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Linear Maps Preserving Invertibility or Spectral Radius on Some C^*-algebras

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Abstract. Let A be a unital C^*-algebra which has a faithful state. If $\varphi : A \to A$ is a unital linear map which is bijective and invertibility preserving or surjective and spectral radius preserving, then φ is a Jordan isomorphism. Also, we discuss other types of linear preserver maps on A.

1. Introduction

Let A and B be unital Banach algebras. A linear map $\varphi : A \longrightarrow B$ is said to be invertibility preserving if a is invertible in A, then $\varphi(a)$ is invertible in B, and φ is called Jordan homomorphism if $\varphi(a^2) = (\varphi(a))^2$ for all $a \in A$. Also φ is said to be spectrum preserving if $\sigma(\varphi(a)) = \sigma(a)$ and spectral radius preserving (spectral isometry) if $r(\varphi(a)) = r(a)$, for all $a \in A$. The map φ said to be unital if $\varphi(1) = 1$.

A Hilbert C^*-module over a C^*-algebra A is a right A-module E equipped with an A-valued inner product $\langle ., . \rangle$ which is A-conjugate linear in the first variable and A-linear in the second variable, such that E is a Banach space with the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ $(x \in E)$. Every C^*-algebra A is a Hilbert A-module if we define the inner product $\langle a, b\rangle = a^*b$, for all $a, b \in A$. Let E, F be Hilbert modules over the C^*-algebra A. A map $T : E \to F$ is said to be adjointable if there exists a map $T^* : F \to E$ satisfying $\langle x, T(y)\rangle = \langle T^*(x), y\rangle$, for all $x \in F$ and $y \in E$. Such a map T^* is called the adjoint of T. We denote the set of all adjointable maps from E to F by $\mathbb{B}(E, F)$, which is the most important

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class of operators on Hilbert C^*-modules. It is well known that with the operator norm $\|T\| = \sup\{\|T(x)\| : x \in E, \|x\| \leq 1\}$, $\mathbb{B}(E) = \mathbb{B}(E, E)$ is a C^*-algebra. The topology on $\mathbb{B}(E, F)$ generated by the seminorms $T \to\| Tx \|$ $(x \in E)$ and $T \to\| T^*y \|$ $(y \in F)$ is called the strict topology on $\mathbb{B}(E, F)$. A net $(T_\lambda)_{\lambda\in\Lambda}$ converges strictly to an operator $T \in \mathbb{B}(E, F)$ if and only if $\| (T_\lambda - T)(x) \|\to 0$ and $\| (T^*_\lambda - T^*)(y) \|\to 0$, for all $x \in E$ and $y \in F$, [6].

Let A and B be C^*-algebras. An element $a \in A$ is said to be positive and denoted by $a \geq 0$, if a is self-adjoint and $\sigma(a) \subseteq R^+$. Also a linear map $\varphi : A \to B$ is called positive if $\varphi(a) \geq 0$ for all $a \geq 0$.

Let A be a C^*-algebra and let B be a strictly closed C^*-subalgebra of $M(A)$(the multiplier algebra of A). A retraction from A to B is a positive linear map $\varphi : A \to B$ satisfying the following conditions:

(i) $\varphi(ab) = \varphi(a)b \quad (a \in A, b \in B)$,
(ii) $\varphi(A)$ is strictly dense in B,
(iii) There exists an approximate unit (u_i) of A such that $(\varphi(u_i))$ converges strictly in $M(A)$ to a projection $p \in B$.

If q is a projection in $M(A)$, then the map $a \to pap$ is a retraction from A to $pM(A)p$. A retraction φ on A is said to be faithful if $a \in A^+$ and $\varphi(a) = 0$, then $a = 0$, see [6]. A state on a C^*-algebra A is a positive linear functional on A of norm one. Any positive linear unital functional on A is a state. Similarly a state ψ on A is said to be faithful if $a \in A^+$ and $\psi(a) = 0$, then $a = 0$. Any state on C^*-algebra A is a retraction from A to $\mathbb{C}$ (identified with the subalgebra $\mathbb{C}1$ of $M(A)$).

Let A be a $C^* - algebra$ and let p and q be projections in A. Then q is said to be a subprojection of p, if $p - q$ is a projection in A, also p and q is called orthogonal if $pq = 0$. We know that if q is a subprojection of p, then $pq = qp = q$. We say a projection p in a von Neumann algebra is countably decomposable if any collection of mutually orthogonal nonzero subprojections of p is countable. A von Neumann algebra A is countably decomposable if 1_A is countably decomposable. Since by [11, Proposition 2.1.10] every von Neumann algebra on a separable Hilbert space has a separable predual and by [11, Proposition 2.1.9] every von Neumann algebra with separable predual is a countably decomposable, one may conclude that every von Neumann algebra on a separable Hilbert space is countably decomposable.

The study of linear preserving maps has attracted the attention of many mathematicians in recent decades. For example, Sourour and Bresar studied invertibility preserving linear maps, and Jafarian, Semrl and Aupetit studied spectrum preserving linear maps [1, 4, 5, 12]. The spectral radius preserving linear maps has been studied by Bresar and Mathieu [3, 8], also Marcus and Moyls studied eigenvalue preserving linear maps on matrix algebra [7]. In this paper, we consider linear preserver maps on the algebra of adjointable operators on a Hilbert C^*-module, and we will show that every unital bijective and inevitability preserving or surjective and spectral radius preserving linear map on a unital C^*-algebra which has a faithful state, is a Jordan isomorphism.

2. Main Results

Let A be a C^*-algebra, B be a strictly closed C^*-subalgebra of $M(A)$ and E be a Hilbert A-module. Suppose that φ is a retraction from A to B. We define a B-valued semi-inner product $\langle x, y\rangle_\varphi$ on E by $\langle x, y\rangle_\varphi = \varphi(\langle x, y\rangle)$, for all $x, y \in E$. Let $N_\varphi = \{x \in E : \langle x, x\rangle_\varphi = 0\}$ and $(x + N, y + N)_\varphi = \langle x, y\rangle_\varphi$, for all $x, y \in E$. Then $(.,.)_\varphi$ defines an inner product on $\frac{E}{N_\varphi}$ and $E_\varphi = \overline{(\frac{E}{N_\varphi})}$ is a B-Hilbert module (the closure with respect to the norm defined by $\|x + N\| = \|(x + N, x + N)_\varphi\|^{\frac{1}{2}}$). If φ is a state on A, then E_φ is a Hilbert space. It is well known that the map $\pi_\varphi : \mathbb{B}(E) \to \mathbb{B}(E_\varphi)$ defined by $\pi_\varphi(T)(x + N_\varphi) = T(x) + N_\varphi$ is a unital $*$- homomorphism and is an isomorphism, when φ is faithful, see [6, 13].

We need the following lemma to prove our main results.

Lemma 2.1. *Let A be a unital C^*-algebra which has a faithful state and E be a Hilbert A-module. Then there exists a Hilbert space H such that $\mathbb{B}(E)$ is isomorphic to $B(H)$.*

Proof. Let ρ be a faithful state on A. Then ρ is a faithful retraction from A to $\mathbb{C}$, so $H = E_\rho$ is a Hilbert space and $\pi_\rho : \mathbb{B}(E) \to \mathbb{B}(E_\rho)$ is a $*$-isomorphism. □

In the following, theorem we characterize the linear map φ which is bijective and invertibility preserving.

Theorem 2.2. *Let A and B be unital C^*-algebras which have a faithful state. Let E and F be Hilbert A-module and B-module respectively. If $\varphi : \mathbb{B}(E) \to \mathbb{B}(F)$ is a bijective and invertibility preserving unital linear map, then φ is a Jordan isomorphism.*

Proof. Let ρ and ζ be faithful states on A and B respectively. Since ρ and ζ are retractions, there exist Hilbert spaces E_ρ and F_ς such that maps $\pi_\rho : \mathbb{B}(E) \to \mathbb{B}(E_\rho)$ and $\pi_\zeta : \mathbb{B}(F) \to \mathbb{B}(F_\zeta)$ are unital $*$-isomorphisms, by Lemma 2.1. If $\psi = \pi_\zeta \varphi \pi_\rho^{-1}$, then ψ is a linear, bijective and invertibility preserving map, so by [12, Theorem 1.1] ψ is a Jordan isomorphism. Therefore, φ is a Jordan isomorphism. □

In the following theorems, we review a map φ with different conditions.

Theorem 2.3. *Let A be a unital C^*-algebra which has a faithful state and let E be a Hilbert A-module. If $\varphi : \mathbb{B}(E) \to \mathbb{B}(E)$ is a unital surjective and spectral radius preserving linear map, then either*

(1) *there exists an invertible operator $P \in \mathbb{B}(E)$ such that*

$$\varphi(T) = PTP^{-1}, \quad T \in \mathbb{B}(E).$$

or

(2) *there exists an invertible operator $Q \in \mathbb{B}(E)$ such that*

$$\varphi(T) = QT^*Q^{-1}, \quad T \in \mathbb{B}(E).$$

Proof. Let ρ be a faithful state on A. Then $\pi_\rho : \mathbb{B}(E) \to \mathbb{B}(E_\rho)$ is a unital $*$-isomorphism, by Lemma 2.1. Let $\psi = \pi_\zeta \varphi \pi_\rho^{-1}$. Since ψ is a surjective and spectral radius preserving linear map, by [3, Theorem 1] there exists $c \in \mathbb{C}$ such that $|c| = 1$ and either

(1) there exists a bijective $V \in B(E_\rho)$ such that $\psi(S) = cVSV^{-1}$, for all $S \in B(E_\rho)$, or
(2) there exists a bijective $W \in B(E_\rho)$ such that $\psi(S) = cWS^*W^{-1}$, for all $S \in B(E_\rho)$.

Now let $P = \pi_\rho^{-1}(V)$ and $Q = \pi_\rho^{-1}(W)$. Then P is an invertible element of $\mathbb{B}(E)$ and we have

$$\begin{aligned} P\pi_\rho^{-1}(V^{-1}) &= \pi_\rho^{-1}(V)\pi_\rho^{-1}(V^{-1}) \\ &= \pi_\rho^{-1}(VV^{-1}) \\ &= I, \end{aligned}$$

and

$$\begin{aligned} \pi_\rho^{-1}(V^{-1})P &= \pi_\rho^{-1}(V^{-1})\pi_\rho^{-1}(V) \\ &= \pi_\rho^{-1}(V^{-1}V) \\ &= I, \end{aligned}$$

so $P^{-1} = \pi_\rho^{-1}(V^{-1})$. Similarly we can show that Q is an invertible element of $\mathbb{B}(E)$ and $Q^{-1} = \pi_\rho^{-1}(W^{-1})$. Also we have $\varphi(T) = cPTP^{-1}$ or $\varphi(T) = cQT^*Q^{-1}$. Since φ is a unital map, $c = 1$ and therefore

$$\varphi(T) = PTP^{-1} \quad \text{or} \quad \varphi(T) = QT^*Q^{-1}.$$

□

Theorem 2.4. *Let A be a unital C^*-algebra which has a faithful state, E be an infinite dimensional Hilbert A-module and $\varphi : \mathbb{B}(E) \to \mathbb{B}(E)$ is a surjective linear map.*

(i) *If $\sigma(\varphi(T)\varphi(S)) = \sigma(TS)$, for all $T, S \in \mathbb{B}(E)$, then there exists an invertible $C \in \mathbb{B}(E)$ such that for all $T \in \mathbb{B}(E)$ either*

$$\varphi(T) = CTC^{-1} \quad \textit{or} \quad \varphi(T) = -CT^*C^{-1}.$$

(ii) *If* $\sigma((\varphi(T))^*\varphi(S)) = \sigma(\varphi(T^*S))$, *for all* $T, S \in \mathbb{B}(E)$, *then there exist unitary operators* $U, V \in \mathbb{B}(E)$ *such that* $\varphi(T) = UTV$ *for all* $T \in \mathbb{B}(E)$.

Proof. Let ρ be a faithful state on A. Then $\pi_\rho : \mathbb{B}(E) \to \mathbb{B}(E_\rho)$ is a unital $*$-isomorphism by Lemma 2.1. Let $\psi = \pi_\rho \varphi \pi_\rho^{-1}$. Then ψ is a surjective linear map on $\mathbb{B}(E_\rho) = B(E_\rho)$. If $A, B \in B(E_\rho)$, then

$$\begin{aligned}\sigma(\psi(A)\psi(B)) &= \sigma(\pi_\rho\varphi\pi_\rho^{-1}(A)\pi_\rho\varphi\pi_\rho^{-1}(B))\\ &= \sigma(\pi_\rho(\varphi\pi_\rho^{-1}(A)\varphi\pi_\rho^{-1}(B)))\\ &= \sigma(\varphi\pi_\rho^{-1}(A)\varphi\pi_\rho^{-1}(B))\\ &= \sigma(\pi_\rho^{-1}(A)\pi_\rho^{-1}(B))\\ &= \sigma(\pi_\rho^{-1}(AB))\\ &= \sigma(AB).\end{aligned}$$

On the other hand ρ is a faithful state, so $N_\rho = \{0\}$, thus $E_\rho = \overline{(\frac{E}{N_\rho})}$ is isomorph with E, so E_ρ is an infinite dimensional Hilbert space. Therefore, by [9, Theorem 3] there exists an invertible $P \in B(E_\rho)$ such that for all $T \in B(E_\rho)$ either $\psi(T) = PTP^{-1}$ or $\psi(T) = -PT^*P^{-1}$. Let $C = \pi_\rho^{-1}(P)$. Then C is invertible and $C^{-1} = \pi_\rho^{-1}(P^{-1})$. Also we have

$$\begin{aligned}\varphi(T) = (\pi_\rho^{-1}\psi\pi_\rho)(T) &= \pi_\rho^{-1}(P\pi_\rho(T)P^{-1})\\ &= \pi_\rho^{-1}(P)T\pi_\rho^{-1}(P^{-1})\\ &= CTC^{-1}.\end{aligned}$$

The proof of $\varphi(T) = -CTC^{-1}$ is similar.
To prove (ii), if $A, B \in B(E_\rho)$, then

$$\begin{aligned}\sigma((\psi(A))^*\psi(B)) &= \sigma(((\pi_\rho\varphi\pi_\rho^{-1})(A))^*(\pi_\rho\varphi\pi_\rho^{-1})(B))\\ &= \sigma(\pi_\rho((\varphi\pi_\rho^{-1}(A))^*\varphi\pi_\rho^{-1}(B)))\\ &= \sigma((\varphi\pi_\rho^{-1}(A))^*\varphi\pi_\rho^{-1}(B)))\\ &= \sigma(\varphi((\pi_\rho^{-1}(A))^*\pi_\rho^{-1}(B)))\\ &= \sigma((\pi_\rho^{-1}(A^*B))\\ &= \sigma(A^*B).\end{aligned}$$

Therefore, by [9, Theorem 4] there exist unitary operators $P, Q \in B(E_\rho)$ such that $\psi(T) = PTQ$, for all $T \in B(E_\rho)$. Let $U = \pi_\rho^{-1}(P)$ and

$V = \pi_\rho^{-1}(Q)$, then

$$\begin{aligned} UU^* &= \pi_\rho^{-1}(P)(\pi_\rho^{-1}(P))^* \\ &= \pi_\rho^{-1}(P)\pi_\rho^{-1}((P)^*) \\ &= \pi_\rho^{-1}(PP^*) \\ &= \pi_\rho^{-1}(I) \\ &= I. \end{aligned}$$

The proof of $V^*V = VV^* = I$ is similar. Also

$$\begin{aligned} \varphi(T) &= (\pi_\rho^{-1}\psi\pi_\rho)(T) \\ &= \pi_\rho^{-1}(P\pi_\rho(T)Q) \\ &= \pi_\rho^{-1}(P)T\pi_\rho^{-1}(Q) \\ &= UTV. \end{aligned}$$

□

Lemma 2.5. *Let A be a unital C^*-algebra which has a faithful state and $\varphi : A \to A$ be a unital linear map. If φ is bijective and invertibility preserving or surjective and spectral radius preserving, then φ is a Jordan isomorphism.*

Proof. Since A is a unital C^*-algebra, a linear map $\psi : \mathbb{K}(A) \to A$ defined by $\psi(\Sigma\lambda_i\theta_{a_ib_i}) = \Sigma\lambda_i a_i b_i^*$ for all $a_i, b_i \in A$ and $\lambda_i \in \mathbb{C}$ is an isomorphism and by [13, Theorem 15.2.12], $M(\mathbb{K}(A)) \simeq \mathbb{B}(A)$, so $\mathbb{B}(A) \simeq A$ and therefore φ is a Jordan isomorphism, by Theorems 2.2 and 2.3. □

Remark 2.6. If A is a unital finite dimension C^*-algebra, A has a faithful state by [10, Exercise 6.2], so by Lemma 2.5 every unital surjective and spectral radius preserving linear map on A is a Jordan isomorphism which is a result of [8, Corollary 5].

Remark 2.7. If A is a unital von Neumann algebra on a separable Hilbert space, then A is countably decomposable, so by [2, Proposition III.2.2.27] A has a faithful state and by Lemma 2.5 every surjective and spectrum preserving linear map on A is a Jordan isomorphism which is a result of [1, Theorem 1.3].

Theorem 2.8. *Let A be an infinite dimension unital C^*-algebra which has a faithful state. If $\varphi : A \to A$ is a surjective linear map, then*

(i) *if $\sigma(\varphi(a)\varphi(b)) = \sigma(ab)$, for all $a, b \in A$, then there exists $c \in A$ such that for all $a \in A$, $\varphi(a) = c^{-1}ac$ or $\varphi(a) = -cac^{-1}$,*
(ii) *if $\sigma((\varphi(a))^*\varphi(b)) = \sigma(a^*b)$, for all $a, b \in A$, then there exist unitary elements $u, v \in A$ such that for all $a \in A$, $\varphi(a) = uav$.*

Proof. Since A is a unital C^*-algebra, $\mathbb{B}(A) \simeq A$ and the result follows by Theorem 2.4. □

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