On Some Coupled Fixed Point Theorems with Rational Expressions in Partially Ordered Metric Spaces

N. Seshagiri Rao and K. Kalyani

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 18
Number: 1
Pages: 123-136

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2020.120323.739

Sahand Communications in Mathematical Analysis (SCMA) Vol. 18 No. 1 (2021), 123-136
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2020.120323.739

On Some Coupled Fixed Point Theorems with Rational Expressions in Partially Ordered Metric Spaces

N. Seshagiri Rao[1*] and K. Kalyani[2]

Abstract. The aim of this paper is to prove some coupled fixed point theorems of a self mapping satisfying a certain rational type contraction along with strict mixed monotone property in an ordered metric space. Further, a result is presented for the uniqueness of a coupled fixed point under an order relation in a space. These results generalize and extend known existing results in the literature.

1. Introduction

The celebrated Banach contraction principle plays a pivotal role in obtaining a unique solution of many existing results of analysis. It is the most important and prominent implementation in acquiring solutions of several problems in linear and nonlinear analysis. Later, this principle is extended and generalized by many researchers using various forms of contractive conditions in a space, see [16, 18, 20, 33, 36, 38, 40] and references therein. Furthermore, numerous generalizations of this theorem have been obtained, by weakening its hypotheses on various spaces such as rectangular, cone, quasi, quasi semi, fuzzy, pseudo, probabilistic metric spaces and also in D, G, F-metric spaces etc. Lakzian et al. [25] generalized $\alpha-\psi$ contractive maps by introducing a w-distance to obtain fixed point theorems in a metric space. In [17] Gopal et al. used $\alpha-F$ contractive map in a complete metric space to acquire a unique fixed point to the map and also gave some applications to nonlinear fractional differential equations. Also, Shukla et al. [37] introduced the

2010 *Mathematics Subject Classification.* 47H10, 55M20, 54H25, 26A42.
Key words and phrases. Ordered metric space, Monotone property, Rational type contraction, Coupled fixed point.
Received: 18 January 2020, Accepted: 19 December 2020.
* Corresponding author.

notion of set valued $\alpha - F$-contractions and proved corresponding fixed point theorems in complete metric space. Further, their results are applied to nonlinear integral type equations for the existence of unique solution.

First, the generalized result of contraction principle in the direction of ordered sets was obtained by Wolk [39], afterwards Monjardet [27]. While, Ran and Reurings [32] found fixed point results in ordered metric spaces and then presented relevant applications of matrix linear equations therein. Subsequently, the extended result of [32] was given by Nieto et al. [28–30] taking nondecreasing mappings and applied to first order ordinary differential equations with periodic boundary conditions for the unique solution. Agarwal et al. [1] investigated fixed point results in ordered metric space of generalized contractive conditions. Extensive works on acquiring fixed points of various contractive conditions in ordered metric spaces, can be found [2–7, 9, 10, 15, 19, 21, 31, 34, 41]. Bhaskar and Lakshmikantham [11] introduced the concept of coupled fixed points in ordered spaces and then applied to boundary value problems for the unique solution. Lakshmikanthm and Cirić [24] introduced the concept of coupled coincidence, common fixed points to nonlinear contractions in ordered metric spaces. More results on coupled coincidence, common fixed points in ordered metric spaces, can be found [8, 12–14, 22, 23, 26, 35].

The work presented in this paper, mainly extends and generalizes a result of Sharma and Yuel [36] in ordered metric space by incorporating a strict mixed monotone property for a mapping in acquiring a coupled fixed point and its uniqueness.

2. Mathematical Preliminaries

The following definitions are frequently used in our study.

Definition 2.1 ([12]). Let $(P, \leq)$ be a partially ordered set. A map $S : P \to P$ is called strictly increasing or strictly decreasing, if $S(\tau) < S(\xi)$ or $S(\tau) > S(\xi)$ for all $\tau, \xi \in P$ with $\tau < \xi$.

Definition 2.2 ([12]). A map $S : P \times P \to P$ is said to have strict mixed monotone property on $(P, \leq)$, if $S(\tau, \xi)$ is strictly increasing in τ and strictly decreasing in ξ. i.e.,

$$\text{for all } \tau_1, \tau_2 \in P \text{ with } \tau_1 < \tau_2 \quad \Rightarrow \quad S(\tau_1, \xi) < S(\tau_2, \xi)$$

and

$$\text{for all } \xi_1, \xi_2 \in P \text{ with } \xi_1 < \xi_2 \quad \Rightarrow \quad S(\tau, \xi_1) > S(\tau, \xi_2).$$

Definition 2.3 ([12])**.** Let $S : P \times P \to P$ be a map on $(P, \leq)$. A point $(c^*, d^*) \in P \times P$ is called a coupled fixed point to S, if $S(c^*, d^*) = c^*$ and $S(d^*, c^*) = d^*$.

Definition 2.4 ([7])**.** A metric d on P together with a partially ordered relation $\leq$ is called a partially ordered metric space. It is represented by $(P, d, \leq)$.

Definition 2.5 ([7])**.** If the metric d is complete then $(P, d, \leq)$ is known as a complete partially ordered metric space.

Definition 2.6 ([7])**.** If every convergent sequence $\{\tau_n\}_{n=0}^{\infty}$ in P satisfies the following property then $(P, d, \leq)$ is known as an ordered complete (OC):

(i). If $\{\tau_n\}$ is a nondecreasing sequence in P with $\tau_n \to \tau$ then $\tau_n \leq \tau$, for all $n \in \mathbb{N}$, i.e., $\tau = \sup\{\tau_n\}$ or,

(ii). suppose $\{\tau_n\}$ is a non increasing sequence in P with $\tau_n \to \tau$ then $\tau \leq \tau_n$, for all $n \in \mathbb{N}$, i.e., $\tau = \inf\{\tau_n\}$.

3. Main Results

We begin this section with the following theorem.

Theorem 3.1. *Let $(P, \leq)$ be a partially ordered set and d a complete metric on P. Let $S : P \times P \to P$ be a mapping which has the strict mixed monotone property on P satisfying*

$$\begin{aligned}(3.1)\quad d(S(\tau,\xi), S(\mu,\upsilon)) \leq{}& \alpha \frac{d(\tau, S(\tau,\xi))\,[1 + d(\mu, S(\mu,\upsilon))]}{1 + d(\tau,\mu)} \\ &+ \beta[d(\tau, S(\tau,\xi)) + d(\mu, S(\mu,\upsilon))] \\ &+ \gamma[d(\tau, S(\mu,\upsilon)) + d(\mu, S(\tau,\xi))] + \delta d(\tau,\mu),\end{aligned}$$

for all $\tau, \xi, \mu, \upsilon \in P$ such that $\tau \geq \mu$ and $\xi \leq \upsilon$ and there exist $\alpha, \beta, \gamma, \delta \geq 0$ such that $0 \leq \alpha + 2(\beta + \gamma) + \delta < 1$. Suppose either S is continuous or P has an ordered complete property (OC), then S has a coupled fixed point $(\tau, \xi) \in P \times P$, if there exist two points $\tau_0, \xi_0 \in P$ with $S(\tau_0, \xi_0) > \tau_0$ and $S(\xi_0, \tau_0) < \xi_0$.

Proof. We suppose that S is continuous on P. Let τ_0, ξ_0 be two points in P such that $S(\tau_0, \xi_0) > \tau_0$ and $S(\xi_0, \tau_0) < \xi_0$, then define two sequences $\{\tau_n\}, \{\xi_n\}$ for $n \geq 0$ respectively as

$$(3.2)\qquad \tau_{n+1} = S(\tau_n, \xi_n), \qquad \xi_{n+1} = S(\xi_n, \tau_n).$$

Now, using mathematical induction, we prove the following for any n,

$$(3.3)\qquad \tau_n < \tau_{n+1},$$

and

$$\xi_n > \xi_{n+1}. \tag{3.4}$$

Assume $n = 0$, then from equation (3.2), we get $\tau_1 = S(\tau_0, \xi_0) > \tau_0$ and $\xi_1 = S(\xi_0, \tau_0) < \xi_0$ as $S(\tau_0, \xi_0) > \tau_0$ and $S(\xi_0, \tau_0) < \xi_0$ and hence the above inequalities hold. Next, we assume that for some $n > 0$, the inequalities (3.3) and (3.4) hold. Then from strict mixed monotone of S, we have

$$\begin{aligned} \tau_{n+1} &= S(\tau_n, \xi_n) \\ &< S(\tau_{n+1}, \xi_n) \\ &< S(\tau_{n+1}, \xi_{n+1}) \\ &= \tau_{n+2}, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \xi_{n+1} &= S(\xi_n, \tau_n) \\ &> S(\xi_{n+1}, \tau_n) \\ &> S(\xi_{n+1}, \tau_{n+1}) \\ &= \xi_{n+2}. \end{aligned} \tag{3.6}$$

Hence, we conclude that for all $n \geq 0$, the inequalities (3.3) and (3.4) are satisfied. Thus,

$$\tau_0 < \tau_1 < \cdots < \tau_n < \tau_{n+1} < \cdots, \tag{3.7}$$

and

$$\xi_0 > \xi_1 > \cdots > \xi_n > \xi_{n+1} > \cdots. \tag{3.8}$$

Then from (3.1) and (3.2), we get

$$\begin{aligned} d(\tau_n, \tau_{n-1}) &= d(S(\tau_{n-1}, \xi_{n-1}), S(\tau_{n-2}, \xi_{n-2})) \\ &\leq \alpha \frac{d(\tau_{n-1}, S(\tau_{n-1}, \xi_{n-1}))\,[1 + d(\tau_{n-2}, S(\tau_{n-2}, \xi_{n-2}))]}{1 + d(\tau_{n-1}, \tau_{n-2})} \\ &\quad + \beta[d(\tau_{n-1}, S(\tau_{n-1}, \xi_{n-1})) + d(\tau_{n-2}, S(\tau_{n-2}, \xi_{n-2}))] \\ &\quad + \gamma[d(\tau_{n-1}, S(\tau_{n-2}, \xi_{n-2})) + d(\tau_{n-2}, S(\tau_{n-1}, \xi_{n-1}))] \\ &\quad + \delta d(\tau_{n-1}, \tau_{n-2}), \end{aligned}$$

which implies that

$$\begin{aligned} d(\tau_n, \tau_{n-1}) \leq \alpha\ & \frac{d(\tau_{n-1}, \tau_n)\,[1 + d(\tau_{n-2}, \tau_{n-1})]}{1 + d(\tau_{n-1}, \tau_{n-2})} \\ & + \beta[d(\tau_{n-1}, \tau_n) + d(\tau_{n-2}, \tau_{n-1})] \\ & + \gamma[d(\tau_{n-1}, \tau_{n-1}) + d(\tau_{n-2}, \tau_n)] + \delta d(\tau_{n-1}, \tau_{n-2}). \end{aligned}$$

Finally, we arrive at

$$d(\tau_n, \tau_{n-1}) \leq h d(\tau_{n-1}, \tau_{n-2}), \tag{3.9}$$

where $h = \frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma} \in (0, 1]$. Similarly using the same argument as above, we obtain

$$d(\xi_n, \xi_{n-1}) \leq h d(\xi_{n-1}, \xi_{n-2}). \tag{3.10}$$

Therefore, from inequalities (3.9) and(3.10), we get

$$d(\tau_n, \tau_{n-1}) + d(\xi_n, \xi_{n-1}) \leq h \left[d(\tau_{n-1}, \tau_{n-2}) + d(\xi_{n-1}, \xi_{n-2})\right].$$

Now, let us define a sequence $\{\sigma_{n-1}\} = \{d(\tau_n, \tau_{n-1}) + d(\xi_n, \xi_{n-1})\}$. By induction we obtain

$$0 \leq \sigma_{n-1} \leq h\sigma_{n-2} \leq h^2\sigma_{n-3} \leq h^3\sigma_{n-4} \leq \cdots \leq h^{n-1}\sigma_0.$$

Thus,

$$\begin{aligned}
\lim_{n\to+\infty} \sigma_{n-1} &= \lim_{n\to+\infty} \left[d(\tau_n, \tau_{n-1}) + d(\xi_n, \xi_{n-1})\right] \\
&= 0.
\end{aligned}$$

Consequently, we get $\lim_{n\to+\infty} d(\tau_n, \tau_{n-1}) = 0$ and $\lim_{n\to+\infty} d(\xi_n, \xi_{n-1}) = 0$. Further, using triangular inequality of d for $n \leq m$, we get

$$d(\tau_m, \tau_n) \leq d(\tau_m, \tau_{m-1}) + d(\tau_{m-1}, \tau_{m-2}) + \cdots + d(\tau_{n+1}, \tau_n),$$

and

$$d(\xi_m, \xi_n) \leq d(\xi_m, \xi_{m-1}) + d(\xi_{m-1}, \xi_{m-2}) + \cdots + d(\xi_{n+1}, \xi_n).$$

Therefore,

$$\begin{aligned}
d(\tau_m, \tau_n) + d(\xi_m, \xi_n) &\leq \sigma_{m-1} + \sigma_{m-2} + \cdots + \sigma_n \\
&\leq \left(h^{m-1} + h^{m-2} + \cdots + h^n\right)\sigma_0 \\
&\leq \frac{h^n}{1-h}\sigma_0,
\end{aligned}$$

as $m, n \to \infty$, $d(\tau_m, \tau_n) + d(\xi_m, \xi_n) \to 0$. Hence, the sequences $\{\tau_n\}$ and $\{\xi_n\}$ are Cauchy in P and thus converges to a point $(\tau, \xi) \in P \times P$ so that $\tau_n \to \tau$ and $\xi_n \to \xi$. Also, by the continuity of S, we get

$$\begin{aligned}
\tau &= \lim_{n\to+\infty} \tau_{n+1} \\
&= \lim_{n\to+\infty} S(\tau_n, \xi_n) \\
&= S\left(\lim_{n\to+\infty} \tau_n, \lim_{n\to+\infty} \xi_n\right) \\
&= S(\tau, \xi),
\end{aligned}$$

and

$$\begin{aligned}\xi &= \lim_{n\to+\infty} \xi_{n+1} \\ &= \lim_{n\to+\infty} S(\xi_n, \tau_n) \\ &= S\left(\lim_{n\to+\infty} \xi_n, \lim_{n\to+\infty} \tau_n\right) \\ &= S(\xi, \tau).\end{aligned}$$

Therefore, S has a coupled fixed point in $P \times P$.

Another way, suppose that P has an ordered complete property (OC). From above, a non-decreasing Cauchy sequence $\{\tau_n\} \subset P$ exists and tends to τ. Hence, from (OC) property of P, we have $\tau = \sup\{\tau_n\}$, i.e., $\tau_n \leq \tau$, for all $n \in \mathbb{N}$. Therefore, we claim that $\tau_n < \tau$, for all n except there are some $n_0 \in \mathbb{N}$ with $\tau_{n_0} = \tau$, and thence $\tau < \tau_{n_0} \leq \tau_{n_0+1} = \tau$ which is a contradiction. Thus, from the strict monotone increasing property of S on first variable, we get

$$S(\tau_n, \xi_n) < S(\tau, \xi_n). \tag{3.11}$$

Similarly, from the above discussion there is a nonincreasing Cauchy sequence $\{\xi_n\}$ in P, which tends to a point $\xi \in P$. Thus, by (OC) property of P, we have $\xi = \inf\{\xi_n\}$, i.e., $\xi_n \geq \xi$, for all $n \in \mathbb{N}$. Using a similar argument as above, we have $\xi_n > \xi$, for all n. Also, from the strict monotone decreasing property of S on second variable, we get

$$S(\tau, \xi_n) < S(\tau, \xi). \tag{3.12}$$

Therefore, from inequalities (3.11) and (3.12), we get

$$S(\tau_n, \xi_n) < S(\tau, \xi) \quad \Rightarrow \quad \tau_{n+1} < S(\tau, \xi), \text{ for all } n. \tag{3.13}$$

But $\tau_n < \tau_{n+1} < S(\tau, \xi)$, for all n and $\tau = \sup\{\tau_n\}$, implies that $\tau \leq S(\tau, \xi)$. Now, construct a sequence $\{z_n\}$ in P as $z_{n+1} = S(z_n, \xi_n)$ with $z_0 = \tau$. Then using a similar argument as above, $\{z_n\}$ is a nondecreasing Cauchy sequence and tends to z in P, which implies that $z = \sup\{z_n\}$. As $\tau_n \leq \tau = z_0 \leq S(z_0, \xi_0) \leq z_n \leq z$ for all n, thus from (3.1), we get

$$\begin{aligned} d(\tau_n, z_n) &= d(S(\tau_{n-1}, \xi_{n-1}), S(z_{n-1}, \xi_{n-1})) \\ &\leq \alpha \frac{d(\tau_{n-1}, S(\tau_{n-1}, \xi_{n-1}))\,[1 + d(z_{n-1}, S(z_{n-1}, \xi_{n-1}))]}{1 + d(\tau_{n-1}, z_{n-1})} \\ &\quad + \beta[d(\tau_{n-1}, S(\tau_{n-1}, \xi_{n-1})) + d(z_{n-1}, S(z_{n-1}, \xi_{n-1}))] \\ &\quad + \gamma[d(\tau_{n-1}, S(z_{n-1}, \xi_{n-1})) + d(z_{n-1}, S(\tau_{n-1}, \xi_{n-1}))] \\ &\quad + \delta d(\tau_{n-1}, z_{n-1}). \end{aligned} \tag{3.14}$$

Letting $n \to \infty$ in (3.14), we acquire

$$d(\tau, z) \leq (2\gamma + \delta) d(\tau, z).$$

Consequently, we get $d(\tau, z) = 0$ as $2\gamma + \delta < 1$. Thus, $\tau = z = \sup\{\tau_n\}$ and hence $\tau \leq S(\tau, \xi) \leq \tau$. Therefore, $S(\tau, \xi) = \tau$. By the similar discussion as above, we have $S(\xi, \tau) = \xi$, which shows the existence of a coupled fixed to S in P. □

If P has the following partial order relation, then S attains a unique coupled fixed point in P.

$$(c^*, d^*) \leq (\tau, \xi) \quad \Leftrightarrow \quad \tau \geq c^*, \qquad \xi \leq d^*, \text{ for } (\tau, \xi), (c^*, d^*) \in P \times P.$$

Theorem 3.2. *In addition to Theorem 3.1, if for all $(\tau, \xi), (r, s) \in P \times P$, there exists $(c^*, d^*) \in P \times P$ such that $(S(c^*, d^*), S(d^*, c^*))$ is comparable to $(S(\tau, \xi), S(\xi, \tau))$ and $(S(r, s), S(s, r))$ then S has a unique coupled fixed point in $P \times P$.*

Proof. By Theorem 3.1, S has a non-empty set of coupled fixed points. Assume (τ, ξ), (r, s) are coupled fixed points of S, i.e., $S(\tau, \xi) = \tau, S(\xi, \tau) = \xi, S(r, s) = r$ and $S(s, r) = s$. Then, for uniqueness we have to prove that $\tau = r, \xi = s$.

From hypotheses, there exists a point $(c^*, d^*) \in P \times P$ such that $(S(c^*, d^*), S(d^*, c^*))$ is comparable to both the points $(S(\tau, \xi), S(\xi, \tau))$ and $(S(r, s), S(s, r))$. Let $c^* = c_0^*$ and $d^* = d_0^*$, then choose $c_1^*, d_1^* \in P$ such that $c_1^* = S(c_0^*, d_0^*)$ and $d_1^* = S(d_0^*, c_0^*)$. Then, by following the proof of Theorem 3.1, we can construct the sequences $\{c_n^*\}$, $\{d_n^*\}$ from $c_{n+1}^* = S(c_n^*, d_n^*)$ and $d_{n+1}^* = S(d_n^*, c_n^*)$ for all $n \in \mathbb{N}$. Similarly, construct the sequences $\{\tau_n\}$, $\{\xi_n\}$ and $\{r_n\}$, $\{s_n\}$ by setting $\tau = \tau_0$, $\xi = \xi_0$, $r = r_0$ and $s = s_0$. Then from Theorem 3.1, we get $\tau_n \to \tau = S(\tau, \xi)$, $\xi_n \to \xi = S(\xi, \tau)$ and $r_n \to r = S(r, s)$, $s_n \to s = S(s, r)$ for all $n \geq 1$. But $(\tau, \xi) = (S(\tau, \xi), S(\xi, \tau))$ and $(c_1^*, d_1^*) = (S(c_0^*, d_0^*), S(d_0^*, c_0^*))$ are comparable, implies that $\tau \geq c_1^*$ and $\xi \leq d_1^*$. Next we claim that (τ, ξ) and (c_n^*, d_n^*) are comparable, i.e., $\tau \geq c_n^*$ and $\xi \leq d_n^*$ for all n. Assume that for some $n \geq 0$ it is true, thus by strict mixed monotone property of S, we have $c_{n+1}^* = S(c_n^*, d_n^*) \leq S(\tau, \xi) = \tau$ and $d_{n+1}^* = S(d_n^*, c_n^*) \geq S(\xi, \tau) = \xi$, implies that $\tau \geq c_n^*$ and $\xi \leq d_n^*$ hold for all n.
By contraction condition (3.1),

$$\begin{aligned}
d(\tau, c_{n+1}^*) &= d(S(\tau, \xi), S(c_n^*, d_n^*)) \\
&\leq \alpha \frac{d(\tau, S(\tau, \xi))\,[1 + d(c_n^*, S(c_n^*, d_n^*))]}{1 + d(\tau, c_n^*)} \\
&\quad + \beta[d(\tau, S(\tau, \xi)) + d(c_n^*, S(c_n^*, d_n^*))] \\
&\quad + \gamma[d(\tau, S(c_n^*, d_n^*)) + d(c_n^*, S(\tau, \xi))] + \delta d(\tau, c_n^*),
\end{aligned}$$

from which we obtain

$$d(\tau, c_{n+1}^*) \leq \left(\frac{\beta + \gamma + \delta}{1 - \beta - \gamma}\right) d(\tau, c_n^*). \tag{3.15}$$

In a similar fashion, one can obtain that

$$d(\xi, d^*_{n+1}) \leq \left(\frac{\beta+\gamma+\delta}{1-\beta-\gamma}\right) d(\xi, d^*_n). \tag{3.16}$$

Suppose $D = \frac{\beta+\gamma+\delta}{1-\beta-\gamma} < 1$. Combining above inequalities (3.15) and (3.16), we get

$$\begin{aligned} d(\tau, c^*_{n+1}) + d(\xi, d^*_{n+1}) &\leq D\left[d(\tau, c^*_n) + d(\xi, d^*_n)\right] \\ &\leq D^2\left[d(\tau, c^*_{n-1}) + d(\xi, d^*_{n-1})\right] \\ &\ \ \vdots \\ &\leq D^n\left[d(\tau, c^*_0) + d(\xi, d^*_0)\right]. \end{aligned}$$

Thus, $d(\tau, c^*_{n+1}) + d(\xi, d^*_{n+1}) \to 0$ as $n \to \infty$. Consequently, we get $d(\tau, c^*_{n+1}) \to 0$ and $d(\xi, d^*_{n+1}) \to 0$ as $n \to \infty$.

Similarly, one can prove that $d(r, c^*_n) \to 0$ and $d(s, d^*_n) \to 0$ when $n \to \infty$. Using triangular inequality of d, we get

$$d(\tau, r) \leq d(\tau, c^*_n) + d(c^*_n, r) \quad \text{and} \quad d(\xi, s) \leq d(\xi, d^*_n) + d(d^*_n, s). \tag{3.17}$$

Taking $n \to \infty$ in (3.17), we obtain $d(\tau, r) = d(\xi, s) = 0$. Thus $\tau = r$ and $\xi = s$. □

Theorem 3.3. *If τ_0, ξ_0 are comparable under the hypotheses of Theorem 3.1, then S has a coupled fixed point in $P \times P$.*

Proof. By proof of Theorem 3.1, we have the sequences $\{\tau_n\}$, $\{\xi_n\}$ in P such that $\tau_n \to \tau$ and $\xi_n \to \xi$. Note that (τ, ξ) is a coupled fixed point of S.

Suppose $\tau_0 \leq \xi_0$, we have to prove that $\tau_n \leq \xi_n$, for all $n \geq 0$. Suppose it holds for some $n \geq 0$ and from strict monotone property of S, we obtain that $\tau_{n+1} = S(\tau_n, \xi_n) \leq S(\xi_n, \tau_n) = \xi_{n+1}$. Therefore, from (3.1),

$$\begin{aligned} d(\tau_n, \xi_n) &= d(S(\tau_{n-1}, \xi_{n-1}), S(\xi_{n-1}, \tau_{n-1})) \\ &\leq \alpha\ \frac{d(\tau_{n-1}, S(\tau_{n-1}, \xi_{n-1}))\left[1 + d(\xi_{n-1}, S(\xi_{n-1}, \tau_{n-1}))\right]}{1 + d(\tau_{n-1}, \xi_{n-1})} \\ &\quad + \beta[d(\tau_{n-1}, S(\tau_{n-1}, \xi_{n-1})) + d(\xi_{n-1}, S(\xi_{n-1}, \tau_{n-1}))] \\ &\quad + \gamma[d(\tau_{n-1}, S(\xi_{n-1}, \tau_{n-1})) + d(\xi_{n-1}, S(\tau_{n-1}, \xi_{n-1}))] \\ &\quad + \delta d(\tau_{n-1}, \xi_{n-1}). \end{aligned} \tag{3.18}$$

Taking $n \to \infty$ in (3.18), we get

$$d(\tau, \xi) \leq (2\gamma + \delta) d(\tau, \xi).$$

Thus, $d(\tau, \xi) = 0$ as $2\gamma + \delta < 1$. Hence, $S(\tau, \xi) = \tau = \xi = S(\xi, \tau)$.

If $\tau_0 \geq \xi_0$ then we have the result from the above argument. □

4. Applications

Applications of the main theorem and its consequences in terms of integral type contractions are carried out in the present section.

Let Ω denote the set of functions $\zeta : [0,+\infty) \to [0,+\infty)$ satisfying the following properties

(i). ζ is a Lebesgue-integrable function on every compact subset in $[0,+\infty)$ and;
(ii). $\int_0^\epsilon \zeta(l)dl > 0$, for all $\epsilon > 0$.

Theorem 4.1. *Suppose $(P,\le)$ is a partially ordered set and d is a complete metric on P. Let $S : P\times P \to P$ be a mapping which has the strict mixed monotone property on P such that*

$$\begin{aligned}\int_0^{d(S(\tau,\xi),S(\mu,\upsilon))}\kappa(l)dl \le{}& \alpha\int_0^{\frac{d(\tau,S(\tau,\xi))[1+d(\mu,S(\mu,\upsilon))]}{1+d(\tau,\mu)}}\kappa(l)dl\\ &+\beta\int_0^{d(\tau,S(\tau,\xi))+d(\mu,S(\mu,\upsilon))}\kappa(l)dl\\ &+\gamma\int_0^{d(\tau,S(\mu,\upsilon))+d(\mu,S(\tau,\xi))}\kappa(l)dl+\delta\int_0^{d(\tau,\mu)}\kappa(l)dl,\end{aligned} \tag{4.1}$$

for all $\tau,\xi,\mu,\upsilon \in P$ such that $\tau \ge \mu$ and $\xi \le \upsilon$, $\kappa \in \Omega$ and where $\alpha,\beta,\gamma,\delta \ge 0$ such that $0 \le \alpha + 2(\beta+\gamma) + \delta < 1$. Suppose that either S is continuous or P has an ordered complete property (OC), then S has a coupled fixed point $(\tau,\xi) \in P \times P$, if there exist two points $\tau_0,\xi_0 \in P$ with $S(\tau_0,\xi_0) > \tau_0$ and $S(\xi_0,\tau_0) < \xi_0$.

If $\beta=\gamma=0$; $\alpha=\gamma=0$; $\alpha=\beta=0$ and $\alpha=\beta=\gamma=0$ in Theorem 4.1, we get the succeeding results.

Theorem 4.2. *Suppose $(P,\le)$ is a partially ordered set and d is a complete metric on P. Let $S : P\times P \to P$ be a mapping which has the strict mixed monotone property on P such that*

$$\int_0^{d(S(\tau,\xi),S(\mu,\upsilon))}\kappa(l)dl \le \alpha\int_0^{\frac{d(\tau,S(\tau,\xi))[1+d(\mu,S(\mu,\upsilon))]}{1+d(\tau,\mu)}}\kappa(l)dl+\delta\int_0^{d(\tau,\mu)}\kappa(l)dl, \tag{4.2}$$

for all $\tau,\xi,\mu,\upsilon \in P$ such that $\tau \ge \mu$ and $\xi \le \upsilon$, $\kappa \in \Omega$ and where $\alpha,\delta \ge 0$ such that $0 \le \alpha+\delta < 1$. Suppose either S is continuous or P has an ordered complete property (OC), then S has a coupled fixed point $(\tau,\xi) \in P \times P$, if there exist two points $\tau_0,\xi_0 \in P$ with $S(\tau_0,\xi_0) > \tau_0$ and $S(\xi_0,\tau_0) < \xi_0$.

Theorem 4.3. *Suppose $(P, \leq)$ is a partially ordered metric space and d is a complete metric on P. Let $S : P \times P \to P$ be a mapping which has the strict mixed monotone property on P such that*

(4.3)
$$\int_0^{d(S(\tau,\xi),S(\mu,\upsilon))} \kappa(l)dl \leq \beta \int_0^{d(\tau,S(\tau,\xi))+d(\mu,S(\mu,\upsilon))} \kappa(l)dl + \delta \int_0^{d(\tau,\mu)} \kappa(l)dl,$$

for all $\tau, \xi, \mu, \upsilon \in P$ such that $\tau \geq \mu$ and $\xi \leq \upsilon$, $\kappa \in \Omega$ and where $\beta, \delta \geq 0$ such that $0 \leq 2\beta + \delta < 1$. Suppose either S is continuous or P has (OC) property, then S has a coupled fixed point $(\tau, \xi) \in P \times P$, if there exist two points $\tau_0, \xi_0 \in P$ with $S(\tau_0, \xi_0) > \tau_0$ and $S(\xi_0, \tau_0) < \xi_0$.

Theorem 4.4. *Suppose $(P, \leq)$ is a partially ordered set and d is a complete metric on P. Let $S : P \times P \to P$ be a mapping which has the strict mixed monotone property on P such that*

(4.4)
$$\int_0^{d(S(\tau,\xi),S(\mu,\upsilon))} \kappa(l)dl \leq \gamma \int_0^{d(\tau,S(\mu,\upsilon))+d(\mu,S(\tau,\xi))} \kappa(l)dl + \delta \int_0^{d(\tau,\mu)} \kappa(l)dl,$$

for all $\tau, \xi, \mu, \upsilon \in P$ such that $\tau \geq \mu$ and $\xi \leq \upsilon$, $\kappa \in \Omega$ and where $\gamma, \delta \geq 0$ such that $0 \leq 2\gamma + \delta < 1$. Suppose that either S is continuous or P has (OC) property, then S has a coupled fixed point $(\tau, \xi) \in P \times P$, if there exist two points $\tau_0, \xi_0 \in P$ with $S(\tau_0, \xi_0) > \tau_0$ and $S(\xi_0, \tau_0) < \xi_0$.

Theorem 4.5. *Let $(P, \leq)$ is a partially ordered set and d is a complete metric on P. Let $S : P \times P \to P$ be a mapping which has the strict mixed monotone property on P such that*

$$\int_0^{d(S(\tau,\xi),S(\mu,\upsilon))} \kappa(l)dl \leq \delta \int_0^{d(\tau,\mu)} \kappa(l)dl, \tag{4.5}$$

for all $\tau, \xi, \mu, \upsilon \in P$ such that $\tau \geq \mu$ and $\xi \leq \upsilon$, $\kappa \in \Omega$ and $\delta \geq 0$ such that $0 \leq \delta < 1$. Suppose that either S is continuous or P has (OC) property, then S has a coupled fixed point $(\tau, \xi) \in P \times P$, if there exist two points $\tau_0, \xi_0 \in P$ with $S(\tau_0, \xi_0) > \tau_0$ and $S(\xi_0, \tau_0) < \xi_0$.

We illustrate the usefulness of the obtained main Theorem 3.1 with the following example.

Example 4.6. Let $P = \{0, 1, 3\}$. Define a metric $d : P \times P \to [0, +\infty)$ by $d(\tau, \xi) = |\tau - \xi|$. Let $S : P \times P \to P$ be defined by

$$S(\tau, \xi) = \begin{cases} 0, & \text{if } (\tau, \xi) \in A, \\ 1, & \text{if } (\tau, \xi) \in B, \end{cases}$$

where $A = \{(\tau, \xi) | \tau = 3 \text{ and } \xi \in P\}$ and $B = \{(\tau, \xi) | \tau \in \{0, 1\} \text{ and } \xi \in P\}$. Then S has a coupled fixed point in P with usual order $\leq$.

From definition of d, $(P, d, \leq)$ is a partially ordered complete metric space, S is continuous and has strict mixed monotone property in P. Furthermore, there are some $\tau_0 = 0$, $\xi_0 = 3$ in P such that $\tau_0 = 0 < S(0,3) = 1$ and $\xi_0 = 3 > S(3,0) = 0$.

Now, consider the following possible cases for any $(\tau, \xi), (\mu, \upsilon) \in P$ such that $\tau \geq \mu$ and $\xi \leq \upsilon$.

Case 1. Suppose $(\tau, \xi), (\mu, \upsilon) \in A$ or B, then $d(S(\tau, \xi), S(\mu, \upsilon)) = 0$. Hence, the contraction condition in Theorem 3.1 holds.

Case 2. Suppose $(\tau, \xi) \in A$ and $(\mu, \upsilon) \in B$, then $d(S(\tau, \xi), S(\mu, \upsilon)) = 1$. Therefore, the contraction condition in Theorem 3.1 holds, because $d(\tau, \mu) \in \{2, 3\}$ and by selecting proper values of $\alpha, \beta, \gamma, \delta$ in $[0, 1)$ such that $0 \leq \alpha + 2(\beta + \gamma) + \delta < 1$.

For example, if $(\tau, \xi) = (3, 0)$ and $(\mu, \upsilon) = (1, 3)$, then

$$\begin{aligned} d(S(3,0), S(1,3)) = d(0,1) & \\ & = 1 \leq \frac{3\alpha}{3} + 3\beta + 3\gamma + 2\delta \\ & = \alpha \frac{d(3, S(3,0))\,[1 + d(1, S(1,3))]}{1 + d(3,1)} \\ & \quad + \beta\,[d(3, S(3,0)) + d(1, S(1,3))] \\ & \quad + \gamma\,[d(3, S(1,3)) + d(1, S(3,0))] + \delta d(3,1). \end{aligned}$$

Hence, $(1, 1)$ in P is a coupled fixed point for S.

Acknowledgment. The authors thank the editor and referees for their precise remarks which improved the presentation of the paper.

References

1. R.P. Agarwal, M.A. El-Gebeily and D. O'Regan, *Generalized contractions in partially ordered metric spaces*, Appl. Anal., 87 (2008), pp. 1-8.
2. J. Ahmad, M. Arshad and C. Vetro, *On a theorem of Khan in a generalized metric space*, Int. J. Anal., 2013, Article ID 852727, (2013).
3. I. Altun, B. Damjanovic and D. Djoric, *Fixed point and common fixed point theorems on ordered cone metric spaces*, Appl. Math. Lett., 23 (2010), pp. 310-316.
4. A. Amini-Harandi and H. Emami, *A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations*, Nonlinear Anal., Theory Methods Appl., 72 (2010), pp. 2238-2242.

5. M. Arshad, A. Azam and P. Vetro, *Some common fixed results in cone metric spaces*, Fixed Point Theory Appl., 2009, Article ID 493965 (2009).
6. M. Arshad, J. Ahmad and E. Karapinar, *Some common fixed point results in rectangular metric spaces*, Int. J. Anal., 2013, Article ID 307234 (2013).
7. M. Arshad, E. Karapinar and J. Ahmad, *Some unique fixed point theorems for rational contractions in partially ordered metric spaces*, Journal of Inequalities and Applications, 2013:248, 2013.
8. H. Aydi, E. Karapinar and W. Shatanawi, *Coupled fixed point results for (ψ, φ)-weakly contractive condition in ordered partial metric spaces*, Comput. Math. Appl., 62(12) (2011), pp. 4449-4460.
9. A. Azam, B. Fisher and M. Khan, *Common fixed point theorems in complex valued metric spaces*, Numer. Funct. Anal. Optim., 32(3) (2011), pp. 243–253.
10. I. Beg and A.R. Butt, *Fixed point for set-valued mappings satisfying an implicit relation in partially ordered metric spaces*, Nonlinear Anal., 71 (2009), pp. 3699-3704.
11. T.G. Bhaskar and V. Lakshmikantham, *Fixed point theory in partially ordered metric spaces and applications*, Nonlinear Anal., Theory Methods Appl., 65 (2006), pp. 1379-1393.
12. S. Chandok, T.D. Narang and M.A. Taoudi, *Some coupled fixed point theorems for mappings satisfying a generalized contractive condition of rational type*, Palestine Journal of Mathematics, 4(2) (2015), pp. 360-366.
13. B.S. Choudhury and A. Kundu, *A coupled coincidence point result in partially ordered metric spaces for compatible mappings*, Nonlinear Anal., Theory Methods Appl., 73 (2010), pp. 2524–2531.
14. L. Cirić, M.O. Olatinwo, D. Gopal and G. Akinbo, *Coupled fixed point theorems for mappings satisfying a contractive condition of rational type on a partially ordered metric space*, Advances in Fixed Point Theory, 2(1) (2012), pp. 1-8.
15. Z. Dricia, F.A. McRaeb and J.V. Devi, *Fixed-point theorems in partially ordered metric spaces for operators with PPF dependence*, Nonlinear Anal., Theory Methods Appl., 67 (2007), pp. 641-647.
16. M. Edelstein, *On fixed points and periodic points under contraction mappings*, J. Lond. Math. Soc., 37 (1962), pp. 74-79.
17. D. Gopal, M. Abbas, D. K. Patel and C. Vetro, *Fixed points of α-type F-contractive mappings with an application to nonlinear fractional differential equation*, Acta Math. Sci., 36(3) (2016), pp. 957-970.
18. G.C. Hardy and T. Rogers, *A generalization of fixed point theorem of S. Reich*, Can. Math. Bull., 16 (1973), pp. 201-206.

19. S. Hong, *Fixed points of multivalued operators in ordered metric spaces with applications*, Nonlinear Anal., Theory Methods Appl., 72 (2010), pp. 3929-3942.
20. R. Kannan, *Some results on fixed points-II*, Am. Math. Mon., 76 (1969), pp. 71-76.
21. E. Karapinar and N.V. Luong, *Quadruple fixed point theorems for nonlinear contractions*, Comput. Math. Appl., 64(6) (2012), pp. 1839-1848.
22. E. Karapinar, *Coupled fixed point on cone metric spaces*, Gazi Univ. J. Sci., 24(1) (2011), pp. 51-58.
23. P. Kumam, F. Rouzkard, M. Imdad and D. Gopal, *Fixed Point Theorems on Ordered Metric Spaces through a Rational Contraction*, Abstract and Applied Analysis, 2013, Article ID 206515, 9 pages http://dx.doi.org/10.1155/2013/206515.
24. V. Lakshmikantham and L.B. Cirić, *Couple fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., Theory Methods Appl., 70 (2009), pp. 4341-4349.
25. H. Lakzian, D. Gopal and W. Sintunavarat, *New fixed point results for mappings of contractive type with an application to nonlinear fractional differential equations*, J. Fixed Point Theory Appl., 18(2) (2016), pp. 251-266.
26. N.V. Luong and N.X. Thuan, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal., Theory Methods Appl., 74 (2011), pp. 983-992.
27. B. Monjardet, *Metrics on partially ordered sets-a survey*, Discrete Math., 35 (1981), pp. 173-184.
28. J.J. Nieto and R.R. Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, 22 (2005), pp. 223-239.
29. J.J. Nieto and R.R. Lopez, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equation*, Acta Math. Sin. Engl. Ser., 23(12) (2007), pp. 2205-2212.
30. J.J. Nieto, L. Pouso and R. Rodrguez-Lpez, *Fixed point theorems in ordered abstract spaces*, Proc. Am. Math. Soc., 135 (2007), pp. 2505-2517.
31. M. Ozturk and M. Basarir, *On some common fixed point theorems with rational expressions on cone metric spaces over a Banach algebra*, Hacet. J. Math. Stat., 41(2) (2012), pp. 211-222.
32. A.C.M. Ran and M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some application to matrix equations*, Proc. Am. Math. Soc., 132 (2004), pp. 1435-1443.

33. S. Reich, *Some remarks concerning contraction mappings*, Can. Math. Bull., 14 (1971), pp. 121-124.
34. F. Rouzkard and M. Imdad, *Some common fixed point theorems on complex valued metric spaces*, Comput. Math. Appl., (2012). doi:10.1016/j.camwa.2012.02.063.
35. B. Samet, *Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces*, Nonlinear Anal., 74(12) (2010), pp. 4508-4517.
36. P.L. Sharma and A.K. Yuel, *A unique fixed point theorem in metric space*, Bull. Cal. Math. Soc., 76 (1984), pp. 153-156.
37. S. Shukla, D. Gopal and J. Martinez-Moreno, *Fixed Points of set valued $\alpha - F$-contractions and its application to nonlinear integral equations*, Filomat, 31 (11) (2017), pp. 3377-3390.
38. D.R. Smart, *Fixed Point Theorems*, Cambridge University Press, Cambridge, 1974.
39. E.S. Wolk, *Continuous convergence in partially ordered sets*, Gen. Topol. Appl., 5 (1975), pp. 221-234.
40. C.S. Wong, *Common fixed points of two mappings*, Pac. J. Math., 48 (1973), pp. 299-312.
41. X. Zhang, *Fixed point theorems of multivalued monotone mappings in ordered metric spaces*, Appl. Math. Lett., 23 (2010), pp. 235-240.

---

[1] Department of Applied Mathematics, School of Applied Natural Sciences, Adama Science and Technology University, Post Box No.1888, Adama, Ethiopia.

E-mail address: seshu.namana@gmail.com

[2] Department of Mathematics, Vignan's Foundation for Science, Technology & Research, Vadlamudi-522213, Andhra Pradesh, India.

E-mail address: kalyani.namana@gmail.com