin{aligned}\frac{\tau_f^{[m]}}{\Delta_g^{[n]}} &\leq \liminf_{\sigma\to\infty}\frac{\log^{[m-1]}M(\sigma,f)}{\log^{[n-1]}M(\sigma,g)}\\ &\leq \frac{\tau_f^{[m]}}{\overline{\Delta}_g^{[n]}}\\ &\leq \limsup_{\sigma\to\infty}\frac{\log^{[m-1]}M(\sigma,f)}{\log^{[n-1]}M(\sigma,g)}\\ &\leq \frac{\overline{\tau}_f^{[m]}}{\overline{\Delta}_g^{[n]}}.\end{aligned}$$

**Theorem 2.11.** *If $f$, $g$ be any two entire vector valued Dirichlet series such that $0<\overline{\tau}_f^{[m]}<\infty$, $0<\Delta_g^{[n]}<\infty$ and $\lambda_f^{[m]}=\rho_g^{[n]}$ where $m$ and $n$ are any two pisitive integers, then*

$$\begin{aligned}\liminf_{\sigma\to\infty}\frac{\log^{[m-1]}M(\sigma,f)}{\log^{[n-1]}M(\sigma,g)} &\leq \frac{\overline{\tau}_f^{[m]}}{\Delta_g^{[n]}}\\ &\leq \limsup_{\sigma\to\infty}\frac{\log^{[m-1]}M(\sigma,f)}{\log^{[n-1]}M(\sigma,g)}.\end{aligned}$$

**Theorem 2.12.** *If $f$, $g$ be any two entire vector valued Dirichlet series with $0<\tau_f^{[m]}\leq\overline{\tau}_f^{[m]}<\infty$, $0<\overline{\Delta}_g^{[n]}\leq\Delta_g^{[n]}<\infty$ and $\lambda_f^{[m]}=\rho_g^{[n]}$ where* *m and n are any two pisitive integers, then*

$$\begin{aligned}\liminf_{\sigma\to\infty}\frac{\log^{[m-1]} M(\sigma,f)}{\log^{[n-1]} M(\sigma,g)} &\leq \min\left\{\frac{\tau_f^{[m]}}{\overline{\Delta}_g^{[n]}},\frac{\overline{\tau}_f^{[m]}}{\Delta_g^{[n]}}\right\}\\ &\leq \max\left\{\frac{\tau_f^{[m]}}{\overline{\Delta}_g^{[n]}},\frac{\overline{\tau}_f^{[m]}}{\Delta_g^{[n]}}\right\}\\ &\leq \limsup_{\sigma\to\infty}\frac{\log^{[m-1]} M(\sigma,f)}{\log^{[n-1]} M(\sigma,g)}.\end{aligned}$$

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[1] Department of Mathematics, University of Kalyani, P.O.-Kalyani, Dist-Nadia, PIN- 741235, West Bengal, India.

*E-mail address*: `sanjib_kr_datta@yahoo.co.in`

[2] Rajbari, Rabindrapalli, R. N. Tagore Road, P.O.-Krishnagar, Dist-Nadia, PIN- 741101, West Bengal, India.

*E-mail address*: `tanmaybiswas_math@rediffmail.com`

[3] Department of Mathematics, University of Kalyani, P.O.-Kalyani, Dist-Nadia, PIN- 741235, West Bengal, India.

*E-mail address*: `jnrlhqshkh188@gmail.com`