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SOME NOTES FOR TOPOLOGICAL CENTERS ON THE DUALS OF BANACH ALGEBRAS

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ABSTRACT. We introduce the weak topological centers of left and right module actions and we study some of their properties. We investigate the relationship between these new concepts and the topological centers of of left and right module actions with some results in the group algebras.

1. INTRODUCTION

For a Banach algebra A, Arens showed that its second dual A^{**} endowed with the either Arens multiplications is a Banach algebra, see [1]. The constructions of the two Arens multiplications in A^{**} lead us to definition of topological centers for A^{**} with respect both Arens multiplications. The topological centers of Banach algebras, module actions and their applications have been introduced and discussed in [3–5, 7, 8]. Suppose that A is a Banach algebra and B is a Banach $A-bimodule$. It is clear that, B^{**} is a Banach $A^{**}-bimodule$, where A^{**} is equipped with the first Arens product. We define the topological centers of the left and right module actions of A^{**} on B^{**} as follows:

$$Z^{\ell}_{A^{**}}(B^{**}) = Z(\pi_r) = \{b'' \in B^{**} : a'' \to \pi_r^{***}(b'', a'') : A^{**} \to B^{**} \text{ is weak}^* - \text{weak}^* \text{ continuous}\},$$

$$Z^{\ell}_{B^{**}}(A^{**}) = Z(\pi_\ell) = \{a'' \in A^{**} : b'' \to \pi_\ell^{***}(a'', b'') : B^{**} \to B^{**} \text{ is weak}^* - \text{weak}^* \text{ continuous}\},$$

$$Z^{r}_{A^{**}}(B^{**}) = Z(\pi_\ell^t) = \{b'' \in B^{**} : a'' \to \pi_\ell^{t***}(b'', a'') : A^{**} \to B^{**} \text{ is weak}^* - \text{weak}^* \text{ continuous}\},$$

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$$Z^r_{B^{**}}(A^{**}) = Z(\pi_r^t) = \{a'' \in A^{**} : b'' \to \pi_r^{t***}(a'', b'') : B^{**} \to B^{**} \text{ is weak}^* - \text{weak}^* \text{ continuous}\},$$

where $\pi_\ell : A \times B \to B$ and $\pi_r : B \times A \to B$ are the left and right module actions of A on B. We define the weak topological centers of left and right module actions as follows:

$$\tilde{Z}^\ell_{A^{**}}(B^{**}) = \{b'' \in B^{**} : a'' \to b''a'' \text{ is weak}^* - \text{weak continuous}\},$$
$$\tilde{Z}^r_{A^{**}}(B^{**}) = \{b'' \in B^{**} : a'' \to a''b'' \text{ is weak}^* - \text{weak continuous}\},$$
$$\tilde{Z}^\ell_{B^{**}}(A^{**}) = \{a'' \in A^{**} : a'' \to b''a'' \text{ is weak}^* - \text{weak continuous}\},$$
$$\tilde{Z}^r_{B^{**}}(A^{**}) = \{a'' \in A^{**} : a'' \to a''b'' \text{ is weak}^* - \text{weak continuous}\}.$$

It is clear that $B \subseteq \tilde{Z}^\ell_{A^{**}}(B^{**}) \subseteq Z^\ell_{A^{**}}(B^{**})$ and in the next section, we show that the topological center and weak topological center of A^{**} on B^{**}, in general, is not equal.

Let A be a Banach algebra and A^*, A^{**}, respectively, be the first and second dual of A. For $a \in A$ and $a' \in A^*$, we denote by $a'a$ and aa' respectively, functionals in A^* defined by $\langle a'a, b\rangle = \langle a', ab\rangle = a'(ab)$ and $\langle aa', b\rangle = \langle a', ba\rangle = a'(ba)$ for all $b \in A$. The Banach algebra A is embedded in its second dual via the identification $\langle a, a'\rangle$ - $\langle a', a\rangle$ for every $a \in A$ and $a' \in A^*$.

2. Main results

Let $A^{(n)}$ and $B^{(n)}$ be nth dual of B and A, respectively where $n \geq 0$ is even number. By [10], $B^{(n)}$ is a Banach $A^{(n)} - bimodule$ and we may therefore define the topological centers and weak topological centers of the right and left module action of $A^{(n)}$ on $B^{(n)}$ to similar. Then we define $B^{(n)}B^{(n-1)}$ as a subspace of $A^{(n-1)}$, that is, for every $b^{(n)} \in B^{(n)}$, $b^{(n-1)} \in B^{(n-1)}$ and $a^{(n-2)} \in A^{(n-2)}$, we have

$$\langle b^{(n)}b^{(n-1)}, a^{(n-2)}\rangle = \langle b^{(n)}, b^{(n-1)}a^{(n-2)}\rangle.$$

If n is odd number, we define $B^{(n)}B^{(n-1)}$ as a subspace of $A^{(n)}$, that is, for all $b^{(n)} \in B^{(n)}$, $b^{(n-1)} \in B^{(n-1)}$ and $a^{(n-1)} \in A^{(n-1)}$, we define

$$\langle b^{(n)}b^{(n-1)}, a^{(n-1)}\rangle = \langle b^{(n)}, b^{(n-1)}a^{(n-1)}\rangle,$$

and if $n = 0$, we take $A^{(0)} = A$ and $B^{(0)} = B$. Let B be a left Banach $A - module$ and e be a left unit element of A. Then we say that e is a left unit for B, if $eb = b$ where $b \in B$. The definition of the right unit is similar. We say that a Banach $A - bimodule$ B is unital, if B has the same left and right unit.

Let B be a left Banach $A - module$ and $(e_\alpha)_\alpha \subseteq A$ be a left approximate identity for A. Then, we say that $(e_\alpha)_\alpha$ is a left approximate identity (LAI) for B, if for all $b \in B$, we have $e_\alpha b \to b$. The definition of the right approximate identity (RAI) is similar. Thus, $(e_\alpha)_\alpha \subseteq A$ is an approximate identity (AI) for B, if B has the same left and right approximate identity. The notation $BLAI$ refer to Bounded left approximate identity. The definitions of $BRAI$ and BAI are similar .

Theorem 2.1. *Let B be a Banach $A-bimodule$. Then we have the following assertions.*

(i) *If $B^{(n)}B^{(n-1)} \subseteq A^{(n-2)}$, then $Z^{\ell}_{A^{(n-1)}}(B^{(n-1)}) = \tilde{Z^{\ell}}_{A^{(n-1)}}(B^{(n-1)})$ where $n \geq 2$ is an odd number.*

(ii) *Assume $n \geq 2$ is an odd number. If $B^{(n)}A^{(n-1)} \subseteq B^{(n-2)}$ and $B^{(n-1)}$ has a left unite $A^{(n-1)}-module$, then B is reflexive.*

Proof. (i) It is clear that $\tilde{Z^{\ell}}_{A^{(n-1)}}(B^{(n-1)}) \subseteq Z^{\ell}_{A^{(n-1)}}(B^{(n-1)})$. We show that for all $b^{(n-1)} \in Z^{\ell}_{A^{(n-1)}}(B^{(n-1)})$, the mapping $a^{(n-1)} \to b^{(n-1)}a^{(n-1)}$ is weak$^*-$weak continuous. Suppose that $(a_{\alpha}^{(n-1)})_{\alpha} \subseteq A^{(n-1)}$ and $a_{\alpha}^{(n-1)} \overset{w^*}{\to} a^{(n-1)}$. Since $b^{(n-1)} \in Z^{\ell}_{A^{(n-1)}}(B^{(n-1)})$, $b^{(n-1)}a_{\alpha}^{(n-1)} \overset{w^*}{\to} b^{(n-1)}a^{(n-1)}$. Take $b^{(n)} \in B^{(n)}$. Since $B^{(n)}B^{(n-1)} \subseteq A^{(n-2)}$, we have

$$\begin{aligned}\langle b^{(n)}, b^{(n-1)}a_{\alpha}^{(n-1)}\rangle &= \langle b^{(n)}b^{(n-1)}, a_{\alpha}^{(n-1)}\rangle \\ &= \langle a_{\alpha}^{(n-1)}, b^{(n)}b^{(n-1)}\rangle \to \langle a^{(n-1)}, b^{(n)}b^{(n-1)}\rangle \\ &= \langle b^{(n)}b^{(n-1)}, a^{(n-1)}\rangle \\ &= \langle b^{(n)}, b^{(n-1)}a^{(n-1)}\rangle.\end{aligned}$$

(ii) Let $e^{(n-1)}$ be a left unit $A^{(n-1)}-module$ for $B^{(n-1)}$ and let $(b_{\alpha}^{(n-1)})_{\alpha} \subseteq B^{(n-1)}$ such that $b_{\alpha}^{(n-1)} \overset{w^*}{\to} b^{(n-1)}$. Suppose that $b^{(n)} \in B^{(n)}$. Since $b^{(n)}e^{(n-1)} \in B^{(n-2)}$, we have

$$\begin{aligned}\langle b^{(n)}, b_{\alpha}^{(n-1)}\rangle &= \langle b^{(n)}, e^{(n-1)}b_{\alpha}^{(n-1)}\rangle \\ &= \langle b^{(n)}e^{(n-1)}, b_{\alpha}^{(n-1)}\rangle \\ &= \langle b_{\alpha}^{(n-1)}, b^{(n)}e^{(n-1)}\rangle \to \langle b^{(n-1)}, b^{(n)}e^{(n-1)}\rangle \\ &= \langle b^{(n-1)}, b^{(n)}\rangle.\end{aligned}$$

Consequently, the weak topology coincide with weak* topology on $B^{(n-1)}$, and so $B^{(n-1)}$ is reflexive which implies that B is reflexive.

□

Corollary 2.2. *Suppose that A is a Banach algebra. Then, for every odd number $n \geq 3$, we have the following assertions.*

(i) *If* $A^{(n)}A^{(n-1)} \subseteq A^{(n-2)}$, *then* $Z_1^{\ell}(A^{(n-1)})) = \tilde{Z}_1^{\ell}(A^{(n-1)})$.

(ii) *If* $A^{(n)}A^{(n-1)} \subseteq A^{(n-2)}$ *and* $A^{(n-3)}$ *has a BLAI, then* A *is reflexive.*

Theorem 2.3. *Let* B *be a Banach* $A-$*bimodule. Then for every even number* $n \geq 2$, *we have the following assertions.*

(i) $Z^{\ell}_{A^{(n)}}(B^{(n+1)}) = B^{(n+1)}$ *if and only if* $\tilde{Z}^r_{A^{(n)}}(B^{(n)}) = B^{(n)}$.

(ii) $Z^{\ell}_{A^{(n)}}(A^{(n+1)}) = A^{(n+1)}$ *if and only if* $\tilde{Z}^r_1(A^{(n)}) = A^{(n)}$.

(iii) $Z^{r}_{A^{(n)}}(A^{(n+1)}) = A^{(n+1)}$ *if and only if* $\tilde{Z}^{\ell}_1(A^{(n)}) = A^{(n)}$.

(iv) $Z^{r}_{B^{(n)}}(A^{(n+1)}) = A^{(n+1)}$ *if and only if* $\tilde{Z}^{\ell}_{A^{(n)}}(B^{(n)}) = B^{(n)}$.

Proof. (i) Suppose that $Z^{\ell}{}_{A^{(n)}}(B^{(n+1)}) = B^{(n+1)}$ and $b^{(n)} \in B^{(n)}$. We show that the mapping $a^{(n)} \to a^{(n)}b^{(n)}$ is weak* $-$ weak continuous. Assume that $(a_\alpha^{(n)})_\alpha \subseteq A^{(n)}$ such that $a_\alpha^{(n)} \xrightarrow{w^*} a^{(n)}$. Then for all $b^{(n+1)} \in B^{(n+1)}$, we have $b^{(n+1)}a_\alpha^{(n)} \xrightarrow{w^*} b^{(n+1)}a^{(n)}$. It follows that

$$\langle b^{(n+1)}, a_\alpha^{(n)}b^{(n)}\rangle = \langle b^{(n+1)}a_\alpha^{(n)}, b^{(n)}\rangle,$$
$$\to \langle b^{(n+1)}a^{(n)}, b^{(n)}\rangle = \langle b^{(n+1)}, a^{(n)}b^{(n)}\rangle.$$

Thus we conclude that $b^{(n)} \in \tilde{Z}^r_{A^{(n)}}(B^{(n)})$.
The converse is similar.

Proofs of (ii), (iii), (iv) are similar to (i).

□

In the following examples, we show that the weak topological centers of a Banach algebra A, in general is not A or A^{**}.

Example 2.4. Let A be a non-reflexive Banach space and let $\langle f, x\rangle = 1$ and $\|f\| \leq 1$ for some $f \in A^*$ and $x \in A$. We define the product on A by $ab = \langle f, b\rangle a$. It is clear that A is a Banach algebra with this product and it has right identity x, see [7]. By easy calculation, for all $a' \in A^*$, $a'' \in A^{**}$ and $a''' \in A^{***}$, we have

$$a'a = \langle a', a\rangle f,$$
$$a''a' = \langle a'', f\rangle a',$$
$$a'''a'' = \langle a'', a''\rangle \langle ., f\rangle.$$

Therefore we have $Z^{\ell}{}_{A^{**}}(A^{***}) \neq A^{***}$. So by Theorem 2.3, we have

$$\tilde{Z}^r_{A^{**}}(A^{**}) \neq A^{**}.$$

Similarly, if we define the product on A as $ab = \langle f, a\rangle b$ for all $a, b \in A$, then we have ${Z^\ell}_{A^{**}}(A^{***}) = A^{***}$. By using Theorem 2.3, it follows that

$$\tilde{Z}^r_{A^{**}}(A^{**}) = A^{**}.$$

Theorem 2.5. *Let $n > 0$ be an even number and let B be a left (resp. right) Banach $A-$module such that*

$$A^{(n-2)}B^{(n)} \subseteq B^{(n-2)}, \qquad (resp.\ B^{(n)}A^{(n-2)} \subseteq B^{(n-2)}).$$

(i) *Then $A^{(n-2)} \subseteq \tilde{Z}^\ell_{B^{(n)}}(A^{(n)})$, (resp. $A^{(n-2)} \subseteq \tilde{Z}^r_{B^{(n)}}(A^{(n)})$).*
(ii) *If $B^{(n)}$ has a left (resp. right) unit element in $\tilde{Z}^\ell_{B^{(n)}}(A^{(n)})$ (resp. $\tilde{Z}^r_{B^{(n)}}(A^{(n)})$), then A is reflexive.*
(iii) *If $A^{(n-2)} \subset B^{(n-2)}$ and $A^{(n-2)}$ is a left (resp. right) Arens irregular, then*

$$A^{(n-2)} = \tilde{Z}^\ell_{B^{(n)}}(A^{(n)}), \qquad (resp.\ A^{(n-2)} = \tilde{Z}^r_{B^{(n)}}(A^{(n)})).$$

Proof. (i) Assume that $(b^{(n)}_\alpha)_\alpha \subseteq B^{(n)}$ such that $b^{(n)}_\alpha \overset{w^*}{\to} b^{(n)}$ in $B^{(n)}$. Let $b^{(n+1)} \in B^{(n+1)}$. Since $B^{(n+1)} = B^{(n-1)} \oplus B^\perp$, there are $b^{(n-1)} \in B^{(n-1)}$ and $t \in B^\perp$ such that $b^{(n+1)} = (b^{(n-1)}, t)$. Then

$$\begin{aligned}\langle b^{(n+1)}, a^{(n-2)}b^{(n)}_\alpha\rangle &= \langle (b^{(n-1)}, t), a^{(n-2)}b^{(n)}_\alpha\rangle \\ &= \langle a^{(n-2)}b^{(n)}_\alpha, b^{(n-1)}\rangle \to \langle a^{(n-2)}b^{(n)}, b^{(n-1)}\rangle \\ &= \langle b^{(n+1)}, a^{(n-2)}b^{(n)}\rangle,\end{aligned}$$

where $a^{(n-2)} \in A^{(n-2)}$. It follows that $A^{(n-2)} \subset \tilde{Z}^\ell_{B^{(n)}}(A^{(n)})$.
(ii) Proof is clear.
(iii) Since $A^{(n-2)} \subset B^{(n-2)}$, $\tilde{Z}^\ell_{B^{(n)}}(A^{(n)}) \subset Z_1(A^{(n)}) = A^{(n-2)}$. Thus by part (i), since $\tilde{Z}^\ell_{B^{(n)}}(A^{(n)})) \subseteq Z^\ell_{B^{(n)}}(A^{(n)}))$, we are done. □

Example 2.6. Let G be a compact group. We know that $L^1(G) \subseteq M(G)$ and $L^1(G)$ is an ideal in $M(G)^{**}$. Since $L^1(G)$ is strongly Arens irregular, see [3], by preceding theorem, we conclude

$$\begin{aligned}\tilde{Z}^\ell_{M(G)^{**}}(L^1(G)^{**}) &\subseteq Z^\ell_{M(G)^{**}}(L^1(G)^{**}) \\ &\subseteq Z^\ell_1(L^1(G)^{**}) \\ &= L^1(G).\end{aligned}$$

By Theorem 2.5, we have $L^1(G) \subseteq \tilde{Z}^\ell_{M(G)^{**}}(L^1(G)^{**})$. Thus we conclude that

$$L^1(G) = \tilde{Z}^\ell_{M(G)^{**}}(L^1(G)^{**}).$$

Similarly

$$L^1(G) = \tilde{Z}^r_{M(G)^{**}}(L^1(G)^{**}).$$

Theorem 2.7. *Let B be a Banach $A-bimodule$. Then for an even number $n \geq 0$, we have the following assertions.*

(i) *If $B^{(n+1)}B^{(n)} = A^{(n+1)}$ and $\tilde{Z}^{\ell}_{A^{(n)}}(B^{(n)}) = B^{(n)}$, then A is reflexive.*

(ii) *Let $e^{(n)} \in A^{(n)}$ be a left unit for $B^{(n)}$ and $e^{(n)} \in \tilde{Z}^{\ell}_{B^{(n)}}(A^{(n)})$. Then B is reflexive.*

Proof. (i) Let $(a_\alpha^{(n)})_\alpha \subseteq A^{(n)}$ and $a_\alpha^{(n)} \xrightarrow{w^*} a^{(n)}$. Let $a^{(n+1)} \in A^{(n+1)}$. Since $B^{(n+1)}B^{(n)} \subseteq A^{(n+1)}$, there are $b^{(n+1)} \in B^{(n+1)}$ and $b^{(n)} \in B^{(n)}$ such that $a^{(n+1)} = b^{(n+1)}b^{(n)}$. Thus, we have

$$\begin{aligned}\langle a^{(n+1)}, a_\alpha^{(n)}\rangle &= \langle b^{(n+1)}b^{(n)}, a_\alpha^{(n)}\rangle \\ &= \langle b^{(n+1)}, b^{(n)}a_\alpha^{(n)}\rangle \to \langle b^{(n+1)}, b^{(n)}a^{(n)}\rangle \\ &= \langle a^{(n+1)}, a^{(n)}\rangle.\end{aligned}$$

We conclude that $A^{(n)}$ is reflexive, and so A is reflexive.

(ii) Let $(b_\alpha^{(n)})_\alpha \subseteq B^{(n)}$ and $b_\alpha^{(n)} \xrightarrow{w^*} b^{(n)}$. Let $b^{(n+1)} \in B^{(n+1)}$. Then

$$\begin{aligned}\langle b^{(n+1)}, b_\alpha^{(n)}\rangle &= \langle b^{(n+1)}, e^{(n)}b_\alpha^{(n)}\rangle \to \langle b^{(n+1)}, e^{(n)}b^{(n)}\rangle \\ &= \langle b^{(n+1)}, b^{(n)}\rangle.\end{aligned}$$

Thus B is reflexive.

□

Theorem 2.8. *Assume that A is a Banach algebra. Then for $n \geq 0$, we have the following assertions.*

(i) *If $A^{(n)}$ has a bounded right approximate identity, then*

$$\tilde{Z}^{\ell}{}_{A^{(n+2)}}(A^{(n+3)}) \subseteq A^{(n+1)}.$$

Moreover, for every odd number n, if $A^{(n+3)}A^{(n+1)} \subseteq A^{(n+1)}$, then $\tilde{Z}^{\ell}{}_{A^{(n+2)}}(A^{(n+3)}) = A^{(n+1)}$.

(ii) *If $A^{(n+2)}$ has a left unit, then $\tilde{Z}^{r}{}_{A^{(n+3)}}(A^{(n+2)}) \subseteq A^{(n)}$. Moreover, for every even number n, if $A^{(n+4)}A^{(n)} \subseteq A^{(n+1)}$, then*

$$\tilde{Z}^{r}{}_{A^{(n+3)}}(A^{(n+2)}) = A^{(n)}.$$

Proof. (i) Assume that $A^{(n)}$ has a $BRAI$ as $(e_\alpha^{(n)})_\alpha \subseteq A^{(n)}$ such that $e_\alpha^{(n)} \xrightarrow{w^*} e^{(n+2)}$ in $A^{(n+2)}$ where $e^{(n+2)}$ is a right unite in $A^{(n+2)}$. Let $a^{(n+3)} \in \tilde{Z}^{\ell}{}_{A^{(n+2)}}(A^{(n+3)})$ and $(a_\alpha^{(n+2)})_\alpha \subseteq A^{(n+2)}$

such that $a_\alpha^{(n+2)} \xrightarrow{w^*} a^{(n+2)}$ in $A^{(n+2)}$. Then we have

$$\begin{aligned}\langle a^{(n+3)}, a_\alpha^{(n+2)}\rangle &= \langle a^{(n+3)}, a_\alpha^{(n+2)}e^{(n+2)}\rangle \\ &= \langle a^{(n+3)}a_\alpha^{(n+2)}, e^{(n+2)}\rangle \\ &= \langle e^{(n+2)}, a^{(n+3)}a_\alpha^{(n+2)}\rangle \to \langle e^{(n+2)}, a^{(n+3)}a^{(n+2)}\rangle \\ &= \langle a^{(n+3)}, a^{(n+2)}\rangle.\end{aligned}$$

It follows that $a^{(n+3)} : A^{(n+3)} \to \mathbb{C}$ is weak* continuous, and so $a^{(n+3)} \in A^{(n+1)}$. Consequently, we have

$$\tilde{Z}^\ell{}_{A^{(n+2)}}(A^{(n+3)}) \subseteq A^{(n+1)}.$$

Now let $A^{(n+3)}A^{(n+1)} \subseteq A^{(n+1)}$. We show that the mapping $a^{(n+2)} \to a^{(n+1)}a^{(n+2)}$ is weak*−weak continuous for all $a^{(n+1)} \in A^{(n+1)}$. Assume that $(a_\alpha^{(n+2)})_\alpha \subseteq A^{(n+2)}$ such that $a_\alpha^{(n+2)} \xrightarrow{w^*} a^{(n+2)}$ in $A^{(n+2)}$. Let $a^{(n+3)} \in A^{(n+3)}$. Then we have

$$\begin{aligned}\langle a^{(n+3)}, a^{(n+1)}a_\alpha^{(n+2)}\rangle &= \langle a^{(n+3)}a^{(n+1)}, a_\alpha^{(n+2)}\rangle \\ &= \langle a_\alpha^{(n+2)}, a^{(n+3)}a^{(n+1)}\rangle \to \langle a^{(n+2)}, a^{(n+3)}a^{(n+1)}\rangle \\ &= \langle a^{(n+3)}, a^{(n+1)}a^{(n+2)}\rangle.\end{aligned}$$

It follows that $a^{(n+1)} \in \tilde{Z}^\ell{}_{A^{(n+2)}}(A^{(n+3)})$, and so

$$\tilde{Z}^\ell{}_{A^{(n+2)}}(A^{(n+3)}) = A^{(n+1)}.$$

(ii) The proof is similar to (i).

□

Corollary 2.9. *Assume that A is a Banach algebra.*

(i) *If A has a BRAI and* $\tilde{Z}^\ell{}_{A^{**}}(A^{***}) = A^{***}$, *then A is reflexive.*
(ii) *If A has a left unit and* $\tilde{Z}^r{}_{A^{***}}(A^{**}) = A^{**}$, *then A is reflexive.*

Example 2.10. (i) If G is finite group, then

$$\begin{aligned}\tilde{Z}^\ell{}_{L^1(G)^{**}}(L^1(G)^{***}) &= \tilde{Z}^r{}_{L^1(G)^{***}}(L^1(G)^{**}) \\ &= L^1(G).\end{aligned}$$

(ii) Let G be a locally compact infinite group. Then

$$\tilde{Z}^\ell{}_{L^1(G)^{**}}(L^1(G)^{***}) \subseteq L^1(G)^*, \qquad \tilde{Z}^r{}_{L^1(G)^{***}}(L^1(G)^{**}) \subseteq L^1(G).$$

(iii) Since $c_0^* = \ell^1$, we have $\tilde{Z}^\ell{}_{\ell^\infty}((\ell^\infty)^*) \subseteq \ell^\infty$ and $\tilde{Z}^r{}_{(\ell^\infty)^*}(\ell^\infty) \subseteq c_0$ where both multiplications in c_0^{**} and ℓ^∞ coincide.

Theorem 2.11. *Let A be a Banach algebra and* $n > 0$ *be an even number.*

(i) *Assume that* $A^{(n)}$ *has a right (resp. left) unit. Then*

$$Z^\ell_{A^{(n)}}(A^{(n+1)}) = A^{(n-1)}, \qquad (\textit{resp. } Z^r_{A^{(n)}}(A^{(n+1)}) = A^{(n-1)}).$$

(ii) *Assume that* $A^{(n-1)}$ *has a right unit as* $A^{(n-2)}$*-module. Then*

$$Z^r_{A^{(n-1)}}(A^{(n)}) = A^{(n-2)},$$

where multiplication is the first Arens product.

Proof. (i) Assume that $(a^{(n)}_\alpha)_\alpha \subseteq A^{(n)}$ such that $a^{(n)}_\alpha \xrightarrow{w^*} a^{(n)}$ in $A^{(n)}$. Let $a^{(n+1)} \in Z^\ell_{A^{(n)}}(A^{(n+1)})$ and $e^{(n)} \in A^{(n)}$ be a left unit for $A^{(n)}$. Then

$$\begin{aligned}\langle a^{(n+1)}, a^{(n)}_\alpha\rangle &= \langle a^{(n+1)}, a^{(n)}_\alpha e^{(n)}\rangle \\ &= \langle a^{(n+1)} a^{(n)}_\alpha, e^{(n)}\rangle \to \langle a^{(n+1)} a^{(n)}, e^{(n)}\rangle \\ &= \langle a^{(n+1)}, a^{(n)}\rangle.\end{aligned}$$

It follows that $a^{(n+1)} \in (A^{(n+1)}, \text{weak}^*)^* = A^{(n-1)}$.
The next part is similar.

(ii) The proof is clear.

□

Corollary 2.12. *Let A be a Banach algebra with a right bounded approximate identity (resp. left approximate identity). Then*

$$Z^\ell_{A^{**}}(A^{***}) = A^*, \qquad (\textit{resp. } Z^r_{A^{**}}(A^{***}) = A^*).$$

Example 2.13. Let G be a locally compact group. By [6], we know that

$$Z^\ell_{L^1(G)^{**}}(L^1(G)^{**}) = L^1(G).$$

Now by proceeding corollary, we have the following statements.

(i) $Z^\ell_{L^1(G)^{**}}(L^1(G)^{***}) = L^1(G)^*$ (resp. $Z^r_{L^1(G)^{**}}(L^1(G)^{***}) = L^1(G)^*$).

(ii) Let $n > 0$ be an even number. Then $Z^\ell_{M(G)^{(n)}}(M(G)^{(n+1)}) = M(G)^{(n-1)}$ (resp. $Z^r_{M(G)^{(n)}}(M(G)^{(n+1)}) = M(G)^{(n-1)}$).

Theorem 2.14. *Assume that B is a Banach $A-bimodule$. For $n \geq 1$, we have the following assertions:*

(i) *If* $B^{(n)}B^{(n-1)} = A^{(n-1)}$, *then* $Z^\ell_{B^{(n)}}(A^{(n)}) \subseteq Z_1(A^{(n)})$.

(ii) *If* $B^{(n+1)}B^{(n)} = A^{(n+1)}$, *then* $Z^r_{B^{(n+1)}}(A^{(n)}) \subseteq Z_1(A^{(n)})$.

Proof. (i) Let $a^{(n)} \in Z_{B^{(n)}}(A^{(n)})$. We show that the mapping $x^{(n)} \to a^{(n)}x^{(n)}$ from $A^{(n)}$ into $A^{(n)}$ is weak* – weak* continuous. Suppose that $(x^{(n)}_\alpha)_\alpha \subseteq A^{(n)}$ such that $x^{(n)}_\alpha \xrightarrow{w^*} x^{(n)}$ holds in $A^{(n)}$. First we show that $x_\alpha^{(n)}b^{(n)} \overset{w^*}{\to} x^{(n)}b^{(n)}$ holds in $B^{(n)}$. Let $b^{(n-1)} \in B^{(n-1)}$. Then

$$\begin{aligned}\langle x_\alpha^{(n)}b^{(n)}, b^{(n-1)}\rangle &= \langle x_\alpha^{(n)}, b^{(n)}b^{(n-1)}\rangle \to \langle x^{(n)}, b^{(n)}b^{(n-1)}\rangle \\ &= \langle x^{(n)}b^{(n)}, b^{(n-1)}\rangle.\end{aligned}$$

Now let $a^{(n-1)} \in A^{(n-1)}$. Then we have

$$\begin{aligned}\langle a^{(n)}x_\alpha^{(n)}, a^{(n-1)}\rangle &= \langle a^{(n)}x_\alpha^{(n)}, b^{(n)}b^{(n-1)}\rangle \\ &= \langle a^{(n)}x_\alpha^{(n)}b^{(n)}, b^{(n-1)}\rangle \to \langle a^{(n)}x^{(n)}b^{(n)}, b^{(n-1)}\rangle \\ &= \langle a^{(n)}x^{(n)}, b^{(n)}b^{(n-1)}\rangle \\ &= \langle a^{(n)}x^{(n)}, a^{(n-1)}\rangle.\end{aligned}$$

It follows that $a^{(n)} \in Z_1(A^{(n)})$.

(ii) The proof is similar to (i).

□

Corollary 2.15. *Assume that B is a Banach $A-$bimodule with left and right module actions π_ℓ and π_r, respectively.*

(i) *If π_ℓ^{**} is surjective, then $Z^\ell_{B^{**}}(A^{**}) \subseteq Z_1(A^{**})$.*
(ii) *If π_r^{t**} is surjective, then $Z^r_{B^{**}}(A^{**}) \subseteq Z_2(A^{**})$.*

Example 2.16. Let G be a locally compact group and $n \geq 3$. Then the following statements hold.

(i) $Z^\ell_{L^1(G)^{(n)}}(L^1(G)^{**}) = L^1(G)$, see [3].
(ii) $Z^\ell_{M(G)^{(n)}}(L^1(G)^{**}) = L^1(G)$.
(iii) $Z^\ell_{M(G)^{(n)}}(M(G)^{**}) = M(G)$.

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