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A Fixed Point Theorem for Weakly Contractive Mappings

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Abstract. In this paper, we generalize the concepts of weakly Kannan, weakly Chatterjea and weakly Zamfirescu for fuzzy metric spaces. Also, we investigate Banach's fixed point theorem for the mentioned classes of functions in these spaces. Moreover, we show that the class of weakly Kannan and weakly Chatterjea maps are subclasses of the class of weakly Zamfirescu maps.

1. Introduction

Banach's fixed point theorem is one of the most well-known analytical theorems. The theorem shows that every contractive map on complete metric space has a unique fixed point, where the contractive map is defined as follows:

Definition 1.1. Let (X, d) be a metric space, $D \subseteq X$, and $f : D \longrightarrow X$. the function f is said to be contractive, if there exists $\alpha \in [0, 1)$ such that

$$d\left(f(x), f(y)\right) \leq \alpha d\left(x, y\right), \tag{1.1}$$

for all $x, y \in D$.

Banach's fixed point theorem has been also proved under different contraction conditions. For instance, Kannan [11], Chatterjea [4], Ćirić [5] and Zamfirescu [21] established the theorem by replacing the condition (1.1) with the following conditions, respectively,

- $d\left(f(x), f(y)\right) \leq (\alpha/2)\left[d\left(x, f(x)\right) + d\left(y, f(y)\right)\right]$,
- $d\left(f(x), f(y)\right) \leq (\alpha/2)\left[d\left(x, f(y)\right) + d\left(y, f(x)\right)\right]$,

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- $d\left(f(x), f(y)\right) \leq \alpha \max\{d\left(x, y\right), d\left(x, f(x)\right), d\left(y, f(y)\right), (1/2)\left[d\left(x, f(y)\right) + d\left(y, f(x)\right)\right]\}$.
- $d\left(f(x), f(y)\right) \leq \alpha \max\{d\left(x, y\right), (1/2)\left[d\left(x, f(x)\right) + d\left(y, f(y)\right)\right], (1/2)\left[d\left(x, f(y)\right) + d\left(y, f(x)\right)\right]\}$.

We comment that the scalar α, in each of the above considered contraction conditions, can be replaced by a function. More precisely, given a nonincreasing function $\alpha(t)$, Rakotch [12] developed an extension of Banach's fixed point theorem. In addition, Banach's fixed point theorem extended for the weakly contractive maps, weakly Kannan, weakly Chatterjea and weakly Zamfirescu, for more details see [2, 3, 6].

Moreover, some fixed point theorems were studied in fuzzy metric spaces, L-fuzzy metric spaces, intuitionistic fuzzy metric spaces and Menger probabilistic metric space [1, 8, 13–20].

In this paper, we first introduce the concept of weakly Zamfirescu in fuzzy metric space. Then we show that every weakly Zamfirescu map has a unique fixed point and it is continuous at the point. Finally, we define the concepts of weakly Kannan and weakly Chatterjea in fuzzy metric spaces. Also, it is shown that these classes of maps are subclasses of the class of weakly Zamfirescu maps which implies the existence of fixed point for each of maps.

2. Preliminaries

In this section, we give some basic necessary preliminaries for this paper.

Definition 2.1 ([7]). A fuzzy metric space is a triple $(X, M, *)$, where X is a nonempty set, $*$ is a continuous t-norm and $M : X \times X \times (0, \infty) \longrightarrow [0, 1]$ is a mapping satisfying the following axioms:

- (FM1) $M\left(x, y, t\right) > 0$, for all $x, y \in X$;
- (FM2) $M\left(x, y, t\right) = 1$, for all $t > 0$ iff $x = y$;
- (FM3) $M\left(x, y, t\right) = M\left(y, x, t\right)$, for all $x, y \in X$ and $t > 0$;
- (FM4) $M\left(x, y, .\right) : (0, \infty) \longrightarrow [0, 1]$ is continuous, for all $x, y \in X$;
- (FM5) $M\left(x, z, t + s\right) \geq M\left(x, y, t\right) * M\left(y, z, s\right)$, for all $x, y, z \in X$, and $t, s > 0$.

We assume that
(FM6) $\lim\limits_{t \to \infty} M\left(x, y, t\right) = 1$, for all $x, y \in X$.

Lemma 2.2 ([7]). *Let $(X, M, *)$ be a fuzzy metric space. Then $M\left(x, y, .\right)$ is nondecreasing for all $x, y \in X$.*

Definition 2.3 ([10]). Let $(X, M, *)$ be a fuzzy metric space.

(i) A sequence $\{x_n\}$ in X is said to be Cauchy if for each $\epsilon \in (0,1)$ and $t > 0$ there exists $N \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1$ for all $m > n \geq N$.

(ii) The sequence $\{x_n\}$ in X is convergent if there exists $x \in X$ such that $\lim \lim_{n\to\infty} M(x_n, x, t) = 1$, for all $t > 0$.

Lemma 2.4 ([9]). *Let $(X, M, *)$ be a fuzzy metric space and $\{x_n\}$, $\{y_n\}$ be sequences in X. If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then*

$$\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t), \quad \textit{for all } t > 0.$$

3. Fixed Point Theorms

In this section, we first introduce weakly Zamfirescu maps on fuzzy metric spaces.

Definition 3.1. Let $(X, M, \min)$ be a fuzzy metric space, $D \subseteq X$, and $f : D \longrightarrow X$. The function f is said to be a weakly Zamfirescu map, if for every $\alpha \in (0,1)$, there exists $\zeta_\alpha : D \times D \longrightarrow (0,1]$ such that

$$\sup\left\{\zeta_\alpha(x,y) : a \leq \inf_{M(x,y,t)\geq\alpha} t \leq b\right\} < 1, \quad \text{for all } 0 < a \leq b.$$

Moreover, if

$$\begin{array}{lll} M(x,y,t_1) \geq \alpha, & M(x,f(x),t_2) \geq \alpha, & M(y,f(y),t_3) \geq \alpha, \\ M(y,f(x),t_4) \geq \alpha, & M(x,f(y),t_5) \geq \alpha, & \end{array}$$

then

$$M\left(f(x), f(y), \zeta_\alpha(x,y)\max\left(t_1, (1/2)(t_2+t_3), (1/2)(t_4+t_5)\right)\right) \geq \alpha,$$

for all $x, y \in D$ and all $t_1, \ldots, t_5 > 0$.

Example 3.2. Let (X, d) be a metric space, $D \subseteq X$ and $\zeta : D \times D \longrightarrow (0,1]$ be a function such that

$$\sup\{\zeta(x,y) : a \leq d(x,y) \leq b\} < 1, \quad \text{for every } 0 < a \leq b.$$

Moreover, let $f : D \longrightarrow X$ be a function such that, for all $x, y \in D$,

$$d(f(x), f(y)) \leq \zeta(x,y)\max\{s_1, s_2, s_3\},$$

where

$$\begin{aligned} s_1 &= d(x,y), \\ s_2 &= (1/2)\left[d(x,f(x)) + d(y,f(y))\right], \\ s_3 &= (1/2)\left[d(x,f(y)) + d(y,f(x))\right]. \end{aligned}$$

Define a fuzzy metric M as follows:

$$M(x,y,t) = \begin{cases} t/d(x,y), & 0 < t \leq d(x,y), \\ 1, & d(x,y) < t, \end{cases}$$

for all $x, y \in X$ and all $t > 0$.

Let $0 < a \leq b$ and $\alpha \in (0,1)$. We have

$$\sup\left\{\zeta(x,y) : a \leq \inf_{M(x,y,t)\geq\alpha} t \leq b\right\} = \sup\{\zeta(x,y) : a \leq \alpha d(x,y) \leq b\} < 1.$$

Suppose that $x, y \in X$, $t_1, \ldots, t_5 > 0$,

$$\begin{array}{lll} M(x,y,t_1) \geq \alpha, & M(x,f(x),t_2) \geq \alpha, & M(y,f(y),t_3) \geq \alpha, \\ M(y,f(x),t_4) \geq \alpha & M(x,f(y),t_5) \geq \alpha. & \end{array}$$

Hence

$$\begin{array}{lll} t_1/d(x,y) \geq \alpha, & t_2/d(f(x),x) \geq \alpha, & t_3/d(f(y),y) \geq \alpha, \\ t_4/d(f(x),y) \geq \alpha & t_5/d(f(y),x) \geq \alpha. & \end{array}$$

Therefore

$$\begin{array}{lll} t_1 \geq \alpha d(x,y), & t_2 \geq \alpha d(f(x),x), & t_3 \geq \alpha d(f(y),y), \\ t_4 \geq \alpha d(f(x),y) & t_5 \geq \alpha d(f(y),x). & \end{array}$$

Thus

$$\begin{aligned} \alpha d(f(x),(y)) &\leq \zeta(x,y)\max\{r_1,r_2,r_3\} \\ &\leq \zeta(x,y)\max\{t_1, 1/2[t_2+t_3], (1/2)[t_4+t_5]\}, \end{aligned}$$

where

$$\begin{aligned} r_1 &= \alpha d(x,y), \\ r_2 &= (1/2)[\alpha d(x,f(x)) + \alpha d(y,f(y))], \\ r_3 &= (1/2)[\alpha d(x,f(y)) + \alpha d(y,f(x))]. \end{aligned}$$

This implies that

$$M(f(x), f(y), \zeta(x,y)\max(t_1, (1/2)(t_2+t_3), (1/2)(t_4+t_5))) \geq \alpha.$$

Then $f : X \longrightarrow X$ is a weakly Zamfirescu map.

Example 3.3. We define $f : [0,1] \longrightarrow [0,1]$, fuzzy metric M and $\zeta : [0,1]\times[0,1] \longrightarrow (0,1]$ as follows:

$$f(x) = \begin{cases} (2/3)x, & 0 \leq x < 1, \\ 0, & x = 1, \end{cases}$$

$$\zeta(x,y) = \begin{cases} 4x/(3+x), & x \neq 0 \text{ and } y = 1, \\ 1/2, & x = 0 \text{ and } y = 1, \\ (4(y-x))/(x+y), & 0 < (2/3)y \leq x < y < 1, \\ 4/5, & 0 \leq x < (2/3)y, \end{cases}$$

$$M(x,y,t) = \begin{cases} t/|x-y|, & 0 < t \leq |x-y|, \\ 1, & |x-y| < t, \end{cases}$$

for all $x, y \in [0,1]$ and all $t > 0$.

In [3], it is shown that

$$|f(x) - f(y)| \leq (\zeta(x,y)/2)[|x - f(y)| + |y - f(x)|],$$

for all $x, y \in X$. Hence, for all $x, y \in X$,

$$|f(x) - f(y)| \leq \zeta(x, y) \max\{s_1, s_2, s_3\},$$

where

$$s_1 = |x - y|,$$
$$s_2 = (1/2)\left[|x - f(x)| + |y - f(y)|\right],$$
$$s_3 = (1/2)\left[|x - f(y)| + |y - f(x)|\right].$$

By Example 3.2, the function $f : [0,1] \longrightarrow [0,1]$ is a weakly Zamfirescu map.

Example 3.4. Let (X, d) be a metric space, $D \subseteq X$ and $\zeta : D \times D \longrightarrow (0,1]$ be a function such that

$$\sup\{\zeta(x, y) : a \leq d(x, y) \leq b\} < 1, \quad \text{for every } 0 < a \leq b.$$

Moreover, let $f : D \longrightarrow X$ be a function such that, for all $x, y \in D$,

$$d(f(x), f(y)) \leq \zeta(x, y) \max\{s_1, s_2, s_3\},$$

where

$$s_1 = d(x, y),$$
$$s_2 = (1/2)\left[d(x, f(x)) + d(y, f(y))\right],$$
$$s_3 = (1/2)\left[d(x, f(y)) + d(y, f(x))\right].$$

Define a fuzzy metric M as follows:

$$M(x, y, t) = t/\left(t + d(x, y)\right), \quad \text{for all } x, y \in X \text{ and all } t > 0.$$

Let $0 < a \leq b$ and $\alpha \in (0,1)$. We have

$$\begin{aligned} 1 &> \sup\{\zeta(x, y) : ((1-\alpha)/\alpha)\, a \leq d(x, y) \leq ((1-\alpha)/\alpha)\, b\} \\ &= \sup\{\zeta(x, y) : a \leq (\alpha/(1-\alpha))\, d(x, y) \leq b\} \\ &= \sup\left\{\zeta(x, y) : a \leq \inf_{M(x,y,t) \geq \alpha} t \leq b\right\}. \end{aligned}$$

Suppose that $x, y \in X$, $t_1, \ldots, t_5 > 0$,

$$\begin{array}{lll} M(x, y, t_1) \geq \alpha, & M(x, f(x), t_2) \geq \alpha, & M(y, f(y), t_3) \geq \alpha, \\ M(y, f(x), t_4) \geq \alpha, & M(x, f(y), t_5) \geq \alpha. & \end{array}$$

Hence

$$\begin{array}{ll} t_1/\left(t_1 + d(x, y)\right) \geq \alpha, & t_4/\left(t_4 + d(f(x), y)\right) \geq \alpha, \\ t_2/\left(t_2 d(f(x), x)\right) \geq \alpha, & t_5/\left(t_5 + d(f(y), x)\right) \geq \alpha. \\ t_3/\left(t_3 + d(f(y), y)\right) \geq \alpha, & \end{array}$$

Therefore

$$\begin{array}{ll} t_1 \geq (\alpha/(1-\alpha))\, d(x, y), & t_4 \geq (\alpha/(1-\alpha))\, d(f(x), y), \\ t_2 \geq (\alpha/(1-\alpha))\, d(f(x), x), & t_5 \geq (\alpha/(1-\alpha))\, d(f(y), x), \\ t_3 \geq (\alpha/(1-\alpha))\, d(f(y), y). & \end{array}$$

Thus

$$\begin{aligned}(\alpha/(1-\alpha))\, d(f(x),(y)) &\leq \zeta(x,y) \max\{s_1, s_2, s_3\} \\ &\leq \zeta(x,y) \max\{r_1, r_2, r_3\},\end{aligned}$$

where

$$\begin{aligned}&s_1 = (\alpha/(1-\alpha))\, d(x,y),\\ &s_2 = (1/2)[(\alpha/(1-\alpha))\, d(x,f(x)) + (\alpha/(1-\alpha))\, d(y,f(y))],\\ &s_3 = (1/2)[(\alpha/(1-\alpha))\, d(x,f(y)) + (\alpha/(1-\alpha))\, d(y,f(x))],\\ &r_1 = t_1,\\ &r_2 = (1/2)[t_2 + t_3],\\ &r_3 = (1/2)[t_4 + t_5].\end{aligned}$$

This implies that

$$M(f(x), f(y), \zeta(x,y) \max(t_1, (1/2)(t_2+t_3), (1/2)(t_4+t_5))) \geq \alpha.$$

Then $f : X \longrightarrow X$ is a weakly Zamfirescu map.

In the next Theorem, we show that if a weakly Zamfirescu map has a fixed point then it is unique.

Theorem 3.5. *Let* $(X, M, \min)$ *be a fuzzy metric space such that* M *satisfies (FM6),* $D \subseteq X$ *and* $f : D \longrightarrow X$ *be a weakly Zamfirescu map. Then* f *has at most one fixed point in* D.

Proof. Let $u, v \in D$ be fixed points of f, with $u \neq v$. Then there exists $t > 0$ such that $M(u,v,t) < 1$. Since $M(u,v,.) : (0,\infty) \longrightarrow [0,1]$ is continuous and $\lim_{t\to\infty} M(u,v,t) = 1$, there exists $\alpha \in (0,1)$ such that

$$\begin{aligned}\inf\{t > 0 : M(u,v,t) \geq \alpha\} &= t_0 \\ &> 0.\end{aligned}$$

Since $M(u,v,.) : (0,\infty) \longrightarrow [0,1]$ is continuous, $M(u,v,t_0) \geq \alpha$. We have

$$\begin{aligned}M(u,v,t_0) &= M(u,f(v),t_0) \\ &= M(v,f(u),t_0) \\ &\geq \alpha,\end{aligned}$$

and

$$\begin{aligned}M(u,f(u),t_0) &= M(v,f(v),t_0) \\ &= 1 \\ &\geq \alpha.\end{aligned}$$

Since f is a weakly Zamfirescu map, $M(f(u), f(v), \zeta_\alpha(u,v)\, t_0) \geq \alpha$.

Suppose that

$$\Gamma = \sup\left\{\zeta_\alpha(x,y) : t_0 \leq \inf_{M(x,y,t)\geq\alpha} t \leq t_0+1\right\}$$
$$< 1.$$

By lemma 2.2, we have

$$\begin{aligned} M(u,v,\Gamma t_0) &\geq M(u,v,\zeta_\alpha(u,v)\,t_0) \\ &= M(f(u),f(v),\zeta_\alpha(u,v)\,t_0) \\ &\geq \alpha, \end{aligned}$$

which is impossible. □

Theorem 3.6. *Let $(X,M,\min)$ be a fuzzy metric space such that M satisfies (FM6) and $f : X \longrightarrow X$ is a weakly Zamfirescu map. Then $\lim_{n\to\infty} M\left(f^n(x_0), f^{n+1}(x_0), t\right) = 1$, for all $t>0$ and all $x_0 \in X$.*

Proof. Let $x_0 \in X$, and $x_n = f(x_{n-1})$ for $n = 1,2,....$ We define

$$d_{\alpha,n} = \inf\{t>0 : M(x_n,x_{n-1},t)\geq\alpha\}, \quad \text{for all } n>0 \text{ and } \alpha\in(0,1).$$

We show that

$$d_{\alpha,n+1} \leq \zeta_\alpha(x_n,x_{n-1})\,d_{\alpha,n}, \quad \text{for all } n>0 \text{ and } \alpha\in(0,1).$$

Let $n>0$, $\alpha\in(0,1)$ and $\epsilon>0$. Then there exist $0<t_1\leq d_{\alpha,n}+\epsilon$ and $0<t_2\leq d_{\alpha,n+1}+\epsilon$ such that

$$M(x_n,x_{n-1},t_1)\geq\alpha, \qquad M(x_n,x_{n+1},t_2)\geq\alpha.$$

By (FM5), we have

$$\begin{aligned} M(x_{n-1},x_{n+1},t_1+t_2) &\geq \min\{M(x_n,x_{n-1},t_1), M(x_n,x_{n+1},t_2)\} \\ &\geq \alpha. \end{aligned}$$

Since f is a weakly Zamfirescu map,

$$M(x_n,x_{n+1},\zeta_\alpha(x_n,x_{n-1})\max(t_1,(1/2)(t_1+t_2)))\geq\alpha.$$

Thus

$$d_{\alpha,n+1}\leq\zeta_\alpha(x_n,x_{n-1})\max(t_1,(1/2)(t_1+t_2)).$$

If $\max(t_1,(1/2)(t_1+t_2)) = t_1$, then

$$\begin{aligned} d_{\alpha,n+1} &\leq \zeta_\alpha(x_n,x_{n-1})\,t_1 \\ &\leq \zeta_\alpha(x_n,x_{n-1})\,(d_{\alpha,n}+\epsilon). \end{aligned}$$

As $\epsilon\longrightarrow 0$, we obtain that $d_{\alpha,n+1}\leq\zeta_\alpha(x_n,x_{n-1})\,d_{\alpha,n}$. If

$$\max(t_1,(1/2)(t_1+t_2)) = (1/2)(t_1+t_2),$$

then

$$d_{\alpha,n+1}\leq(\zeta_\alpha(x_n,x_{n-1})/2)(t_1+t_2)$$

$$\leq \left(\zeta_\alpha\left(x_n, x_{n-1}\right)/2\right)\left(d_{\alpha,n} + \epsilon + d_{\alpha,n+1} + \epsilon\right).$$

As $\epsilon \longrightarrow 0$, we obtain that $d_{\alpha,n+1} \leq \left(\zeta_\alpha\left(x_n, x_{n-1}\right)/2\right)\left(d_{\alpha,n} + d_{\alpha,n+1}\right)$. So

$$d_{\alpha,n+1} \leq \zeta_\alpha\left(x_n, x_{n-1}\right) d_{\alpha,n}, \quad \text{for all } n > 0 \text{ and } \alpha \in (0,1).$$

This implies that the sequence $\{d_{\alpha,n}\}$ is nonincreasing, for all $\alpha \in (0,1)$. Let $d_\alpha = \inf\{d_{\alpha,n} : n > 0\}$, for all $\alpha \in (0,1)$. Now we show that $d_\alpha = 0$, for all $\alpha \in (0,1)$. Suppose that $\alpha \in (0,1)$ and $d_\alpha > 0$. Assume that

$$\begin{aligned}\Gamma_\alpha &= \sup\left\{\zeta_\alpha(x,y) : d_\alpha \leq \inf_{M(x,y,t)\geq\alpha} t \leq d_{\alpha,1}\right\} \\ &< 1.\end{aligned}$$

We have $d_{\alpha,n+1} \leq \zeta_\alpha\left(x_n, x_{n-1}\right) d_{\alpha,n} \leq \Gamma_\alpha d_{\alpha,n}$, for all $n > 0$. Therefore $d_\alpha \leq d_{\alpha,n+1} \leq \Gamma_\alpha^n d_{\alpha,1}$, which is impossible. Hence

$$\begin{aligned}\lim_{n\to\infty} d_{\alpha,n} &= \inf_{n>0} d_{\alpha,n} \\ &= d_\alpha \\ &= 0,\end{aligned}$$

for all $\alpha \in (0,1)$.

Let $\alpha \in (0,1)$ and $t > 0$. Since $\lim_{n\to\infty} d_{\alpha,n} = 0$, there exists $N > 0$ such that $d_{\alpha,n} < t$, for all $n \geq N$. Hence $M\left(x_n, x_{n+1}, t\right) \geq \alpha$, for all $n \geq N$. This implies that $\lim_{n\to\infty} M\left(x_n, x_{n+1}, t\right) = 1$. Therefore

$$\lim_{n\to\infty} M\left(x_n, f(x_n), t\right) = 1.$$

□

Theorem 3.7. *Let $(X, M, \min)$ be a fuzzy metric space such that M satisfies (FM6) and $f : X \longrightarrow X$ a weakly Zamfirescu map. If f has a fixed point $u \in X$. Then f is continuous at u.*

Proof. Let $\{x_n\}$ be a sequence converging to $u = f(u)$. We define

$$d_{\alpha,x_n,u} = \inf\left\{t > 0 : M\left(x_n, u, t\right) \geq \alpha\right\}, \quad \text{for all } n > 0 \text{ and } \alpha \in (0,1).$$

Now we show that

$$d_{\alpha,f(x_n),f(u)} \leq d_{\alpha,x_n,u}, \quad \text{for all } n > 0 \text{ and } \alpha \in (0,1).$$

Let $n > 0$, $\alpha \in (0,1)$ and $\epsilon > 0$. Then there exist $t_1 \leq d_{\alpha,x_n,u} + \epsilon$ and $t_2 \leq d_{\alpha,f(x_n),f(u)} + \epsilon$ such that $M\left(x_n, u, t_1\right) \geq \alpha$ and

$$\begin{aligned}M\left(f(x_n), u, t_2\right) &= M\left(f(x_n), f(u), t_2\right) \\ &\geq \alpha.\end{aligned}$$

By (FM5), we obtain that

$$\begin{aligned} M(x_n, f(x_n), t_1+t_2) &\geq \min\{M(x_n, u, t_1), M(f(x_n), u, t_2)\} \\ &\geq \alpha. \end{aligned}$$

Since f is a weakly Zamfirescu map,

$$M(f(x_n), f(u), \zeta_\alpha(x_n, x_{n-1}) \max(t_1, (1/2)(t_1+t_2))) \geq \alpha.$$

Thus

$$\begin{aligned} d_{\alpha, f(x_n), f(u)} &\leq \zeta_\alpha(x_n, x_{n-1}) \max(t_1, (1/2)(t_1+t_2)) \\ &\leq \max(t_1, (1/2)(t_1+t_2)). \end{aligned}$$

If $\max(t_1, (1/2)(t_1+t_2)) = t_1$, then $d_{\alpha, f(x_n), f(u)} \leq t_1 \leq d_{\alpha, x_n, u} + \epsilon$. As $\epsilon \longrightarrow 0$, we obtain that $d_{\alpha, f(x_n), f(u)} \leq d_{\alpha, x_n, u}$.

If $\max(t_1, (1/2)(t_1+t_2)) = (1/2)(t_1+t_2)$, then

$$\begin{aligned} d_{\alpha, f(x_n), f(u)} &\leq (1/2)(t_1+t_2) \\ &\leq (1/2)\left(d_{\alpha, x_n, u} + \epsilon + d_{\alpha, f(x_n), f(u)} + \epsilon\right). \end{aligned}$$

As $\epsilon \longrightarrow 0$, we obtain that $d_{\alpha, f(x_n), f(u)} \leq (1/2)\left(d_{\alpha, x_n, u} + d_{\alpha, f(x_n), f(u)}\right)$. Therefore

$$d_{\alpha, f(x_n), f(u)} \leq d_{\alpha, x_n, u}, \quad \text{for all } n > 0 \text{ and } \alpha \in (0,1).$$

Since $\lim_{n\to\infty} x_n = u$, it follows that $\lim_{n\to\infty} M(x_n, u, t) = 1$, for all $t > 0$. So $\lim_{n\to\infty} d_{\alpha, x_n, u} = 0$. Hence $\lim_{n\to\infty} d_{\alpha, f(x_n), f(u)} = 0$.

Let $\alpha \in (0,1)$ and $t > 0$. Since $\lim_{n\to\infty} d_{\alpha, f(x_n), f(u)} = 0$, there exists $N > 0$ such that $d_{\alpha, f(x_n), f(u)} < t$, for all $n \geq N$. Hence

$$M(f(x_n), f(u), t) \geq \alpha, \quad \text{for all } n \geq N.$$

This implies that $\lim_{n\to\infty} M(f(x_n), f(u), t) = 1$. Thus the sequence $\{f(x_n)\}$ converges to $f(u)$. Therefore, f is continuous at u. □

Now we show that every weakly Zamfirescu map on a fuzzy metric space has a fixed point.

Theorem 3.8. *Let $(X, M, \min)$ be a fuzzy metric space such that M satisfies (FM6) and $f : X \longrightarrow X$ a weakly Zamfirescu map. Then f has a fixed point $u \in X$. Moreover, for each $x_0 \in X$, the sequence $\{f^n(x_0)\}$ converges to u.*

Proof. Let $x_0 \in X$, and $x_n = f(x_{n-1})$ for $n = 1, 2, \ldots$. We define

$$d_{\alpha, n, m} = \inf\{t > 0 : M(x_n, x_m, t) \geq \alpha\},$$

for all $m, n > 0$ and $\alpha \in (0,1)$.

We show that

$$d_{\alpha, n+1, n+k+1} \leq \zeta_\alpha(x_n, x_{n+k})(d_{\alpha, n, n+k} + d_{\alpha, n, n+1}),$$

for all $k, n > 0$ and $\alpha \in (0, 1)$.

Let $k, n > 0$, $\alpha \in (0,1)$ and $\epsilon > 0$. Then there exist

$$0 < t_1 \leq d_{\alpha,n,n+1} + \epsilon, \qquad 0 < t_2 \leq d_{\alpha,n,n+k} + \epsilon$$

such that

$$M\left(x_n, x_{n+1}, t_1\right) \geq \alpha, \qquad M\left(x_n, x_{n+k}, t_2\right) \geq \alpha.$$

By proof of Theorem 3.6, the sequence $\{d_{\alpha,n,n+1}\}$ is nonincreasing. Hence $d_{\alpha,n+k,n+k+1} \leq d_{\alpha,n,n+1} \leq t_1$. Thus $M\left(x_{n+k}, x_{n+k+1}, t_1\right) \geq \alpha$. By (FM5), we obtain that

$$\begin{aligned} M\left(x_n, x_{n+k+1}, t_1 + t_2\right) &\geq \min\left\{M\left(x_n, x_{n+k}, t_2\right), M\left(x_{n+k}, x_{n+k+1}, t_1\right)\right\} \\ &\geq \alpha, \end{aligned}$$

and

$$\begin{aligned} M\left(x_{n+1}, x_{n+k}, t_1 + t_2\right) &\geq \min\left\{M\left(x_n, x_{n+1}, t_1\right), M\left(x_{n+k}, x_n, t_2\right)\right\} \\ &\geq \alpha. \end{aligned}$$

Since f is a weakly Zamfirescu map,

$$\begin{aligned} \alpha &\leq M\left(x_{n+1}, x_{n+k+1}, \zeta_\alpha\left(x_n, x_{n+k}\right) \max\left(t_1, t_2, t_1 + t_2\right)\right) \\ &= M\left(x_{n+1}, x_{n+k+1}, \zeta_\alpha\left(x_n, x_{n+k}\right)\left(t_1 + t_2\right)\right). \end{aligned}$$

Thus $d_{\alpha,n+k+1,n+1} \leq \zeta_\alpha\left(x_n, x_{n+k}\right)\left(t_1 + t_2\right)$. Therefore

$$\begin{aligned} d_{\alpha,n+k+1,n+1} &\leq \zeta_\alpha\left(x_n, x_{n+k}\right)\left(t_1 + t_2\right) \\ &\leq \zeta_\alpha\left(x_n, x_{n+k}\right)\left(d_{\alpha,n,n+k} + \epsilon + d_{\alpha,n,n+1} + \epsilon\right). \end{aligned}$$

As $\epsilon \longrightarrow 0$, we obtain that

$$d_{\alpha,n+1,n+k+1} \leq \zeta_\alpha\left(x_n, x_{n+k}\right)\left(d_{\alpha,n,n+k} + d_{\alpha,n,n+1}\right),$$

for all $k, n > 0$ and $\alpha \in (0, 1)$.

Now we show that

$$d_{\alpha,n,n+k+1} \leq d_{\alpha,n+1,n+k+1} + d_{\alpha,n,n+1}, \qquad \text{for all } k, n > 0 \text{ and } \alpha \in (0,1).$$

Let $k, n > 0$, $\alpha \in (0,1)$ and $\epsilon > 0$. Then there exist

$$0 < t_1 \leq d_{\alpha,n,n+1} + \epsilon, \qquad 0 < t_2 \leq d_{\alpha,n+1,n+k+1} + \epsilon$$

such that $M\left(x_n, x_{n+1}, t_1\right) \geq \alpha$ and $M\left(x_{n+1}, x_{n+k+1}, t_2\right) \geq \alpha$. By (FM5), we obtain that

$$\begin{aligned} M\left(x_n, x_{n+k+1}, t_1 + t_2\right) &\geq \min\left\{M\left(x_n, x_{n+1}, t_1\right), M\left(x_{n+k+1}, x_{n+1}, t_2\right)\right\} \\ &\geq \alpha. \end{aligned}$$

Thus

$$\begin{aligned} d_{\alpha,n,n+k+1} &\leq t_1 + t_2 \\ &\leq d_{\alpha,n+1,n+k+1} + d_{\alpha,n,n+1} + 2\epsilon. \end{aligned}$$

As $\epsilon \longrightarrow 0$, we get

$$d_{\alpha,n,n+k+1} \leq d_{\alpha,n+1,n+k+1} + d_{\alpha,n,n+1} \quad \text{for all } k,n>0 \text{ and } \alpha \in (0,1).$$

This implies that

$$\begin{aligned} d_{\alpha,n,n+k+1} &\leq d_{\alpha,n+1,n+k+1} + d_{\alpha,n,n+1} \\ &\leq \zeta_\alpha(x_n, x_{n+k}) d_{\alpha,n,n+k} + (\zeta_\alpha(x_n, x_{n+k}) + 1) d_{\alpha,n,n+1}, \end{aligned}$$

for all $k,n>0$ and $\alpha \in (0,1)$. Let $t_0>0$, $\alpha \in (0,1)$ and

$$\begin{aligned} \Gamma_\alpha &= \sup\left\{\zeta_\alpha(x,y) : t_0/2 \leq \inf_{M(x,y,t)\geq\alpha} t \leq t_0\right\} \\ &< 1. \end{aligned}$$

By proof of Theorem 3.6, $\lim_{n\to\infty} d_{\alpha,n,n+1} = 0$. Hence there exists $N>0$ such that $d_{\alpha,n,n+1} < (1-\Gamma_\alpha) t_0/4$, for all $n \geq N$. Assume that $n \geq N$. We will prove inductively that $d_{\alpha,n,n+k} < t_0$, for all $k \geq 1$. It is obvious for $k=1$, and assuming $d_{\alpha,n,n+k} < t_0$.

If $d_{\alpha,n,n+k} \leq t_0/2$. Therefore

$$\begin{aligned} d_{\alpha,n,n+k+1} &\leq \zeta_\alpha(x_n, x_{n+k}) d_{\alpha,n,n+k} + (\zeta_\alpha(x_n, x_{n+k}) + 1) d_{\alpha,n,n+1} \\ &< t_0. \end{aligned}$$

If $d_{\alpha,n,n+k} > t_0/2$. Then

$$\begin{aligned} d_{\alpha,n,n+k+1} &\leq \zeta_\alpha(x_n, x_{n+k}) d_{\alpha,n,n+k} + (\zeta_\alpha(x_n, x_{n+k}) + 1) d_{\alpha,n,n+1} \\ &< \Gamma_\alpha d_{\alpha,n,n+k} + (\Gamma_\alpha + 1)(1-\Gamma_\alpha) t_0/4 \\ &< \Gamma_\alpha t_0 + (1-\Gamma_\alpha) t_0/2 \\ &< t_0. \end{aligned}$$

Hence $d_{\alpha,n,m} < t_0$, for all $m,n \geq N$. Thus $M(x_n, x_m, t_0) \geq \alpha$, for all $m,n \geq N$. This implies that $\{x_n\}$ is a Cauchy sequence. Since $(X, M, \min)$ is complete, $\{x_n\}$ is convergent, say to $u \in X$.

Now we define

$$d_{\alpha,x,y} = \inf\{t>0 : M(x,y,t) \geq \alpha\},$$

for all $n>0, x,y \in X$ and $\alpha \in (0,1)$.

Let $\alpha \in (0,1)$. Now we show that there exists $N>0$ such that

$$d_{\alpha,f(x_n),f(u)} \leq (3/4) d_{\alpha,u,f(u)}, \text{ for all } n>N.$$

Let $\epsilon>0$. Then there exists $0<t_1 \leq d_{\alpha,f(u),u} + \epsilon$ such that

$$M(f(u), u, t_1) \geq \alpha.$$

Since $\lim_{n\to\infty} x_n = u$, it follows that

$$\begin{aligned} \lim_{n\to\infty} M(x_n, x_{n+1}, t_1/4) &= M(u,u,t_1/4) \\ &= 1, \end{aligned}$$

and $\lim_{n\to\infty} M(x_n, u, t_1/4) = 1$. Therefore there exists $N > 0$ such that $M(x_n, x_{n+1}, t_1/4) \geq \alpha$ and $M(x_n, u, t_1/4) \geq \alpha$, for all $n > N$. By (FM5), we have

$$\begin{aligned} M(x_n, f(u), t_1 + t_1/4) &\geq \min\{M(x_n, u, t_1/4), M(u, f(u), t_1)\} \\ &\geq \alpha, \end{aligned}$$

for all $n > N$. Since f is a weakly Zamfirescu map, for all $n > N$,

$$\begin{aligned} M(f(x_n), f(u), (3/4)t_1) &= M(f(x_n), f(u), s) \\ &\geq M(f(x_n), f(u), \zeta_\alpha(x_n, u)s) \\ &\geq \alpha, \end{aligned}$$

where $s = \max(t_1/4, (1/2)(t_1/4 + t_1/4), (1/2)(t_1/4 + t_1 + t_1/4))$. Thus $d_{\alpha,f(x_n),f(u)} \leq (3/4)t_1 \leq (3/4)d_{\alpha,u,f(u)} + \epsilon$, for all $n > N$. As $\epsilon \longrightarrow 0$, we obtain that

$$d_{\alpha,f(x_n),f(u)} \leq (3/4)d_{\alpha,u,f(u)}, \qquad \text{for all } n > N.$$

Therefore $M\left(x_{n+1}, f(u), (3/4)d_{\alpha,u,f(u)}\right) \geq \alpha$, for all $n > N$. So

$$\begin{aligned} M\left(u, f(u), (3/4)d_{\alpha,u,f(u)}\right) &= \lim_{n\to\infty} M\left(x_{n+1}, f(u), (3/4)d_{\alpha,u,f(u)}\right) \\ &\geq \alpha. \end{aligned}$$

Hence $d_{\alpha,u,f(u)} \leq (3/4)d_{\alpha,u,f(u)}$. Therefore

$$d_{\alpha,u,f(u)} = 0, \qquad \text{for all } \alpha \in (0,1).$$

This implies that $M(u, f(u), t) \geq \alpha$, for all $t > 0$ and all $\alpha \in (0,1)$. So $u = f(u)$. □

Now we introduce weakly Kannan and weakly Chatterjea maps on fuzzy metric spaces and show that these class of functions are a subclass of the class of weakly Zamfirescu maps. Therefore, the classes of maps have a fixed point.

Definition 3.9. Let $(X, M, *)$ be a fuzzy metric space, $D \subseteq X$, and $f : D \longrightarrow X$. The function f is said to be a weakly Kannan map, if for every $\alpha \in (0,1)$, there exists $\vartheta_\alpha : D \times D \longrightarrow (0,1]$ such that

$$\sup\left\{\vartheta_\alpha(x,y) : a \leq \inf_{M(x,y,t)\geq\alpha} t \leq b\right\} < 1, \qquad \text{for all } 0 < a \leq b.$$

Moreover, if $M(x, f(x), t) \geq \alpha$ and $M(y, f(y), s) \geq \alpha$, then

$$M(f(x), f(y), (\vartheta_\alpha(x,y)/2)(t+s)) \geq \alpha,$$

for all $x, y \in D$ and all $t, s > 0$.

Theorem 3.10. *Let* $(X, M, \min)$ *be a fuzzy metric space,* $D \subseteq X$ *and* $f : D \longrightarrow X$ *be a weakly Kannan map. Then* $f : D \longrightarrow X$ *is a weakly Zamfirescu map.*

Proof. Since $f : D \longrightarrow X$ is a weakly Kannan map, for every $\alpha \in (0, 1)$, there exists $\vartheta_\alpha : D \times D \longrightarrow (0, 1]$ such that

$$\sup \left\{ \vartheta_\alpha (x, y) : a \leq \inf_{M(x,y,t) \geq \alpha} t \leq b \right\} < 1, \quad \text{for all } 0 < a \leq b,$$

moreover, if $M(x, f(x), t) \geq \alpha$, $M(y, f(y), s) \geq \alpha$, then

$$M(f(x), f(y), (\vartheta_\alpha (x, y)/2)(t + s)) \geq \alpha,$$

for all $x, y \in D$ and all $s, t > 0$.

Let $\alpha \in (0, 1)$, $x, y \in D$ and $t_1, \ldots, t_5 > 0$. Assume that

$$M(x, y, t_1) \geq \alpha, \qquad M(x, f(x), t_2) \geq \alpha, \qquad M(y, f(y), t_3) \geq \alpha,$$
$$M(y, f(x), t_4) \geq \alpha, \qquad M(x, f(y), t_5) \geq \alpha.$$

Since f is a weakly Kannan map,

$$M(f(x), f(y), \vartheta_\alpha (x, y)(1/2)(t_2 + t_3)) \geq \alpha.$$

Hence

$$\begin{aligned} \alpha &\leq M(f(x), f(y), \vartheta_\alpha (x, y)(1/2)(t_2 + t_3)) \\ &\leq M(f(x), f(y), \vartheta_\alpha (x, y) \max(t_1, (1/2)(t_2 + t_3), (1/2)(t_4 + t_5))). \end{aligned}$$

This implies that $f : D \longrightarrow X$ is a weakly Zamfirescu map. □

Corollary 3.11. *Let* $(X, M, \min)$ *be a complete fuzzy metric space such that* M *satisfies (FM6) and* $f : X \longrightarrow X$ *be a weakly Kannan map. Then* f *has a unique fixed point* $u \in X$ *and* f *is continuous at* u*. Moreover, for each* $x_0 \in X$*, the sequence* $\{f^n(x_0)\}$ *converges to* u*.*

Definition 3.12. Let $(X, M, *)$ be a fuzzy metric space, $D \subseteq X$, and $f : D \longrightarrow X$. The function f is said to be a weakly Chatterjea map, if for every $\alpha \in (0, 1)$, there exists $\xi_\alpha : D \times D \longrightarrow (0, 1]$ such that

$$\sup \left\{ \xi_\alpha (x, y) : a \leq \inf_{M(x,y,t) \geq \alpha} t \leq b \right\} < 1, \quad \text{for all } 0 < a \leq b.$$

Moreover, if $M(x, f(y), t) \geq \alpha$ and $M(y, f(x), s) \geq \alpha$, then

$$M(f(x), f(y), (\xi_\alpha (x, y)/2)(t + s)) \geq \alpha,$$

for all $x, y \in D$ and all $s, t > 0$.

Theorem 3.13. *Let* $(X, M, \min)$ *be a fuzzy metric space* $D \subseteq X$ *and* $f : D \longrightarrow X$ *be a weakly Chatterjea map. Then* $f : D \longrightarrow X$ *is a weakly Zamfirescu map.*

Proof. Since $f : D \longrightarrow X$ is a weakly Chatterjea map, for every $\alpha \in (0,1)$, there exists $\xi_\alpha : D \times D \longrightarrow (0,1]$ such that

$$\sup\left\{\xi_\alpha(x,y) : a \le \inf_{M(x,y,t)\ge\alpha} t \le b\right\} < 1, \quad \text{for all } 0 < a \le b.$$

Moreover, if $M(x, f(x), t) \ge \alpha$, $M(y, f(y), s) \ge \alpha$, then

$$M\left(f(x), f(y), \left(\xi_\alpha(x,y)/2\right)(t+s)\right) \ge \alpha,$$

for all $x, y \in D$ and all $s, t > 0$.

Let $\alpha \in (0,1)$, $x, y \in D$ and $t_1, \dots, t_5 > 0$. Suppose that

$$M(x,y,t_1) \ge \alpha, \qquad M(x, f(x), t_2) \ge \alpha, \qquad M(y, f(y), t_3) \ge \alpha,$$

$$M(y, f(x), t_4) \ge \alpha, \qquad M(x, f(y), t_5) \ge \alpha.$$

Since f is a weakly Chatterjea map,

$$M\left(f(x), f(y), \xi_\alpha(x,y)(1/2)(t_4+t_5)\right) \ge \alpha.$$

Hence

$$\begin{aligned}\alpha &\le M\left(f(x), f(y), \xi_\alpha(x,y)(1/2)(t_4+t_5)\right)\\ &\le M\left(f(x), f(y), \xi_\alpha(x,y)\max\left(t_1, (1/2)(t_2+t_3), (1/2)(t_4+t_5)\right)\right).\end{aligned}$$

This implies that $f : D \longrightarrow X$ is a weakly Zamfirescu map. □

Corollary 3.14. *Let $(X, M, \min)$ be a complete fuzzy metric space such that M satisfies (FM6) and $f : X \longrightarrow X$ be a weakly Chatterjea map. Then f has a unique fixed point $u \in X$ and f is continuous at u. Moreover, for each $x_0 \in X$, the sequence $\{f^n(x_0)\}$ converges to u.*

4. Conclusions

Nowadays, fixed point and operator theory play an important role in different areas of mathematics and its applications, particularly in mathematics, physics, differential equation, game theory, and dynamic programming. Since fuzzy mathematics and fuzzy physics along with the classical ones are constantly developing, the fuzzy type of the fixed point and operator theory can also play an important role in the new fuzzy area and fuzzy mathematical physics. We think that this paper could be of interest to the researchers working in the field of fuzzy functional analysis in particular, fuzzy approximate fixed point theory. We proved results about Banach's fixed point theorem for classes of functions on fuzzy metric space such as weakly Kannan, weakly Chatterjee, and weakly Zamfirescu.

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