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On New Extensions of Hermite-Hadamard Inequalities for Generalized Fractional Integrals

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Abstract. In this paper, we establish some Trapezoid and Midpoint type inequalities for generalized fractional integrals by utilizing the functions whose second derivatives are bounded . We also give some new inequalities for k-Riemann-Liouville fractional integrals as special cases of our main results. We also obtain some Hermite-Hadamard type inequalities by using the condition $f'(a+b-x) \geq f'(x)$ for all $x \in \left[a, \frac{a+b}{2}\right]$ instead of convexity.

1. Introduction

The Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics. A number of mathematicians have devoted their efforts to generalise, refine, counterpart and extend it for different classes of functions such as using convex mappings.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g.,[23, p.137], [9]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave.

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In [11] and [12], Dragomir et al. proved the following results connected with the Hermite-Hadamard inequality:

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping such that there exists real constants m and M so that $m \le f'' \le M$. Then, the following inequalities hold:*

$$m\frac{(b-a)^2}{24} \le \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \le M\frac{(b-a)^2}{24}, \tag{1.2}$$

and

$$m\frac{(b-a)^2}{24} \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \le M\frac{(b-a)^2}{24}. \tag{1.3}$$

In [22], Minculate and Mitroi proved another refinement of inequalities (1.1). Sarikaya [29] proved some refinement fractional Hermite-Hadamard and Fejer type inequalities using the results of Minculate and Mitroi.

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 1.2. Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f(x)$ and $J_{b-}^{\alpha}f(x)$ are defined as

$$J_{a+}^{\alpha}f(a) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b.$$

Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x)$.

Definition 1.3 ([21])**.** Let $f \in L^1[a, b]$. Then k–fractional integrals of order $\alpha, k > 0$ with $a \ge 0$ are defined by

$$J_{a+,k}^{\alpha}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)\,dt, \quad x > a,$$

and

$$J_{b-,k}^{\alpha}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)\,dt, \quad b > x,$$

where $\Gamma_k(\cdot)$ stands for the k-gamma function. For $k = 1$, the k–fractional integrals yield Riemann–Liouville integrals. For $\alpha = k = 1$, the k–fractional integrals yield classical integrals.

For more details about fractional integrals, see [17, 19].

Theorem 1.4 ([27]). *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

(1.4)
$$f\left(\frac{a+b}{2}\right)\le\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]\le\frac{f(a)+f(b)}{2}.$$

Moreover, Dragomir gives the following another version of Hermite-Hadamard inequality for Riemann-Lioville fractional integrals:

Theorem 1.5 ([10]). *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

(1.5)
$$f\left(\frac{a+b}{2}\right)\le\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)+J^{\alpha}_{b-}f\left(\frac{a+b}{2}\right)\right]\le\frac{f(a)+f(b)}{2}.$$

Over the years, several papers devoted to fractional Hermite-Hadamard inequalities. One can refer to the references ([1–8, 13–16, 18–20, 24–32]) for some of them.

Budak et al. prove the following inequalities in [5].

Theorem 1.6 ([5]). *Let $f:[a,b]\to\mathbb{R}$ be a positive twice differentiable function with $a<b$ and $f\in L_1[a,b]$. If f'' is bounded, i.e. $m\le f''(t)\le M$, $t\in[a,b]$, $m,M\in\mathbb{R}$, then we have the inequalities*

$$\begin{aligned}(1.6)\quad &\frac{m(b-a)^2}{4(\alpha+1)(\alpha+2)}\\ &\le\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-f\left(\frac{a+b}{2}\right)\\ &\le\frac{M(b-a)^2}{4(\alpha+1)(\alpha+2)},\end{aligned}$$

and

$$\begin{aligned}(1.7)\quad &\frac{m(b-a)^2\alpha(\alpha+3)}{8(\alpha+1)(\alpha+2)}\\ &\le\frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]\\ &\le\frac{m(b-a)^2\alpha(\alpha+3)}{8(\alpha+1)(\alpha+2)},\end{aligned}$$

for $\alpha>0$.

Theorem 1.7 ([5]). *Let $f:[a,b]\to\mathbb{R}$ be a positive twice differentiable function with $a<b$ and $f\in L_1[a,b]$ If f'' is bounded i.e. $m\le f''(t)\le$*

M, $t\in[a,b]$, $m,M\in\mathbb{R}$, then we have the inequalities

$$\begin{aligned}&\frac{m\alpha(b-a)^2}{8(\alpha+2)}\\&\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}f\left(\frac{a+b}{2}\right)+J_{b-}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\\&\leq\frac{M\alpha(b-a)^2}{8(\alpha+2)},\end{aligned}\tag{1.8}$$

and

$$\begin{aligned}&\frac{m(b-a)^2}{4(\alpha+2)}\\&\leq\frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}f\left(\frac{a+b}{2}\right)+J_{b-}f\left(\frac{a+b}{2}\right)\right]\\&\leq\frac{M(b-a)^2}{4(\alpha+2)},\end{aligned}\tag{1.9}$$

for $\alpha>0$.

Let's define a function $\varphi:[0,\infty)\to[0,\infty)$ satisfying the following condition:

$$\int_0^1\frac{\varphi(t)}{t}dt<\infty.$$

Definition 1.8. [28] The following left-sided and right–sided generalized fractional integral operators are defined, respectively, as follows:

$$_{a+}I_{\varphi}f(x)=\int_a^x\frac{\varphi(x-t)}{x-t}f(t)dt,\quad x>a,\tag{1.10}$$

$$_{b-}I_{\varphi}f(x)=\int_x^b\frac{\varphi(t-x)}{t-x}f(t)dt,\quad x<b.\tag{1.11}$$

Recently, some Hermite-Hadamard inequalities for generalized fractional inregrals have been established under the condition of convexity, as follows:

Theorem 1.9 ([28])**.** *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ with $a<b$, then the following inequalities for fractional integral operators hold*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2\Lambda(1)}\left[_{a+}I_{\varphi}f(b)+_{b-}I_{\varphi}f(a)\right]\leq\frac{f(a)+f(b)}{2},\tag{1.12}$$

where the mapping $\Lambda : [0,1] \to \mathbb{R}$ *is defined by*

$$\Lambda(x) = \int_0^x \frac{\varphi((b-a)t)}{t}dt.$$

Theorem 1.10 ([6]). *Let* $f : [a,b] \to \mathbb{R}$ *be a function with* $a < b$ *and* $f \in L_1[a,b]$. *If* f *is a convex function on* $[a,b]$, *then we have the following inequalities for generalized fractional integral operators:*

(1.13)
$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b) + {}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right] \leq \frac{f(a)+f(b)}{2},$$

where the mapping $\Psi : [0,1] \to \mathbb{R}$ *is defined by*

$$\Psi(x) = \int_0^x \frac{\varphi\left(\frac{b-a}{2}t\right)}{t}dt.$$

Theorem 1.11. *Let* $f : [a,b] \to \mathbb{R}$ *be a function with* $a < b$ *and* $f \in L_1[a,b]$. *If* f *is a convex function on* $[a,b]$, *then we have the following inequalities for generalized fractional integral operators:*

(1.14)
$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2\Psi(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right) + {}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2},$$

where the mapping Ψ *is defined as above.*

In this paper we obtain the inequalities (1.13) and (1.14) by using the condition $f'(a+b-x) \geq f'(x)$ for all $x \in \left[a, \frac{a+b}{2}\right]$ instead of convexity.

2. Extension Hermite-Hadamard Type Inequalities

Firstly, we give the following inequalities which give the above and below bounds for the left and right hand sides of inequalities (1.13) and (1.14).

Theorem 2.1. *Let* $f : [a,b] \to \mathbb{R}$ *be a positive, twice differentiable function with* $a < b$ *and* $f \in L_1[a,b]$. *If* f'' *is bounded, i.e.* $m \leq f''(t) \leq M$, $t \in [a,b]$, $m, M \in \mathbb{R}$, *then we have the inequalities*

$$\begin{aligned}(2.1)\qquad &\frac{m}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2\frac{\varphi(x-a)}{(x-a)}dx \\ &\leq \frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b) + {}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right] - f\left(\frac{a+b}{2}\right) \\ &\leq \frac{M}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2\frac{\varphi(x-a)}{(x-a)}dx,\end{aligned}$$

and

$$\frac{m}{2\Psi(1)}\int_a^{\frac{a+b}{2}}(b-x)\varphi(x-a)dx \tag{2.2}$$
$$\le \frac{f(a)+f(b)}{2}-\frac{1}{2\Psi(1)}\left[{}_{(\frac{a+b}{2})+}I_{\varphi}f(b)+{}_{(\frac{a+b}{2})-}I_{\varphi}f(a)\right]$$
$$\le \frac{M}{2\Psi(1)}\int_a^{\frac{a+b}{2}}(b-x)\varphi(x-a)dx,$$

where Ψ *is defined as in Theorem 1.10.*

Proof. By using the change of variables we have

(2.3)

$$\frac{1}{2\Psi(1)}\left[I^{\alpha}_{(\frac{a+b}{2})+}f(b)+I^{\alpha}_{(\frac{a+b}{2})-}f(a)\right]$$
$$=\frac{1}{2\Psi(1)}\left[\int_{\frac{a+b}{2}}^{b}\frac{\varphi(b-x)}{b-x}f(x)dx+\int_a^{\frac{a+b}{2}}\frac{\varphi(x-a)}{x-a}f(x)dx\right]$$
$$=\frac{1}{2\Psi(1)}\left[\int_a^{\frac{a+b}{2}}\frac{\varphi(x-a)}{x-a}f(a+b-x)dx+\int_a^{\frac{a+b}{2}}\frac{\varphi(x-a)}{x-a}f(x)dx\right]$$
$$=\frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right]\frac{\varphi(x-a)}{x-a}dx.$$

By the equality (2.3), we get

(2.4)

$$\frac{1}{2\Psi(1)}\left[I^{\alpha}_{(\frac{a+b}{2})+}f(b)+I^{\alpha}_{(\frac{a+b}{2})-}f(a)\right]-f\left(\frac{a+b}{2}\right)$$
$$=\frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right]\frac{\varphi(x-a)}{x-a}dx-f\left(\frac{a+b}{2}\right)$$
$$=\frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right]\frac{\varphi(x-a)}{x-a}dx.$$

Using the facts that

$$f\left(\frac{a+b}{2}\right)-f(x)=\int_x^{\frac{a+b}{2}}f'(t)dt,$$

and

$$f(a+b-x)-f\left(\frac{a+b}{2}\right)=\int_{\frac{a+b}{2}}^{a+b-x}f'(t)dt,$$

we have

(2.5)

$$f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)=\int_{\frac{a+b}{2}}^{a+b-x}f'(t)dt-\int_{x}^{\frac{a+b}{2}}f'(t)dt$$
$$=\int_{x}^{\frac{a+b}{2}}f'(a+b-u)du-\int_{x}^{\frac{a+b}{2}}f'(t)\,dt$$
$$=\int_{x}^{\frac{a+b}{2}}\left[f'(a+b-t)-f'(t)\right]dt.$$

We also have

$$f'(a+b-t)-f'(t)=\int_{t}^{a+b-t}f''(u)du. \tag{2.6}$$

By using the equality (2.6) and the assumption $m<f''(u)<M$, $u\in[a,b]$, we obtain,

$$m\int_{t}^{a+b-t}du\le\int_{t}^{a+b-t}f''(u)du\le M\int_{t}^{a+b-t}du,$$

i.e.

$$m\,(a+b-2t)\le f'(a+b-t)-f'(t)\le M\,(a+b-2t)\,. \tag{2.7}$$

Integrating inequality (2.7) with respect to t on $\left[x,\frac{a+b}{2}\right]$, we get

$$m\left(\frac{a+b}{2}-x\right)^2\le\int_{x}^{\frac{a+b}{2}}\left[f'(a+b-t)-f'(t)\right]dt\le M\left(\frac{a+b}{2}-x\right)^2.$$

By equality (2.5), we have

(2.8)

$$m\left(\frac{a+b}{2}-x\right)^2\le f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\le M\left(\frac{a+b}{2}-x\right)^2.$$

Multiplying inequality (2.8) by $\frac{\varphi(x-a)}{2\Psi(1)(x-a)}$ and integrating the resultant inequality with respect to x on $\left[a,\frac{a+b}{2}\right]$, we establish

$$\frac{m}{2\Psi(1)}\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2\frac{\varphi(x-a)}{x-a}dx$$
$$\le\frac{1}{2\Psi(1)}\int_{a}^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right]\frac{\varphi(x-a)}{x-a}dx$$
$$\le\frac{M}{2\Psi(1)}\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2\frac{\varphi(x-a)}{x-a}dx.$$

That is, we get

$$\begin{aligned}
&\frac{m}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2\frac{\varphi(x-a)}{x-a}dx \\
&\le \frac{1}{2\Psi(1)}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+I^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-f\left(\frac{a+b}{2}\right) \\
&\le \frac{M}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2\frac{\varphi(x-a)}{x-a}dx,
\end{aligned}$$

which gives inequality (2.1).

On the other hand, by equality (2.3), we have

(2.9)

$$\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{1}{2\Psi(1)}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+I^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right] \\
&=\frac{f(a)+f(b)}{2}-\frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right]\frac{\varphi(x-a)}{x-a}dx \\
&=\frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[f(a)+f(b)-f(x)-f(a+b-x)\right]\frac{\varphi(x-a)}{x-a}dx.
\end{aligned}$$

By using the equalities

$$f(x)-f(a)=\int_a^x f'(t)dt,$$

and

$$f(b)-f(a+b-x)=\int_{a+b-x}^b f'(t)dt,$$

then we get

(2.10)

$$\begin{aligned}
f(a)+f(b)-f(x)-f(a+b-x) &= \int_{a+b-x}^b f'(t)dt-\int_a^x f'(t)dt \\
&= \int_a^x f'(a+b-u)du-\int_a^x f'(t)dt \\
&= \int_a^x \left[f'(a+b-t)dt-f'(t)\right]dt.
\end{aligned}$$

By integrating inequality (2.7) with respect to t on $[a,x]$, we get

$$m\int_a^x (a+b-2t)\,dt\le\int_a^x\left[f'(a+b-t)-f'(t)\right]dt\le M\int_a^x(a+b-2t)\,dt.$$

That is,

$$m(x-a)(b-x) \le f(a)+f(b)-f(x)-f(a+b-x) \le M(x-a)(b-x). \tag{2.11}$$

Multiplying the inequality (2.11) by $\frac{\varphi(x-a)}{2\Psi(1)(x-a)}$ and integrating the resultant inequality with respect to x on $\left[a, \frac{a+b}{2}\right]$, we establish

$$\begin{aligned}
&\frac{m}{2\Psi(1)} \int_a^{\frac{a+b}{2}} (b-x)(x-a) \frac{\varphi(x-a)}{x-a} dx \\
&\quad \le \frac{1}{2\Psi(1)} \int_a^{\frac{a+b}{2}} [f(a)+f(b)-f(x)-f(a+b-x)] \frac{\varphi(x-a)}{x-a} dx \\
&\quad \le \frac{M}{2\Psi(1)} \int_a^{\frac{a+b}{2}} (b-x)(x-a) \frac{\varphi(x-a)}{x-a} dx.
\end{aligned}$$

This completes the proof. □

Remark 2.2. If we choose $\varphi(t)=t$ in Theorem 2.1, then inequality (2.1) reduces to inequality (1.2) and inequality (2.2) reduces to the inequality

$$\frac{m(b-a)^2}{12} \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt \le \frac{M(b-a)^2}{12}, \tag{2.12}$$

which is given by Budak et al. in [5].

Corollary 2.3. *If we choose $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 2.1, then inequalities (2.1) and (2.2) reduce inequalities (1.6) and (1.7), respectively.*

Corollary 2.4. *If we choose $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.1, then we have the following inequality for k-Riemann-Liouville fractional integrals*

$$\begin{aligned}
&\frac{m(b-a)^2}{4\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)} \\
&\quad \le \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[J^\alpha_{\left(\frac{a+b}{2}\right)+,k} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)-,k} f(a)\right] - f\left(\frac{a+b}{2}\right) \\
&\quad \le \frac{M(b-a)^2}{4\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)},
\end{aligned}$$

and

$$\begin{aligned}
&\frac{m(b-a)^2\alpha(\alpha+3)}{8\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)} \\
&\quad \le \frac{2^{\frac{\alpha}{k}-1}\Gamma(\alpha+1)}{(b-a)^{\frac{\alpha}{k}}}\left[J^\alpha_{\left(\frac{a+b}{2}\right)+,k} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)-,k} f(a)\right] - f\left(\frac{a+b}{2}\right)
\end{aligned}$$

$$\leq \frac{m\left(b-a\right)^{2}\alpha\left(\alpha+3\right)}{8\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)}.$$

Now we give the following refinement of inequality (1.13).

Theorem 2.5. *Let $f:[a,b]\to\mathbb{R}$ be a positive, twice differentiable function with $a<b$ and $f\in L_1[a,b]$. If $f'(a+b-x)\geq f'(x)$ for all $x\in\left[a,\frac{a+b}{2}\right]$, then we have the inequalities*

(2.13)
$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right]\leq\frac{f(a)+f(b)}{2}$$

where Ψ is defined as in Theorem 1.10.

Proof. Since $f'(a+b-x)\geq f'(x)$ for all $x\in\left[a,\frac{a+b}{2}\right]$, by the equalities (2.4) and (2.5), we have

$$\begin{aligned}
&\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right]-f\left(\frac{a+b}{2}\right)\\
&=\frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right]\frac{\varphi(x-a)}{x-a}dx\\
&=\frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[\int_x^{\frac{a+b}{2}}\left[f'(a+b-t)-f'(t)\right]dt\right]\frac{\varphi(x-a)}{x-a}dx\\
&\geq 0,
\end{aligned}$$

which gives first inequality in (2.13).

Similarly, by equalities (2.9) and (2.10), we get

$$\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right]\\
&=\frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[f(a)+f(b)-f(x)-f(a+b-x)\right]\frac{\varphi(x-a)}{x-a}dx\\
&=\frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[\int_a^{x}\left[f'(a+b-t)dt-f'(t)\right]dt\right]\frac{\varphi(x-a)}{x-a}dx\\
&\geq 0.
\end{aligned}$$

This completes the proof. □

Now, we establish the following inequalities which give the above and below bounds for the left and right hand sides of inequality (1.14).

Theorem 2.6. *Let $f:[a,b]\to\mathbb{R}$ be a positive, twice differentiable function with $a<b$ and $f\in L_1[a,b]$. If f'' is bounded, i.e. $m\le f''(t)\le M$, $t\in[a,b]$, $m,M\in\mathbb{R}$, then we have the inequalities*

$$\begin{aligned}
(2.14)\quad & \frac{m}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)\varphi\left(\frac{a+b}{2}-x\right)dx \\
&\le \frac{1}{2\Psi(1)}\left[{}_{a+}I_\varphi f\left(\frac{a+b}{2}\right)+{}_{b-}I_\varphi f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right) \\
&\le \frac{M}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)\varphi\left(\frac{a+b}{2}-x\right)dx,
\end{aligned}$$

and

$$\begin{aligned}
(2.15)\quad & \frac{m}{2\Psi(1)}\int_a^{\frac{a+b}{2}}(x-a)(b-x)\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}dx \\
&\le \frac{f(a)+f(b)}{2}-\frac{1}{2\Psi(1)}\left[{}_{a+}I_\varphi f\left(\frac{a+b}{2}\right)+{}_{b-}I_\varphi f\left(\frac{a+b}{2}\right)\right] \\
&\le \frac{M}{2\Psi(1)}\int_a^{\frac{a+b}{2}}(x-a)(b-x)\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}dx,
\end{aligned}$$

where Ψ is defined as in Theorem 1.10.

Proof. From the definition of generalized fractional integrals, we get

(2.16)

$$\begin{aligned}
&\frac{1}{2\Psi(1)}\left[{}_{a+}I_\varphi f\left(\frac{a+b}{2}\right)+{}_{b-}I_\varphi f\left(\frac{a+b}{2}\right)\right] \\
&=\frac{1}{2\Psi(1)}\left[\int_a^{\frac{a+b}{2}}\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}f(x)dx+\int_{\frac{a+b}{2}}^b\frac{\varphi\left(x-\frac{a+b}{2}\right)}{\left(x-\frac{a+b}{2}\right)}f(x)dx\right] \\
&=\frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right]\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}dx.
\end{aligned}$$

By equality (2.16), we have

(2.17)

$$\begin{aligned}
&\frac{1}{2\Psi(1)}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right) \\
&=\frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right]\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}dx-f\left(\frac{a+b}{2}\right) \\
&=\frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right]\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}dx.
\end{aligned}$$

Multiplying inequality (2.8) by $\frac{1}{2\Psi(1)}\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}$ and integrating the resultant inequality with respect to x on $\left[a, \frac{a+b}{2}\right]$, we establish

(2.18)
$$\begin{aligned}
&\frac{m}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}dx\\
&\quad\leq\frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right]\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}dx\\
&\quad\leq\frac{M}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}dx.
\end{aligned}$$

From equality (2.17) and inequalities (2.18), we have the desired result (2.14).

On the other hand, using identity (2.16), we get

(2.19)
$$\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{1}{2\Psi(1)}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\\
&\quad=\frac{f(a)+f(b)}{2}-\frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right]\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}dx\\
&\quad=\frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[f(a)+f(b)-f(x)-f(a+b-x)\right]\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}dx.
\end{aligned}$$

Multiplying inequality (2.11) by $\frac{1}{2\Psi(1)}\frac{\varphi\left(x-\frac{a+b}{2}\right)}{\left(x-\frac{a+b}{2}\right)}$ and integrating the resultant inequality with respect to x on $\left[a, \frac{a+b}{2}\right]$, we establish

(2.20)
$$\begin{aligned}
&\frac{m}{2\Psi(1)}\int_a^{\frac{a+b}{2}}(x-a)(b-x)\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}dx\\
&\quad\leq\frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[f(a)+f(b)-f(x)-f(a+b-x)\right]\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}dx\\
&\quad\leq\frac{M}{2\Psi(1)}\int_a^{\frac{a+b}{2}}(x-a)(b-x)\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}.
\end{aligned}$$

By equality (2.19) and inequalities (2.20), one has the required result (2.15).

This completes the proof of theorem. □

Remark 2.7. If we choose $\varphi(t) = t$ in Theorem 2.6, then inequalities (2.14) and (2.15) reduces to inequalities (1.2) and (2.12), respectively.

Corollary 2.8. *If we choose $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 2.1, then inequalities (2.14) and (2.15) reduce inequalities (1.8) and (1.9), respectively.*

Corollary 2.9. *If we choose $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.6, then we have the following inequalities for k-Riemann-Liouville fractional integrals*

$$\begin{aligned} &\frac{m\alpha(b-a)^2}{8k(\frac{\alpha}{k}+2)} \\ &\quad\leq \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[J_{a+,k}f\left(\frac{a+b}{2}\right)+J_{b-,k}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right) \\ &\quad\leq \frac{M\alpha(b-a)^2}{8k(\frac{\alpha}{k}+2)}, \end{aligned}$$

and

$$\begin{aligned} &\frac{m(b-a)^2}{4(\frac{\alpha}{k}+2)} \\ &\quad\leq \frac{f(a)+f(b)}{2}-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[J_{a+,k}f\left(\frac{a+b}{2}\right)+J_{b-,k}f\left(\frac{a+b}{2}\right)\right] \\ &\quad\leq \frac{m(b-a)^2}{4(\frac{\alpha}{k}+2)}. \end{aligned}$$

Theorem 2.10. *Let $f : [a,b] \to \mathbb{R}$ be a positive, twice differentiable function with $a < b$ and $f \in L_1[a,b]$. If $f'(a+b-x) \geq f'(x)$ for all $x \in \left[a, \frac{a+b}{2}\right]$, then we have the inequalities*

(2.21)
$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2\Psi(1)}\left[{}_{a+}I_\varphi f\left(\frac{a+b}{2}\right) + {}_{b-}I_\varphi f\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2}$$

where Ψ is defined as in Theorem 1.10.

Proof. From equalities (2.17) and (2.18), we have

$$\begin{aligned} &\frac{1}{2\Psi(1)}\left[{}_{a+}I_\varphi f\left(\frac{a+b}{2}\right) + {}_{b-}I_\varphi f\left(\frac{a+b}{2}\right)\right] - f\left(\frac{a+b}{2}\right) \\ &\quad= \frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right]\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}dx \\ &\quad= \frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[\int_x^{\frac{a+b}{2}}\left[f'(a+b-t)-f'(t)\right]dt\right]\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}dx \end{aligned}$$

$$\geq 0$$

which proves, the first inequality in (2.21).

Similarly, by equalities (2.19) and (2.20)

$$\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{1}{2\Psi(1)}\left[{}_{a^+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b^-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right] \\
&=\frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[f(a)+f(b)-f(x)-f(a+b-x)\right]\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}dx \\
&=\frac{1}{2\Psi(1)}\int_a^{\frac{a+b}{2}}\left[\int_a^x\left[f'(a+b-t)dt-f'(t)\right]dt\right]\frac{\varphi\left(\frac{a+b}{2}-x\right)}{\left(\frac{a+b}{2}-x\right)}dx \\
&\geq 0.
\end{aligned}$$

This completes the proof. □

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