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THE ANALYTICAL SOLUTIONS FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITHIN LOCAL FRACTIONAL OPERATORS BY YANG-LAPLACE TRANSFORM

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Abstract. In this paper, we apply the local fractional Laplace transform method (or Yang-Laplace transform) on Volterra integro-differential equations of the second kind within the local fractional integral operators to obtain the analytical approximate solutions. The iteration procedure is based on local fractional derivative operators. This approach provides us with a convenient way to find a solution with less computation as compared with local fractional variational iteration method. Some illustrative examples are discussed. The results show that the methodology is very efficient and a simple tool for solving integral equations.

1. Introduction

The standard order local fractional Volterra integro-differential equation of the second kind is given by [5]:

$$\psi^{(k\alpha)}(x) = f(x) + \frac{1}{\Gamma(1+\alpha)} \int_0^x K(x,t)\psi(t)(dt)^\alpha, \tag{1.1}$$

with the initial conditions

$$\psi^{(m\alpha)}(0) = a_m, \quad m = 0, 1, \dots, k-1,$$

where $K(x,t)$ is the kernel of the local fractional integral equation, and $f(x)$ is a local fractional continuous function.

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Several analytical and numerical techniques were successfully applied to deal with integral equations within local fractional derivative operators such as local fractional variational iteration method [5], Adomian decomposition method [11], Picards successive approximation method [8], and other methods .

The number of applicable mathematical and engineering problems successfully solved by the tools of the fractional calculus is continuously growing in last five decades [3, 4, 6]. Most of the local fractional differential equations have exact analytic solutions, whilst others need either analytical approximations or numerical techniques to be applied, among them: local fractional Fourier and Laplace transforms. The Yang-Laplace transform in fractal space is a generalization of Laplace transforms derived from the local fractional calculus, as well as the Yang-Fourier transform based on the local fractional calculus is a generalization of Fourier transform in fractal space. The Laplace and Fourier transforms are essential mathematical tools for the design, analysis and monitory of systems and show insight into the transient behavior the steady state behavior, and the stability of continuous time systems. However, the classical Laplace-transform does not deal with fractal functions, which are local fractional continuous non-differential functions [7].

Our aims are to present the local fractional Laplace transform method (Yang-Laplace transform) and to use it to solve the Volterra integro-differential equations of the second kind with local fractional derivative operators. The structure of the paper is as follows. In Section 2, the basic mathematical tools are introduced. In Section 3, the local fractional Laplace transform method is analyzed. In Section 4, several examples for Volterra integro-differential equations of the second kind are considered. Finally, in Section 5 the conclusions are given.

2. Mathematical Fundamentals

In this section, we present some basic definitions and notations of the local fractional calculus (see [1, 2, 9, 10]).

Definition 2.1. The local fractional derivative of $\psi(x)$ of order α at the point $x = x_0$ is given by

$$\psi^{(\alpha)}(x_0) = \frac{d^\alpha}{dx^\alpha}\psi(x)\Big|_{x=x_0} = \lim_{x\to x_0}\frac{\triangle^\alpha(\psi(x)-\psi(x_0))}{(x-x_0)^\alpha},$$

where

$$\triangle^\alpha(\psi(x)-\psi(x_0) \cong \Gamma(\alpha+1)(\psi(x)-\psi(x_0)).$$

The formulas of local fractional derivatives of special functions used in the paper are as follows:

$$D_x^{(\alpha)} a\psi(x) = aD_x^{(\alpha)}\psi(x),$$
$$\frac{d^\alpha}{dx^\alpha}\left(\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}\right) = \frac{x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}.$$

Definition 2.2. The local fractional integral of $\psi(x)$ in the interval $[a,b]$ is given by

$$\begin{aligned} {}_aI_b^{(\alpha)}\psi(x) &= \frac{1}{\Gamma(1+\alpha)}\int_a^b \psi(t)(dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)}\lim_{\triangle t\longrightarrow 0}\sum_{j=0}^{N-1}\psi(t_j)(\triangle t_j)^\alpha, \end{aligned}$$

where the partition of the interval $[a,b]$ is denoted as $(t_j, t_{j+1}), j = 0,\ldots,N-1$, $t_0 = a$ and $t_N = b$ with $\triangle t_j = t_{j+1} - t_j$ and $\triangle t = \max\{\triangle t_0, \triangle t_1, \ldots\}$.

The formulas of local fractional integrals of special functions used in the paper are as follows:

$${}_0I_x^{(\alpha)} a\psi(t) = a\,{}_0I_x^{(\alpha)}\psi(t),$$
$${}_0I_x^{(\alpha)}\left(\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}\right) = \frac{x^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)}.$$

Definition 2.3. Let

$$\frac{1}{\Gamma(1+\alpha)}\int_0^\infty |f(x)|(dx)^\alpha < k < \infty.$$

The Yang-Laplace transform of $f(x)$ is given by

$$\begin{aligned} L_\alpha\{\psi(x)\} &= \Psi_s^{L,\alpha}(s) \\ &= \frac{1}{\Gamma(1+\alpha)}\int_0^\infty E_\alpha(-s^\alpha x^\alpha)\psi(x)(dx)^\alpha, \end{aligned}$$

where the latter integral converges and $s^\alpha \in R^\alpha$.

Definition 2.4. The inverse formula of the Yang-Laplace transform of $f(x)$ is given by

$$\begin{aligned} L_\alpha^{-1}\left(\Psi_s^{L,\alpha}(s)\right) &= \psi(x) \\ &= \frac{1}{(2\pi)^\alpha}\int_{\beta-i\omega}^{\beta+i\omega} E_\alpha(s^\alpha x^\alpha)\Psi_s^{L,\alpha}(s)(ds)^\alpha, \end{aligned}$$

where $s^\alpha = \beta^\alpha + i^\alpha\omega^\alpha$; fractal imaginary unit i^α and $\mathrm{Re}(s) = \beta > 0$.

Definition 2.5. The convolution of two functions is defined symbolically by

$$\psi_1(x) * \psi_2(x) = \frac{1}{\Gamma(1+\alpha)} \int_0^x \psi_1(t)\psi_2(x-t)(dt)^\alpha,$$

or

$$\psi_2(x) * \psi_1(x) = \frac{1}{\Gamma(1+\alpha)} \int_0^x \psi_2(t)\psi_1(x-t)(dt)^\alpha.$$

Theorem 2.6 (The convolution theorem)**.** *Let*

$$L_\alpha\left\{\psi_1(x)\right\} = \Psi_{s,1}^{L,\alpha}(s), \qquad L_\alpha\left\{\psi_2(x)\right\} = \Psi_{s,2}^{L,\alpha}(s),$$

then

$$L_\alpha\left\{\psi_1(x) * \psi_2(x)\right\} = \Psi_{s,1}^{L,\alpha}(s)\Psi_{s,2}^{L,\alpha}(s).$$

Theorem 2.7. *Suppose that*

$$L_\alpha\left\{\psi_1(x)\right\} = \Psi_{s,1}^{L,\alpha}(s), \qquad L\alpha\left\{\psi_2(x)\right\} = \Psi_{s,2}^{L,\alpha}(s),$$

then

$$L_\alpha\{\psi^{(k\alpha)}(x)\} = s^{k\alpha}\Psi_s^{L,\alpha}(s) - \cdots - \psi^{((k-1)\alpha)}(0);$$

$$L_\alpha\{E_\alpha(a^\alpha x^\alpha)\} = \frac{1}{s^\alpha - a^\alpha}.$$

3. Local Fractional Laplace Transform Method

In view of the convolution theorem for the Yang-Laplace transform, if the kernel $K(x,t)$ in equation (1.1) be a difference kernel, then the local fractional Volterra integro-differential equation can be written as

$$\psi^{(k\alpha)}(x) = f(x) + \frac{1}{\Gamma(1+\alpha)} \int_0^x K(x-t)\psi(t)(dt)^\alpha. \tag{3.1}$$

By taking Yang-Laplace transform of both sides of (3.1):

$$\begin{aligned} s^{k\alpha}\Psi_s^{L,\alpha}(s) - s^{(k-1)\alpha}\psi(0) - \cdots - \psi^{((k-1)\alpha)}(0) \\ = F_s^{L,\alpha}(s) + K_s^{L,\alpha}(s)\Psi_s^{L,\alpha}(s), \end{aligned} \tag{3.2}$$

where

$$\Psi_s^{L,\alpha}(s) = L_\alpha\left\{\psi(x)\right\}, \quad F_s^{L,\alpha}(s) = L_\alpha\left\{f(x)\right\}, \quad K_s^{L,\alpha}(s) = L_\alpha\left\{K(x)\right\}.$$

Substituting the initial conditions into (3.2), we obtain

$$\begin{aligned} s^{k\alpha}\Psi_s^{L,\alpha}(s) - a_0 s^{(k-1)\alpha} - a_1 s^{(k-2)\alpha} - \cdots - a_{k-1} \\ = F_s^{L,\alpha}(s) + K_s^{L,\alpha}(s)\Psi_s^{L,\alpha}(s). \end{aligned} \tag{3.3}$$

Solving (3.3) for $\Psi_s^{L,\alpha}(s)$ gives

$$\Psi_s^{L,\alpha}(s) = \frac{F_s^{L,\alpha}(s) + a_0 s^{(k-1)\alpha} + a_1 s^{(k-2)\alpha} + \cdots + a_{k-1}}{s^{k\alpha} - K_s^{L,\alpha}(s)}. \tag{3.4}$$

The solution $\psi(x)$ is obtained by applying the inverse Yang-Laplace transform of both sides of (3.4). Therefore, we obtain

$$\psi(x) = L_{\alpha}^{-1}\left[\frac{F_s^{L,\alpha}(s) + a_0 s^{(k-1)\alpha} + a_1 s^{(k-2)\alpha} + \cdots + a_{k-1}}{s^{k\alpha} - K_s^{L,\alpha}(s)}\right].$$

4. Illustrative Examples

Example 4.1. Let us consider the following Volterra integro-differential equation of the second kind involving local fractional derivative:

$$\psi^{(\alpha)}(x) = 1 + \frac{1}{\Gamma(1+\alpha)}\int_0^x \psi(t)(dt)^{\alpha}, \qquad \psi(0) = 1. \tag{4.1}$$

Taking the Yang-Laplace transform of (4.1) gives

$$L_{\alpha}\left\{\psi^{(\alpha)}(x)\right\} = L_{\alpha}\left\{1\right\} + L_{\alpha}\left\{1 * \psi(x)\right\},$$

so that

$$s^{\alpha}\Psi_s^{L,\alpha}(s) - \psi(0) = \frac{1}{s^{\alpha}} + \frac{1}{s^{\alpha}}\Psi_s^{L,\alpha}(s).$$

Using the given initial condition and solving for $\Psi_s^{L,\alpha}(s)$ we find

$$\Psi_s^{L,\alpha}(s) = \frac{1}{s^{\alpha} - 1}. \tag{4.2}$$

By taking the Yang-Laplace inverse of the equation (4.2), the nondifferentiable solution is given by

$$\psi(x) = E_{\alpha}(x^{\alpha}),$$

which is equal to the result based on the local fractional variational iteration method [5].

Example 4.2. Consider the following Volterra integro-differential equation of the second kind involving local fractional derivative:

$$\psi^{(2\alpha)}(x) = 1 + \frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{(x-t)^{\alpha}}{\Gamma(1+\alpha)}\psi(t)(dt)^{\alpha}, \tag{4.3}$$
$$\psi(0) = 1, \qquad \psi^{(\alpha)}(0) = 0.$$

The Yang-Laplace transform of (4.3) yields

$$L_{\alpha}\left\{\psi^{(2\alpha)}(x)\right\} = L_{\alpha}\left\{1\right\} + L_{\alpha}\left\{\frac{x^{\alpha}}{\Gamma(1+\alpha)}\right\}\left\{\psi(x)\right\},$$

so that

$$s^{2\alpha}\Psi_s^{L,\alpha}(s) - s^{\alpha}\psi(0) - \psi^{(\alpha)}(0) = \frac{1}{s^{\alpha}} + \frac{1}{s^{2\alpha}}\Psi_s^{L,\alpha}(s).$$

Using the given initial condition and solving for $\Psi_s^{L,\alpha}(s)$ we find

$$\Psi_s^{L,\alpha}(s) = \frac{1}{2}\frac{1}{s^\alpha - 1} + \frac{1}{2}\frac{1}{s^\alpha + 1}. \tag{4.4}$$

By taking the Yang-Laplace inverse of the equation (4.4), the nondifferentiable solution is given by

$$\begin{aligned}\psi(x) &= \frac{1}{2}E_\alpha(x^\alpha) + \frac{1}{2}E_\alpha(-x^\alpha)\\ &= \cosh_\alpha(x^\alpha),\end{aligned}$$

which is equal to the result based on the local fractional variational iteration method [5].

Example 4.3. Consider the following Volterra integro-differential equation of the second kind with local fractional derivative:

(4.5)

$$\psi^{(4\alpha)}(x) = \sin_\alpha(x^\alpha) + \cos_\alpha(x^\alpha) + \frac{2}{\Gamma(1+\alpha)}\int_0^x \sin_\alpha(x^\alpha - t^\alpha)\psi(t)(dt)^\alpha,$$

with the initial conditions

$$\psi(0) = \psi^{(\alpha)}(0) = \psi^{(2\alpha)}(0) = \psi^{(3\alpha)}(0) = 1.$$

Taking the Yang-Laplace transform of (4.5) gives

$$L_\alpha\left\{\psi^{(4\alpha)}(x)\right\} = L_\alpha\left\{\sin_\alpha(x^\alpha)\right\} + L_\alpha\left\{\cos_\alpha(x^\alpha)\right\} + 2L_\alpha\left\{\sin_\alpha(x^\alpha)\right\}\left\{\psi(x)\right\},$$

so that

$$\begin{aligned}&s^{4\alpha}\Psi_s^{L,\alpha}(s) - s^{3\alpha}\psi(0) - s^{2\alpha}\psi^{(\alpha)}(0) - s^\alpha\psi^{(2\alpha)} - \psi^{(3\alpha)}(0)\\ &\qquad = \frac{1}{s^{2\alpha}+1} - \frac{s^\alpha}{s^{2\alpha}+1} + \frac{2}{s^{2\alpha}+1}\Psi_s^{L,\alpha}(s).\end{aligned}$$

Using the given initial condition and solving for $\Psi_s^{L,\alpha}(s)$ we find

$$\Psi_s^{L,\alpha}(s) = \frac{1}{s^\alpha - 1}. \tag{4.6}$$

By taking the Yang-Laplace inverse of the equation (4.6), the nondifferentiable solution is given by

$$\psi(x) = E_\alpha(x^\alpha).$$

5. Conclusions

In this work, we considered the local fractional Laplace transform method to solve the Volterra integro-differential equations of the second kind within the local fractional operators and their nondifferentiable approximate solutions were obtained. The proposed method is a powerful tool for solving many integral equations within the local fractional derivatives. This method can be applied to various industrial methods. Our goal in the future is to apply the Yang-Laplace transform method to system of coupled PDEs within local fractional derivative operators.

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