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# Some bi-Hamiltonian Systems and their Separation of Variables on 4-dimensional Real Lie Groups

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Abstract. In this work, we discuss bi-Hamiltonian structures on a family of integrable systems on 4-dimensional real Lie groups. By constructing the corresponding control matrix for this family of bi-Hamiltonian structures, we obtain an explicit process for finding the variables of separation and the separated relations in detail.

## 1. Introduction

The study of bi-Hamiltonian systems, i.e., systems with two compatible Poisson structures, started with pioneering works of Magri and Kosmann-Schwarzbach [5, 7] and subsequent fundamental papers of Gelfand and Dorfman [3], Magri and Morosi [6]. These works show that integrability of systems can be closely connected to their bi-Hamiltonian structures. It is proven that all classical systems have the bi-Hamiltonian structure, and also by using the bi-Hamiltonian methods, many new nontrivial and interesting examples of integrable systems can be found. Moreover, bi-Hamiltonian structure is a very important factor not only for finding new examples, but also for integration of systems, constructing separable variables and description of properties of solutions. In [1], the integrable Hamiltonian systems with the symmetry Lie group as a 4-dimensional phase space which has symplectic structure is constructed. The list of symplectic 4-dimensional real Lie groups are classified in [8].

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The aim of this paper is to identify the variables of separation in the framework of the bi-Hamiltonian geometry. This process consists of the following calculation steps:

(i) For the canonical Poisson structures of the Lie groups, the compatible Poisson bi-vectors are obtained.
(ii) The control matrices associated to the Poisson bi-vectors are obtained.
(iii) Variables of separation, eigenvalues of the control matrix, are calculated.
(iv) The canonically conjugated momenta is obtained with respect to the canonical Poisson bracket.
(v) The separated relations are identified.

## 2. Bi-Hamiltonian Structures

In order to obtain the variables of separation based on the Hamiltonian geometry, we calculate the bi-Hamiltonian structure for the given integrable system $H_{1,2}$ on the Poisson manifold $M$ with initial Poisson bi-vector $P$ (see [10, 11, 13] for the complete bibliography).

A bi-Hamiltonian manifold $M$ is a smooth manifold endowed with two compatible bi-vectors $P$, $P'$ such that

$$[P,P]=0, \qquad [P,P^{'}]=0, \qquad [P^{'},P^{'}]=0, \tag{2.1}$$

where $[,]$ is the Schouten bracket.
The bi-vectors $P$, $P'$ determine a pair of compatible Poisson brackets on $M$,

$$\begin{aligned}\{f(z),g(z)\} &= \langle df, Pdg\rangle \\ &= \sum_{i,j=1}^{dimM} P^{i,j}(z)\frac{\partial f(z)}{\partial z_i}\frac{\partial g(z)}{\partial z_j},\end{aligned}$$

for all $f,g \in F(M)$ and similar brackets $\{,\}'$ to $P'$.

Let $H_0, H_1, \ldots, H_n$ be functionally independent functions on $M$ and in involution with respect to this compatible Poisson brackets

$$\{H_i,H_j\}=\{H_i,H_j\}'=0 \qquad i=0,\ldots,n\ ,\ j=0,\ldots,n. \tag{2.2}$$

According to [10, 11, 13], let us suppose that the desired Poisson bi-vector $P'$ is the Lie derivative of $P$ along some unknown Liouville vector field $X$

$$P'=L_X(P), \tag{2.3}$$

which must satisfy the equation

$$[P',P']=[L_X(P),L_X(P)]=0 \quad \Leftrightarrow \quad [L_X^2(P),P]=0, \tag{2.4}$$

with respect to the Schouten bracket $[.,.]$. By (2.3) bi-vector $P'$ is compatible with a given bi-vector $P$, i.e. $[P, P'] = 0$.
Within solutions of the equation (2.4) we choose partial solutions X such that

$$\{H_i, H_j\}' = 0 \qquad i = 0, \dots, n\ ,\ j = 0, \dots, n. \tag{2.5}$$

Obviously enough, in their full generality the system of equations (2.3-2.5) is too difficult to be solved. It is because it has infinitely many solutions labeled by different separated coordinates, see [11] and [12]. In order to get particular solutions, we will use some special ansatze for the Liouville vector field X.

To solve equations (2.3)-(2.5) we will use polynomials of momenta ansatze for the components of the Liouville vector field $X = \sum X^i \partial_i$

$$X^i = \sum_{k=0}^{N} \sum_{m=0}^{k} g^i_{km}(y_1, y_2) y_3^{k-m} y_4^m. \tag{2.6}$$

First, we assume $N = 2$, it means that $X^i$ will be generic second order polynomials in momenta $y_3, y_4$ with coefficients $g^i_{km}(y_1, y_2)$ depending on variables $y_1$ and $y_2$. Substituting this ansatze (2.6) into the equations (2.3)-(2.5) and demanding that all the coefficients at powers of $y_3$ and $y_4$ vanish, one gets the over determined system of equations which can be solved in the modern computer algebra systems.

## 3. Variables of Separation and Separated Relations

In this section we consider new variables of separation and separated relations for mentioned integrable systems. Suppose the canonical variables of separation $(q_1, \dots, q_n, p_1, \dots, p_n)$ and separated relations as

$$\phi_i(q_i, p_i, H_1, \dots, H_n) = 0, \quad i = 1, \dots, n, \text{ with } \det \left[\frac{\partial \phi_i}{\partial H_i}\right] \neq 0, \tag{3.1}$$

connecting single pairs $(q_i, p_i)$ of canonical variables of separation with the $n$ functionally independent Hamiltonian $H_1, \dots, H_n$.

According to [2], the bi-involutivity of the integrals of motion (2.2) is equivalent to the existence of the control matrix $F = (F_{ij})$ defined by

$$P' dH_i = P \sum_{j=1,2}^{2} F_{ij} dH_j \quad i = 1, 2.. \tag{3.2}$$

If this matrix is non-degenerate, then its eigenvalues are desired separated coordinates $q_i$ which coincide with the Darboux-Nijenhuis coordinates (eigenvalues of the recursion operator $N = P'P^{-1}$) on the corresponding symplectic leaves (see for more details [10],[11]).

Separated coordinates $p_i$ are variables conjugated to $q_i$:

$$\begin{aligned}\{q_i, p_j\} &= \delta_{i,j} \\ \{q_i, p_j\}' &= \delta_{i,j} q_i, \\ \{q_i, q_j\} &= \{q_i, q_j\}' = \{p_i, p_j\} = \{p_i, p_j\}' = 0.\end{aligned} \tag{3.3}$$

In order to get explicit information about the separating relations (3.1) we will concentrate on the more precise notion of Stackel separability. Recall that independent integrals of motion $(H_1, \ldots, H_n)$ are Stackel separable if the corresponding separated relations are given by the affine equations in $H_j$, that is,

$$\sum_{j=1}^{n} S_{i,j}(q_i, p_i) H_j - U_i(q_i, p_i) = 0, \quad i = 1, \ldots, n\ , \tag{3.4}$$

where $S$ is an invertible matrix. The functions $S_{i,j}$ and $U_i$ depend only on one pair $(p_j, q_j)$ of canonical variables of separation.

In this case, $S$ is called a Stackel matrix, and $U$ is a Stackel potential. For Stackel separable systems the suitable normalized left eigenvectors of control matrix $F$ form the Stackel matrix $S$

$$F = S^{-1} diag(q_1, \ldots, q_n) S. \tag{3.5}$$

3.1. **A family of Integrable Systems on 4-dimensional Real Lie Groups.** In this section, we study the integrable Hamiltonian systems with the symmetry Lie group as a 4-dimensional phase space having symplectic structure. In other words, we consider non-degenerate Poisson structure and integrable Hamiltonian systems on the Lie groups $\mathbf{A_{4,1}}$, $\mathbf{A_{4,2}^{-1}}$, $\mathbf{A_{4,3}}$, $\mathbf{A_{4,6}^{a,0}}$, $\mathbf{A_{4,7}}$, $\mathbf{A_{4,9}^{1}}$, $\mathbf{A_{4,12}}$ (see [1] for more details). Then for the canonical Poisson structure $P$, calculate the compatible Poisson bracket and bi-Hamiltonian structure for these 4-dimensional real Lie groups as Poisson manifold.

Now, we begin by analyzing the Lie group $\mathbf{A_{4,1}}$. We first introduce the Poisson structure and an integrable Hamiltonian system on this Lie group, following [1]. According to (2.6), we find the Darboux coordinates and the Poisson bi-vector for $\mathbf{A_{4,1}}$. Similar considerations apply to the well-known 4-dimentional Lie groups $\mathbf{A_{4,2}^{-1}}, \mathbf{A_{4,3}}, \mathbf{A_{4,6}^{a,0}}, \mathbf{A_{4,7}}, \mathbf{A_{4,9}^{1}}, \mathbf{A_{4,12}}$. Their nonzero Poisson brackets, Darboux coordinates, integrable Hamiltonian systems and Poisson bi-vectors are summarized in the tables 1 and 2.

3.2. **Lie Group $\mathbf{A_{4,1}}$.** The non-degenerate Poisson structure on $\mathbf{A_{4,1}}$ can be obtained in the following forms (see [1] for more details):

(3.6)

$$\{x_1, x_2\} = -\frac{c}{2} x_4^2, \quad \{x_1, x_3\} = c x_4, \quad \{x_1, x_4\} = -d, \quad \{x_2, x_3\} = -c,$$

where $c$ and $d$ are arbitrary real constants and $x_3, x_4$ are the conjugate momentum of $x_1$ and $x_2$ respectively.

In this Lie group, we have the integrable system with the Hamiltonian $H_1$ and integral of motion $H_2$ as follows [1]:

$$H_1 = -y_3, \tag{3.7}$$

$$H_2 = -\frac{y_2^2 y_3}{2}, \tag{3.8}$$

where the coordinates $y_1, y_2, y_3, y_4$ are Darboux coordinates:

$$\begin{aligned}
y_1 &= \frac{x_3}{c} + \frac{cx_4^2}{8} + \frac{x_4^2}{2d},\\
y_2 &= -x_1 + \frac{x_3^2}{c^2} + \frac{1}{4}cdx_2x_4 - \frac{x_3x_4^2}{4} - \frac{x_3x_4^2}{cd} - \frac{3c^2x_4^4}{64} + \frac{x_4^4}{4d^2} - \frac{cx_4^4}{8d},\\
y_3 &= x_2 - \frac{2x_3x_4}{cd} - \frac{x_4^3}{d^2} - \frac{cx_4^3}{4d},\\
y_4 &= \frac{x_4}{d},
\end{aligned}$$

such that they satisfy the following standard Poisson brackets:

$$\{y_1, y_3\} = 1, \qquad \{y_2, y_4\} = 1. \tag{3.9}$$

In these coordinates $y_1, y_2, y_3, y_4$ the Poisson structure, $P$ can be represented in matrix form as follows:

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \tag{3.10}$$

In other words the coordinate $y_i$ can be used as a coordinates for the phase space $R^4$; such that the $y_1$ and $y_2$ are dynamical variables and $p_{y_1} = y_3$ and $p_{y_2} = y_4$ are their momentum conjugate.

The aim is to find the bi-Hamiltonian structures for given integrable system with integrals of motion $H_1, H_2$ on $A_{4,1}$ and Poisson structure $P$.

Now by (2.6) and solving the related differential equations, the Poisson bi-vector $P'$ is obtained as follows:

$$P' = \begin{bmatrix} 0 & -y_4 & -a_3 + y_1 & 0 \\ * & 0 & 0 & -a_3 + y_1 \\ * & * & 0 & -y_4 \\ * & * & * & 0 \end{bmatrix}, \tag{3.11}$$

where $a_i$ is arbitrary real constants.

According (3.2), for this case, the control matrix $F$ is as follows:

$$F = \begin{bmatrix} -a_3 + y_1 + \dfrac{y_2 y_4}{2y_3} & \dfrac{-y_4}{y_2 y_3} \\ \dfrac{y_2 y_4 (y_2^2 - 4y_3^2)}{4y_3} & -\dfrac{2a_3 y_3 - 2y_1 y_3 + y_2 y_4}{2y_3} \end{bmatrix}. \tag{3.12}$$

The eigenvalues of this matrix are the variables of separation $q_1$ and $q_2$ as follows:

$$\begin{aligned} A(\lambda) &= \det(F - \lambda I) \\ &= (\lambda - q_1)(\lambda - q_2) \\ &= \lambda^2 + \lambda(2a_3 - 2y_1) - 2a_3 y_1 + y_1^2 - y_4^2 + a_3^2, \end{aligned}$$

that is

$$q_1 = y_1 - y_4 - a_3, \qquad q_2 = y_1 + y_4 - a_3.$$

The corresponding momenta are defined by

$$p_1 = \frac{1}{2}(y_3 + y_2), \qquad p_2 = \frac{1}{2}(y_3 - y_2),$$

which satisfy (3.3).

In this case, in the separated variables, the Stackel matrix $S$ is equal to

$$S = \begin{bmatrix} 1 & 1 \\ \dfrac{y_2^2 + 2y_2 y_3}{2} & \dfrac{y_2^2 - 2y_2 y_3}{2} \end{bmatrix}. \tag{3.13}$$

Now considering the initial integrals of motion $H_{1,2}$ (3.7, 3.8), the separated relations is as follows:

$$C : \left(3p_1^2 - 2p_1 p_2 - p_2^2\right) H_1 - H_2 + (p_1 - p_2)(p_1 + p_2)^2 = 0.$$

In the later we present briefly some calculations for remaining Lie groups $\mathbf{A_{4,2}^{-1}}, \mathbf{A_{4,3}}, \mathbf{A_{4,6}^{a,0}}, \mathbf{A_{4,7}}, \mathbf{A_{4,9}^{1}}, \mathbf{A_{4,12}}$. In other words, in the next subsections we find the control matrices, variables of separation and separated relations for integrable Hamiltonian systems which are obtained in [1] on these Lie groups

3.3. **Lie Group $\mathbf{A_{4,2}^{-1}}$.** The Poisson bi-vector for this Lie group has the following form:

(3.14)

$$P' = \begin{bmatrix} 0 & \dfrac{e^{-(2a_1 y_2)} y_3}{a_1} & \dfrac{e^{-(a_1 y_2)} a_2}{a_1} - a_3 & -2y_3(a_4 + f(y_2)) - y_3 g(y_2) \\ * & 0 & 0 & -e^{-(2a_1 y_2)} y_3^2 + \dfrac{e^{-(a_1 y_2)} a_2}{a_1} - a_3 \\ * & * & 0 & -a_2 e^{-(a_1 y_2)} y_3 \\ * & * & * & 0 \end{bmatrix},$$

where $a_i$s are arbitrary real constants and $f(y_2), g(y_2)$ are functions of $y_2$.

In this case, Control matrix for the Poisson bivector $P'$ (3.14) is as follows:

$$F = \begin{bmatrix} \alpha & -\dfrac{\beta}{y_2^3y_3^3a_1} \\ \dfrac{e^{-(2y_2a_1)}y_2y_3^3(y_3^2+y_2y_3^2a_1+e^{(y_2a_1)}y_2^2a_1a_2)}{a_1} & -\dfrac{\gamma}{y_2a_1} \end{bmatrix}, \tag{3.15}$$

where

$$\alpha = -\frac{e^{-(2y_2a_1)}\left(-y_3^2-y_2y_3^2a_1-e^{(y_2a_1)}y_2a_2-2e^{(y_2a_1)}y_2^2a_1a_2+e^{(2y_2a_1)}y_2a_1a_3\right)}{y_2a_1}$$

$$\beta = e^{-(2y_2a_1)}\left(y_3^2+2y_2y_3^2a_1+4e^{(y_2a_1)}y_2^2a_1a_2\right),$$

$$\gamma = e^{-(2y_2a_1)}\left(y_3^2+2y_2y_3^2a_1-e^{(y_2a_1)}y_2a_2+2e^{(y_2a_1)}y_2^2a_1a_2+e^{(2y_2a_1)}y_2a_1a_3\right).$$

Variables of separation $q_{1,2}$ are the roots of the following polynomial

$$\begin{aligned} A(\lambda) &= \det(F-\lambda I) \\ &= \lambda^2+\left(e^{-2y_2a_1}y_3^2-\frac{2e^{-y_2a_1}a_2}{a_1}+2a_3\right)\lambda \\ &\quad+\left(\frac{e^{-2y_2a_1}a_2^2}{a_1^2}+e^{-2y_2a_1}y_3^2a_3-\frac{2e^{-y_2a_1}a_2a_3}{a_1}+a_3^2\right). \end{aligned}$$

In other words, the roots of $A(\lambda)$ are obtained by

$$\begin{aligned} q_1 &= \frac{1}{2y_2^3y_3^3a_1}e^{-2y_2a_1}\Big(-y_2^3y_3^5a_1+2e^{y_2a_1}y_2^3y_3^3a_2 \\ &\quad-\sqrt{y_2^6y_3^10a_1^2-4e^{y_2a_1}y_2^6y_3^8a_1a_2}-2e^{2y_2a_1}y_2^3y_3^3a_1a_3\Big) \\ q_2 &= \frac{1}{2y_2^3y_3^3a_1}e^{-2y_2a_1}\Big(-y_2^3y_3^5a_1+2e^{y_2a_1}y_2^3y_3^3a_2 \\ &\quad+\sqrt{y_2^6y_3^10a_1^2-4e^{y_2a_1}y_2^6y_3^8a_1a_2}-2e^{2y_2a_1}y_2^3y_3^3a_1a_3\Big). \end{aligned}$$

The corresponding momenta are defined by

$$\begin{aligned} p_1 &= -\frac{y_1}{y_2}-\left(\frac{1}{2}\right)\frac{y_2+y_3}{y_2}+\frac{\ln(y_2)a_1}{(y_2+y_3)y_2}-\ln(y_2) \\ &\quad-\left(\frac{1}{2}\right)\frac{a_1\ln(-y_2y_3)}{((y_2+y_3)y_2)}-\frac{2a_2}{y_2}\quad, \\ p_2 &= \frac{1}{4}\frac{2a_1\ln(y3)+(4y_1+2y_4)y_2+4y_3\left(a_2+\frac{y_1}{2}+\frac{y_3}{4}\right)}{y_2^2}. \end{aligned}$$

In this case, in the separated variables, the Stackel matrix $S$ is equal to

(3.16)
$$S = \begin{bmatrix} \dfrac{\begin{matrix}1\\ \beta\end{matrix}}{2y_3^2+4y_2y_3^2a_1+8e^{y_2a_1}y_2^2a_1a_2} & \dfrac{\begin{matrix}1\\ \gamma\end{matrix}}{2y_3^2+4y_2y_3^2a_1+8e^{y_2a_1}y_2^2a_1a_2} \end{bmatrix}$$

where

$$\beta = 2y_2^2y_3^5+3y_2^3y_3^5a_1+4e^{y_2a_1}y_2^4y_3^3a_1a_2+\sqrt{y_2^6y_3^8a_1(y_3^2a_1-4e^{y_2a_1}a_2)},$$

and

$$\gamma = 2y_2^2y_3^5+3y_2^3y_3^5a_1+4e^{y_2a_1}y_2^4y_3^3a_1a_2-\sqrt{y_2^6y_3^8a_1(y_3^2a_1-4e^{y_2a_1}a_2)}.$$

With regards to the integrals of motion $H_{1,2}$ for Lie group $\mathbf{A_{4,2}^{-1}}$, one can calculate separated relations as follows:

$$\begin{aligned} C: &\frac{(q_1-q_2)^2(q_1+q_2)}{\sqrt{q_1}(3q_1+q_2)}H_2+\frac{(q_1-q_2)^2}{2\sqrt{q_1}}H_1 \\ &-\frac{3q_1+2q_1^{7/2}+q_2-4q_1^{5/2}q_2+4\sqrt{q_1}q_2^3-\dfrac{2q_2^4}{\sqrt{q_1}}}{6q_1+2q_2}=0. \end{aligned}$$

3.4. **Lie Group $\mathbf{A_{4,3}}$.** For this Lie group we have the following Poisson bi-vector:

$$P' = \begin{bmatrix} 0 & \dfrac{2a_6-e^{a_1(y_2+a_2)}}{a_1} & -a_7-2a_6y_3 & 0 \\ * & 0 & 0 & \alpha \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}, \tag{3.17}$$

where

$$\alpha = -a_7+2a_6\left(y_4^2a_3+a_4+y_4a_5\right)-e^{a_1(y_2+a_2)}\left(y_3+y_4^2a_3+a_4+y_4a_5\right)$$

and $a_i$s are arbitrary real constants.

In this case, Control matrix for the Poisson bi-vector $P'$ (3.17) is as follows:

$$F = \begin{bmatrix} -a_7-2a_6y_3 & 0 \\ -\dfrac{\alpha}{a_1} & \beta \end{bmatrix}, \tag{3.18}$$

where

$$\begin{aligned} \alpha = &\left(-2a_6+e^{a_1(y_2+a_2)}\right) \\ &\times\left(-y_3+\left(y_3\left(-1+y_2a_1\right)+y_2a_1\left(y_4^2a_3+a_4+y_4a_5\right)\right)\log[y_2]\right), \end{aligned}$$

and

$$\beta = -a_7 + 2a_6\left(y_4^2 a_3 + a_4 + y_4 a_5\right) - e^{a_1(y_2+a_2)}\left(y_3 + y_4^2 a_3 + a_4 + y_4 a_5\right).$$

Bi-vector $P'$ (3.17) gives rise to the following Darboux-Nijenhuis coordinates

$$q_1 = -a_7 - 2a_6 y_3$$
$$q_2 = -a_7 + 2a_6(y_4^2 a_3 + a_4 + y_4 a_5) - e^{a_1(y_2+a_2)}(y_3 + y_4^2 a_3 + a_4 + y_4 a_5).$$

and momenta

$$p_1 = \frac{1}{2}\frac{y_1}{a_6} + \frac{1}{2}\frac{\ln(a_5 y_4 + y_3)}{a_6 a_1 a_5}, \quad p_2 = \frac{1}{2}\frac{-\ln(a_5 y_4 + y_3) + (2a_1 a_5 a_6 - y_2)a_1}{a_6 a_1 a_5}.$$

In the separated variables, the Stackel matrix $S$ is equal to

$$S = \begin{bmatrix} 1 & 1 \\ \gamma & 0 \end{bmatrix}, \tag{3.19}$$

where

$$\gamma = -\frac{y_3 a_1 + y_4^2 a_1 a_3 + a_1 a_4 + y_4 a_1 a_5}{-y_3 - y_3 \ln y_2 + y_2 y_3 a_1 \ln y_2 + y_2 y_4^2 a_1 a_3 \ln y_2 + y_2 a_1 a_4 \ln y_2 + y_2 y_4 a_1 a_5 \ln y_2}.$$

If we come back to initial integrals of motion $H_{1,2}$ for Lie group $\mathbf{A_{4,3}}$, then these separated relations go over to the equation

$$C : -\frac{\sqrt{q_1}(q_1 - q_2 + q_1 ln|q_1|)}{q_1 + q_2} H_2 - \sqrt{q_1} ln|q_1| H_1$$
$$-\frac{(q_1 - q_2)^2 + q_1(q_1 - q_2) ln|q_1| + \frac{1}{4}\sqrt{q_1}(q_1^2 - q_2^2) ln|q_1|^2}{q_1 + q_2} = 0.$$

3.5. **Lie Group $\mathbf{A_{4,7}}$.** For this Lie group we have the following Poisson bi-vector:

$$P' = \begin{bmatrix} 0 & \dfrac{e^{(-2a_1(y_2+a_2))} y_3}{a_1} & -a_3 - 2a_4 y_3 & \dfrac{\alpha}{9a_6} \\ * & 0 & 0 & \beta \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}, \tag{3.20}$$

where $a_i$s are arbitrary real constants and

$$\alpha = e^{(-4y_2 a_1)}(4a_4 e^{(3y_2 a_1)} + 3a_5)(2a_4 e^{(3y_2 a_1)} y_3 - 3(y_3 a_5 + e^{(y_2 a_1)} y_4 a_6)),$$

and

$$\beta = -a_3 - (4a_4 y_3)/3 - e^{(-2a_1(y_2+a_2))} y_3^2 - e^{(-3y_2 a_1)} y_3 a_5 - e^{(-2y_2 a_1)} y_4 a_6.$$

Control matrix $F$ for the Poisson bi-vector $P'$ (3.20) is as follows:

$$F = \begin{bmatrix} \alpha & \beta \\ 0 & -a_3 - 2a_4y_3 \end{bmatrix} \tag{3.21}$$

where

$$\begin{aligned} \alpha &= -a_{310} - \frac{(4a_4y_3)}{3} - e^{(-2a_1(y_2+a_2))}y_3^2 \\ &\quad - e^{(-3y_2a_1)}y_3a_5 - e^{(-2y_2a_1)}y_4a_6, \\ \beta &= -\frac{2}{3}a_4y_2y_3 + \frac{e^{(-2a_1(y_2+a_2))}y_3^2(1+y_2a_1)}{a_1} \\ &\quad + e^{(-3y_2a_1)}y_2y_3a_5 + e^{(-2y_2a_1)}y_2y_4a_6. \end{aligned}$$

Variables of separation $q_{1,2}$ are the roots of the following polynomial

$$\begin{aligned} A(\lambda) &= \det(F - \lambda I) \\ &= \frac{1}{3}\Big(3a_3 + 3\lambda + 4a_4y_3 + 3e^{-2a_1(y_2+a_2)}y_3^2 + 3e^{-3y_2a_1}y_3a_3 \\ &\quad + 3e^{-2y_2a_1}y_4a_4\Big)(a_3 + \lambda + 2a_4y_3) \end{aligned}$$

this is

$$\begin{aligned} q_1 &= -a_3 - 2a_4, \\ q_2 &= -a_3 - \frac{(4a_4y_3)}{3} - e^{(-2a_1(y_2+a_2))}y_3^2 - e^{(-3y_2a_1)}y_3a_5 - e^{(-2y_2a_1)}y_4a_6. \end{aligned}$$

The corresponding momenta are defined by

$$\begin{aligned} p_1 &= \frac{1}{2a_4c_6c_1e^{2c_1c_2+y_2c_1}}(-ln(e^{-3y_2c_1}y_3c_5 + e^{-2y_2c_1}y_4c_6 \\ &\quad + e^{-2c_1(y_2+c_2)}y_3^2)y_3e^{y_2c_1} + (c_6y_1e^{2c_1c_2} - 2y_3y_2)c_1e^{y_2c_1} + c_5e^{2c_1c_2}), \\ p_2 &= \frac{e^{c_1(3y_2+2c_2)}c_6y_4 + y_3c_5e^{2c_1(y_2+c_2)}}{2c_1c_6(c_6y_4e^{c_1(y_2+2c_2)} + e^{y_2c_1}y_3^2 + e^{2c_1c_2}y_3c_5)}. \end{aligned}$$

In the separated variables, the Stackel matrix $S$ is equal to

$$S = \begin{bmatrix} 1 & 1 \\ 0 & \frac{\alpha}{\beta} \end{bmatrix}. \tag{3.22}$$

Where

$$\begin{aligned} \alpha &= a_1\Big(2a_4e^{3y_2a_1+2a_1(y_2+2)}y_3 - 3e^{3y_2a_1}y_3^2 - 3e^{2a_1(y_2+a_2)}y_3a_3 \\ &\quad - 3e^{y_2a_1+2a_1(y_2+a_2)}y_4a_4\Big) \\ \beta &= 2a_4e^{a_1(5y_2+2a_2)}y_2y_3a_1 - 3e^{3y_2a_1}y_3^2(1+y_2a_1) \end{aligned}$$

$$- 3e^{2a_1(y_2+a_2)}y_2y_3a_1a_3 - 3e^{a_1(3y_2+2a_2)}y_2y_4a_1a_4.$$

With regards to the integrals of motion $H_{1,2}$ for Lie group $\mathbf{A_{4,7}}$, we have the following separated relations:

$$C : \frac{q_1+q_2}{\sqrt{q_1}}H_1 + \frac{1}{q_1}H_2 - \frac{(q_1-q_2)(-1+q_1^2+q_1q_2)}{q_1^{3/2}} = 0.$$

3.6. **Lie group $\mathbf{A^{a,0}_{4,6}}$:** For this Lie group we have the following Poisson bi-vector:

$$(3.23) \qquad P' = \begin{bmatrix} 0 & 0 & y_2^2+y_2y_3 & -a_1-2a_2y_3-y_1y_3-y_3^2f(y_2) \\ * & 0 & 0 & y_2^2 \\ * & * & 0 & 2y_2y_3+y_3^2 \\ * & * & * & 0 \end{bmatrix},$$

where $a_i$s are arbitrary real constants and $f(y_2)$ is a function of $y_2$ .

Control matrix for the Poisson bi-vector $P'$ (3.23) is as follows:

$$(3.24) \qquad F = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

where

$$\alpha = y_2(y_2+y_3) - e^{\frac{y_2}{d^2}}\left(-ay_2y_3 + d^2(2y_2+y_3)\right)\cot\left(e^{-\frac{y_2}{d^2}}\right)$$

$$\beta = -2e^{\frac{(1+a)y_2}{d^2}}y_3\left(-ay_2y_3 + d^2\left(2y_2+y_3\right)\right)\csc\left(e^{-\frac{y_2}{d^2}}\right)$$

$$\gamma = \frac{e^{\frac{y_2-2ay_2}{d^2}}\cos\left(e^{-\frac{y_2}{d^2}}\right)\left(-e^{\frac{(-1+a)y_2}{d^2}}y_2y_3 + e^{\frac{ay_2}{d^2}}(-ay_2y_3+d^2(2y_2+y_3))\cot\left(e^{-\frac{y_2}{d^2}}\right)\right)}{2y_3}$$

$$\delta = e^{\frac{y_2-ay_2}{d^2}}\left(e^{\frac{(-1+a)y_2}{d^2}}y_2^2 + e^{\frac{ay_2}{d^2}}\left(-ay_2y_3+d^2(2y_2+y_3)\right)\cot\left(e^{-\frac{y_2}{d^2}}\right)\right).$$

Variables of separation $q_{1,2}$ are eigenvalues of the control matrix for Lie group $\mathbf{A^{a,0}_{4,6}}$, and they are the roots of the following polynomial

$$\begin{aligned} A(\lambda) &= \det(F-\lambda I) \\ &= \lambda^2 - \lambda(2y_2^2+y_2y_3) + y_2^3y_3 + y_2^4, \end{aligned}$$

this is

$$q_1 = y_2^2, \qquad q_2 = y_2^2 + y_2y_3.$$

The corresponding momenta are defined by

$$p_1 = -\frac{y_1}{y_2} - \left(\frac{1}{2}\right)\frac{y_2+y_3}{y_2} + \frac{\ln(y_2)a_1}{(y_2+y_3)y_2} - \ln(y_2) - \left(\frac{1}{2}\right)\frac{a_1\ln(-y_2y_3)}{((y_2+y_3)y_2)} - \frac{2a_2}{y_2},$$

$$p_2 = \frac{1}{4}\frac{2a_1\ln(y3) + (4y_1+2y_4)y_2 + 4y_3(a_2+\frac{y_1}{2}+\frac{y_3}{4})}{y_2^2}.$$

In the separated variables, the Stackel matrix $S$ is equal to

$$S = \begin{bmatrix} 1 & 1 \\ -\dfrac{e^{-\frac{ay_2}{d^2}}\cos(e^{-\frac{y_2}{d^2}})}{2y_3} & \dfrac{\alpha}{2y_3(-ay_2y_3 + d^2(2y_2+y_3))} \end{bmatrix}. \tag{3.25}$$

where

$$\alpha = e^{-\frac{(1+a)y_2}{d^2}} \left( -e^{\frac{y_2}{d^2}} \left( -ay_2y_3 + d^2(2y_2+y_3) \right) \cos\left( e^{-\frac{y_2}{d^2}} \right) + y_2y_3 \sin\left( e^{-\frac{y_2}{d^2}} \right) \right)$$

With regards to the integrals of motion $H_{1,2}$ for Lie group $\mathbf{A}_{\mathbf{4,6}}^{\mathbf{a,0}}$, one can calculate separated relations as follows:

$$\begin{aligned} C : &-\frac{\sqrt{q_1}e^{\frac{a\sqrt{q_1}}{d^2}}\cos(e^{\frac{\sqrt{q_1}}{d^2}})}{2(q_2-q_1)}H_1 + \alpha H_2 - 1/2e^{-a\frac{\sqrt{q_1}}{d^2}}\cos(e^{-\frac{\sqrt{q_1}}{d^2}})(q_1-q_2)/\sqrt{q_1} \\ &+ e^{a\frac{\sqrt{q_1}}{d^2}}\cos(e^{-\frac{\sqrt{q_1}}{d^2}}) + (e^{(-1+a)\frac{\sqrt{q_1}}{d^2}}\sqrt{q_1}(q_1-q_2)\sin(e^{-\frac{\sqrt{q_1}}{d^2}}))/(a\sqrt{q_1}(q_1-q_2) \\ &+ d^2(q_1+q_2))) = 0. \end{aligned}$$

where

$$\alpha = \frac{e^{-((1+a)\sqrt{q_1})/d^2}(e^{\frac{\sqrt{q_1}}{d^2}}\sqrt{q_1}(a\sqrt{q_1}(q_1-q_2)+d^2(q_1+q_2))\cos(e^{-\frac{\sqrt{q_1}}{d^2}}) + q_1(q_1-q_2)\sin(e^{-\frac{\sqrt{q_1}}{d^2}}))}{2(q_1-q_2)(a\sqrt{q_1}(q_1-q_2)+d^2(q_1+q_2))}.$$

3.7. **Lie group $\mathbf{A}_{\mathbf{4,9}}^{\mathbf{1}}$ :** For this Lie group we have the following Poisson bi-vector:

$$P' = \begin{bmatrix} 0 & 2y_2(2-\sqrt{y_2}+y_3) & -a_1 - y_3(2+y_3) & \alpha \\ * & 0 & 0 & -a_1 - \sqrt{y_2}y_3 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix} \tag{3.26}$$

where $\alpha = -a_2 - 2(y_3 + 2y_4 - \sqrt{y_2}y_4 + y_3y_4) - y_3$ and $a_i$- are arbitrary real constants.

In this case, Control matrix for the Poisson bi-vector $P'$ (3.20) is as follows:

$$F = \begin{bmatrix} -a_1 - \sqrt{y_2}y_3 & -a_2 - 2y_3 - 4y_4 + 2\sqrt{y_2}y_4 - 2y_3y_4 - y_3 \\ 0 & -a_1 - 2y_3 - y_3^2 \end{bmatrix}. \tag{3.27}$$

In this case, bi-vector $P'$ (3.26) gives rise to the following Darboux-Nijenhuis coordinates

$$q_1 = -a_1 - y_3(2+y_3), \qquad q_2 = -a_1 - \sqrt{y_2}y_3,$$

and momenta

$$p_1 = \frac{y_1}{2+2y_3} + \frac{y_2y_4}{(1+y_3)y_3}, \qquad p_2 = -\frac{2\sqrt{y_2}y_4}{y_3}.$$

In the separated variables, the Stackel matrix $S$ is equal to

$$S = \begin{bmatrix} 1 & 1 \\ 0 & -\dfrac{-2y_3 + \sqrt{y_2}y_3 - y_3^2}{a_2 + 2y_3 + 4y_4 - 2\sqrt{y_2}y_4 + 2y_3y_4 + y_3} \end{bmatrix}. \tag{3.28}$$

Now considering the initial integrals of motion $H_{1,2}$ for Lie group $\mathbf{A^1_{4,9}}$, the separated relations is as follows:

$$C : \alpha H_1 + H_2 - \beta + \sqrt{1 - a_1 - q_1} + 1 = 0.$$

Where

$$\alpha = \frac{(q_1 - q_2)(a_1 + q_2)}{a_1(a_2 - 3(1 + \sqrt{1 - a_1 - q_1})) + (1 + \sqrt{1 - a_1 - q_1})q_1p_2 - q_2(-a_2 + (1 + \sqrt{1 - a_1 - q_1})(3 + p_2)}\,,$$

$$\beta = \frac{(a_1 - 2(1 + \sqrt{1 - a_1 - q_1}) + q_1)(q_1 - q_2)p_2}{2(a_1(3 - a_2 + 3\sqrt{1 - a_1 - q_1}) - (1 + \sqrt{1 - a_1 - q_1})q_1p_2 + q_2(-a_2 + (1 + \sqrt{1 - a_1 - q_1})(3 + p_2)))}$$

3.8. **Lie group $\mathbf{A_{4,12}}$ :** For this Lie group we have the following Poisson bi-vector:

$$P' = \begin{bmatrix} 0 & 0 & y_2^2 - 2y_1y_3^2 & 2y_1y_2 \\ * & 0 & 0 & y_2^2 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}. \tag{3.29}$$

In this case, Control matrix for the Poisson bi-vector $P'$ (3.29) is as follows:

$$F = \begin{bmatrix} y_2^2 & 4e^{-2y_4}y_1y_2y_3^2 \\ 0 & y_2^2 - 2y_1y_3^2 \end{bmatrix}. \tag{3.30}$$

Variables of separation $q_{1,2}$ are eigenvalues the control matrix. For Lie group $A^{a,0}_{4,6}$, they are the roots of the following polynomial

$$A(\lambda) = \det(F - \lambda I) = \lambda^2 - 2\lambda y_2^2 + y_2^4 + 2\lambda y_1y_3^2 - 2y_1y_2^2y_3^2,$$

this is

$$q_1 = y_2^2, \qquad q_2 = y_2^2 - 2y_1y_3^2.$$

The corresponding momenta are defined by

$$p_1 = \frac{1}{2}\left(\frac{y_4}{y_2} - \frac{1}{y_3}\right), \qquad p_2 = \frac{1}{2}\left(y_2^2 + 2y_1y_3^2 + \frac{1}{y_3}\right).$$

In the separated variables, the Stackel matrix $S$ is equal to

$$S = \begin{bmatrix} 1 & 1 \\ 0 & -\frac{e^{2y_4}}{2y_2} \end{bmatrix}. \tag{3.31}$$

Now considering the initial integrals of motion $H_{1,2}$ for Lie group $\mathbf{A_{4,12}}$, the separated relations is as follows:

$$C : -\frac{e^{2\sqrt{q_1}(2p_1+2p_2-2q_1+q_2)}}{2\sqrt{q_1}}H_1 + H_2 \\ -\frac{1}{2}\left(-2+e^{2\sqrt{q_1}(2p_1+2p_2-2q_1+q_2)}\right)(2p_2-2q_1+q_2)=0$$

## 4. Conclusion

Starting with integrals of motion for integrable systems on 4-dimensional real Lie groups considered as Poisson manifold, we have found the Poisson bi-vectors which are compatible with the canonical Poisson bi-vector on Lie group space. In such Poisson bi-vectors, it is assumed that the Lie derivative of P along some unknown Liouville vector field X. As a result, the system of equations (2.3)-(2.5) has infinitely many solutions.

We are able to obtain a particular answer, assuming that the components of the Liouville vector field are second order polynomials in momenta. An application of the corresponding control matrices allows us to get a framework of the bi-Hamiltonian geometry using two families of variables of separation and of separated relations.

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## Appendix

Table 1. Integrable systems on four dimensional real Lie groups.

| system | nonzero Poisson brackets | Darboux coordinates | $H_1, H_2$ |
|---|---|---|---|
| $A_{4,1}$ | $\{x_1,x_2\} = -\frac{c}{2}x_4^2$<br>$\{x_1,x_3\} = cx_4$<br>$\{x_1,x_4\} = -d$<br>$\{x_2,x_3\} = -c$ | $y_1 = \frac{x_3}{c} + \frac{cx_4^2}{8} + \frac{x_4^2}{2d}$<br>$y_2 = -x_1 + \frac{x_3^2}{c^2} + \frac{1}{4}cdx_2x_4 - \frac{x_3x_4^2}{4}$<br>$-\frac{x_3x_4^2}{cd} - \frac{3c^2x_4^4}{64} + \frac{x_4^4}{4d^2} - \frac{cx_4^4}{8d}$<br>$y_3 = x_2 - \frac{2x_3x_4}{cd} - \frac{x_4^3}{d^2} - \frac{cx_4^3}{4d}$<br>$y_4 = \frac{x_4}{d}$ | $H_1 = -y_3$<br>$H_2 = -\frac{y_2^2y_3}{2}$ |
| $A_{4,2}^{-1}$ | $\{x_1,x_2\} = 2c$<br>$\{x_1,x_3\} = -c$<br>$\{x_2,x_4\} = -d\ e^{-x_4}$ | $y_1 = -\frac{e^{x_4}}{b} + x_3$<br>$y_2 = \frac{-2ae^{x_4} - bx_1 + abx_2}{ab^2}$<br>$y_3 = \frac{2e^{x_4}}{b} + \frac{x_1}{a}$<br>$y_4 = e^{x_4}$ | $H_1 = \frac{1}{y_2y_3^2}$<br>$H_2 = -y_2y_3$ |
| $A_{4,3}$ | $\{x_1,x_2\} = c\ x_4\ e^{-x_4}$<br>$\{x_1,x_3\} = d\ e^{-x_4}$<br>$\{x_1,x_4\} = h\ e^{-x_4}$<br>$\{x_2,x_3\} = f$ | $y_1 = \frac{dx_2}{f} + \frac{chx_3^2}{2df} - \frac{cx_3x_4}{f}$<br>$y_2 = \frac{x_1}{h} - \frac{de^{-x_4}x_2}{fh} - \frac{ce^{-x_4}x_3^2}{2df}$<br>$+\frac{ce^{-x_4}x_3x_4}{fh}$<br>$y_3 = \frac{x_3}{d}$<br>$y_4 = e^{x_4}$ | $H_1 = -y_3$<br>$H_2 = y_2y_3\ln\|y_2\|$ |
| $A_{4,6}^{a,0}$ | $\{x_1,x_4\} = d\ e^{-ax_4}$<br>$\{x_2,x_3\} = c$ | $y_1 = x_3$<br>$y_2 = -\frac{e^{2ax_4}x_1}{ad}$<br>$y_3 = -\frac{x_2}{c}$<br>$y_4 = e^{-ax_4}$ | $H_1 = e^{(\frac{2ay_2}{d^2})}y_3^2$<br>$H_2 =$<br>$-e^{\frac{ay_2}{d^2}}\, y_3\cos(e^{-\frac{y_2}{d^2}})$ |
| $A_{4,7}$ | $\{x_1,x_3\} = -2cx_3e^{-2x_4}$<br>$\{x_1,x_4\} = ce^{-2x_4}$<br>$\{x_2,x_3\} = 2ce^{-2x_4}$ | $y_1 = \frac{e^{2x_4}x_2}{2c}$<br>$y_2 = -\frac{-1-e^{2x_4}+e^{4x_4}x_1+e^{4x_4}x_2x_3}{2c}$<br>$y_3 = x_3$<br>$y_4 = e^{-2x_4}$ | $H_1 = -y_2y_3$<br>$H_2 = -y_3$ |
| $A_{4,9}^{1}$ | $\{x_1,x_3\} = 2cx_3e^{-2x_4}$<br>$\{x_1,x_4\} = -c\ e^{-2x_4}$<br>$\{x_2,x_3\} = -2c\ e^{-2x_4}$ | $y_1 = -\frac{e^{2x_4}x_2}{2c}$<br>$y_2 = \frac{-1-e^{2x_4}+e^{4x_4}x_1+e^{4x_4}x_2x_3}{2c}$<br>$y_3 = x_3$<br>$y_4 = e^{-2x_4}$ | $H_1 = -y_4$<br>$H_2 = -y_3$ |
| $A_{4,12}$ | $\{x_1,x_3\} = -ce^{-x_3}(a\cos(x_4)$<br>$+b\sin(x_4))$<br>$\{x_1,x_4\} = ce^{-x_3}(-b\cos(x_4)$<br>$+a\sin(x_4))$<br>$\{x_2,x_3\} = ce^{-x_3}(b\cos(x_4)$<br>$-a\sin(x_4))$<br>$\{x_2,x_4\} = -ce^{-x_3}(a\cos(x_4)$<br>$+b\sin(x_4))$ | $y_1 = e^{2x_3}(ax_1 - bx_2)\cos(x_4)$<br>$+e^{2x_3}(bx_1 + ax_2)\sin(x_4)$<br>$y_2 = -e^{x_3}(bx_1 + ax_2)\cos(x_4)$<br>$+(-ax_1 + bx_2)\sin(x_4)$<br>$y_3 = e^{x_3}$<br>$y_4 = x_4$ | $H_1 = -\frac{y_2}{y_3}$<br>$H_2 = -\frac{1}{y_3}$ |

TABLE 2. The Poisson bi-vector over four dimensional real Lie groups.

| system | Poisson bi-vector |
|---|---|
| $\mathbf{A_{4,1}}$ | $P' = \begin{bmatrix} 0 & -y_4 & -a_3+y_1 & 0 \\ * & 0 & 0 & -a_3+y_1 \\ * & * & 0 & -y_4 \\ * & * & * & 0 \end{bmatrix}$ |
| $\mathbf{A_{4,2}^{-1}}$ | $P' = \begin{bmatrix} 0 & \frac{e^{-(2a_1y_2)}y_3}{a_1} & \frac{e^{-(a_1y_2)}a_2}{a_1} - a_3 & -2y_3(a_4+f(y_2)) - y_3g(y_2) \\ * & 0 & 0 & -e^{-(2a_1y_2)}y_3^2 + \frac{e^{-(a_1y_2)}a_2}{a_1} - a_3 \\ * & * & 0 & -a_2e^{-(a_1y_2)}y_3 \\ * & * & * & 0 \end{bmatrix}$ |
| $\mathbf{A_{4,3}}$ | $P' = \begin{bmatrix} 0 & \frac{2a_6 - e^{a_1(y_2+a_2)}}{a_1} & -a_7 - 2a_6y_3 & 0 \\ * & 0 & 0 & \alpha \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}$ <br> $\alpha = -a_7 + 2a_6(y_4^2a_3 + a_4 + y_4a_5) - e^{a_1(y_2+a_2)}(y_3 + y_4^2a_3 + a_4 + y_4a_5)$ |
| $\mathbf{A_{4,6}^{a,0}}$ | $P' = \begin{bmatrix} 0 & 0 & y_2^2 + y_2y_3 & -a_1 - 2a_2y_3 - y_1y_3 - y_3^2f(y_2) \\ * & 0 & 0 & y_2^2 \\ * & * & 0 & 2y_2y_3 + y_3^2 \\ * & * & * & 0 \end{bmatrix}$ |
| $\mathbf{A_{4,7}}$ | $P' = \begin{bmatrix} 0 & \frac{e^{(-2a_1(y_2+a_2))}y_3}{a_1} & -a_3 - 2a_4y_3 & \frac{\alpha}{9a_6} \\ * & 0 & 0 & \beta \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}$ <br> $\alpha = e^{(-4y_2a_1)}(4a_{320}e^{(3y_2a_1)} + 3a_5)(2a_4e^{(3y_2a_1)}y_3 - 3(y_3a_5 + e^{(y_2a_1)}y_4a_6))$ <br> $\beta = -a_3 - (4a_4y_3)/3 - e^{(-2a_1(y_2+a_2))}y_3^2 - e^{(-3y_2a_1)}y_3a_5 - e^{(-2y_2a_1)}y_4a_6$ |
| $\mathbf{A_{4,9}^{1}}$ | $P' = \begin{bmatrix} 0 & 2y_2(2-\sqrt{y_2}+y_3) & -a_1 - y_3(2+y_3) & \alpha \\ * & 0 & 0 & -a_1 - \sqrt{y_2}y_3 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}$ <br> $\alpha = -a_2 - 2(y_3 + 2y_4 - \sqrt{y_2}y_4 + y_3y_4) - y_3$ |
| $\mathbf{A_{4,12}}$ | $P' = \begin{bmatrix} 0 & 0 & y_2^2 - 2y_1y_3^2 & 2y_1y_2 \\ * & 0 & 0 & y_2^2 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}$ |

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